DOES DIFFEO-COVARIANT LOOP QUANTIZATION OF KANTOWSKI-SACHS EXIST?

AND DOES IT MATTER?

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Outline

- Introduction
- Kinematical Loop quantization of KS
- Residual diffeos
- Exact Residual-diffeo-cov. forces $\overline{\mu}$ forces wrong classical limit
- AOS has correct classical limit but is not residual-diffeo-cov.
- Summary, lessons?, next steps?

1. **Schwarzschild, inside of the horizon**:

   (a) All symmetries become spatial. r =const provides a foliation into homogeneous spatial slices.

   (b) Hamiltonian framework based on this foliation has finite DOF: Kantowski-Sachs framework. Can use LQC methods to quantize.

   (c) Diffeomorphisms not fixed by the imposition of symmetries: Residual diffeomorphisms

   (d) Loop quantization can answer if mass spectrum is discrete — relevant for 'remnant' after evaporation, possible dark matter candidate (Rovelli and Vidotto (2018)).

2. **Literature**

   (a) First loop quantization of KS (μ_o): Ashtekar, Bojowald (2006)

   (b) Some quantizations covariant under residual diffeomorphisms ($\overline{\mu}$): Chiou (2008); Joe, Singh (2015); Cortez, Cuervo, Morales-Técotl, Ruelas (2017). Most general: Bornhoeft, Dias, Engle (2024).

   (c) Only quantization to match classical theory for all low space-time curvature regimes: Ashtekar, Olmedo, and Singh (2018) (AOS)

Introduction

3. **Definitions**

   (a) "Quantization": A quantum theory that matches a given classical theory where gauge-invariant quantities with dimensions of action are large compared to $\hbar$. $\Rightarrow$ In QG, must match when space-time curv. scalars are small compared to $1/\ell_P^4$.

   (b) "Loop Quantization":

      i. A quantization of a theory of connections in which the basic operators are holonomies and conjugate fluxes. *(Allows for representations in which diffeos act unitarily.)*

      ii. In KS, leads to $\mathcal{H}^2_{\text{Bohr}}$ as kin. Hilbert space, so that $\hat{H}$ must be densely defined on $\mathcal{H}^2_{\text{Bohr}}$.

4. **Conjecture: There exists no residual-diffeo-covariant quantization of KS on $\mathcal{H}^2_{\text{Bohr}}$.** That is, there exists no $\hat{H}$ that simultaneously

   (a) is densely defined on $\mathcal{H}^2_{\text{Bohr}}$

   (b) is exactly covariant under residual diffeos

   (c) matches classical theory in regimes of low space-time curvature.

Introduction

5. **Significance?**

   (a) Can argue: Core problem is that KS is based on a foliation that becomes null at the horizon $\rightarrow$ **is no longer Cauchy, no longer appropriate for canonical quantization**.

   (b) However, the problem occurs also **close** to the horizon where leaves are **still space-like**! So why a problem there?

   (c) By using extended phase space techniques, one can define a quantum KS Hamiltonian on an **extension** of $\mathcal{H}^2_{\text{Bohr}}$ — the EMM framework (Elizaga Navascués, Mena Marugán, and Mínguez-Sánchez (2023)).

      i. No similar technique is used in full LQG — **does it still tell us anything about BH's in LQG**, even heuristically?

      ii. **Can it be modified to be also covariant under residual diffeos?**

   (d) **Can also take attitude: Forget KS**, and impose only spherical symmetry, which allows space-like slices that cross the horizon. Added benefit of inclusion of collapse and Hawking radiation when coupled to matter.

Introduction

BACKGROUND:

Kantowski-Sachs and its kinematical Loop quantization

1. **Kantowski-Sachs in Ashtekar-Barbero variables**

   (a) Canonical 3-slice $M \cong S^2 \times \mathbb{R}$, coordinates (θ, ϕ) and x respectively on S^2 and $\mathbb{R}$ factors, fiducial cell $S^2 \times I \subset M$ with coordinate length L_o in x direction.

   (b) $A = A^i\tau_i = -b\sin\theta\tau_1 d\phi + (\cos\theta\tau_3 - b\tau_2)d\theta + \frac{c}{L_o}\tau_3 \qquad \vec{E} = E^i\tau_i = -\frac{p_b}{L_o}\tau_1\vec{\phi} + \frac{p_b}{L_o}\sin\theta\tau_2\vec{\theta} + p_c\sin\theta\tau_3\vec{x}$

   $$\Rightarrow \quad ds^2 = -N^2 d\tau^2 + \frac{p_b^2}{|p_c|L_o^2}dx^2 + |p_c|d\Omega^2$$

   $$\Rightarrow \quad ds^2 = -\left(\frac{2m}{\tau} - 1\right)^{-1} d\tau^2 + \left(\frac{2m}{\tau} - 1\right) dx^2 + \tau^2 d\Omega^2 \qquad \text{for appropriate lapse.}$$

   N.B. $p_b = 0$ **corresponds to the horizon**, where 3-metric becomes null.

   (c) $\{b, p_b\} = G\gamma, \qquad \{c, p_c\} = 2G\gamma, \qquad H_{cl}[N] = -\frac{N}{2G\gamma^2}\frac{b\,\mathrm{sgn}p_c}{\sqrt{|p_c|}}\left(p_b\left(b + \frac{\gamma^2}{b}\right) + 2cp_c\right).$

2. **Kinematical Quantization**

   (a) Holonomies of A along $\vec{x}$ and along geodesics within S^2 are linear combinations of $e^{i\mu_j b}, e^{i\lambda_j c}$ for appropriate $\mu_j, \lambda_j \in \mathbb{R}$.

   (b) $\psi(b, c) = \sum_j \psi_j e^{i(\mu_j b + \lambda_j c)} \qquad \mathcal{H}^2_{\text{Bohr}} = \left\{\psi \text{ s.t. } ||\psi||^2 = \langle\psi, \psi\rangle = \sum_j |\psi_j|^2 < \infty\right\}$

Background

(c) $\widehat{e^{i\mu b}}, \widehat{e^{i\mu c}}$ well-defined on $\mathcal{H}^2_{\text{Bohr}}$ by multiplication. Can be generalized to let μ, λ depend on (p_b, p_c).

3. **Effective Hamiltonians**

   (a) Given $\hat{H}$, $H_{\text{eff}}(b, c, p_b, p_c)$ is the **leading order term in the asymptotic expansion** of $\langle \psi_{b,c,p_b,p_c} | \hat{H} | \psi_{b,c,p_b,p_c} \rangle$ **for large fiducial cell volume** $V = |p_c^2 p_b|^{1/2}$, where ψ_{b,c,p_b,p_c} is a family of coherent states approximately 'dynamical', i.e., closed under the quantum evolution generated by $\hat{H}$.
   (Taveras (2008), Bojowald and Skirzewski (2006), Ashtekar and Schilling (1997))

   (b) H_{eff} is usually the pre-image of $\hat{H}$ under an appropriately chosen quantization map.

   (c) For most of this talk **we stay at effective level unless otherwise stated**:

      i. All predictions from Loop quantized symmetry reduced models use H_{eff}, **not** $\hat{H}$.
      ii. Can even argue that effective theory is **generally exact** (Rovelli and Wilson-Ewing (2014)).
      iii. **Most** Loop Quantum KS proposals are **only at the effective level**.
      iv. Residual Diffeos are unambiguous and simpler at the effective level.

Background

RESIDUAL DIFFEOMORPHISMS

in Kantowski-Sachs

1. **Definition, solution, and resulting flow on phase space**

$$\Phi^s_{\vec{v}} \triangleright \left(\left(A^i_a, E^a_i\right)(b, c, p_b, p_c)\right) = \left(A^i_a, E^a_i\right)(b(s), c(s), p_b(s), p_c(s))$$

$$\Rightarrow \quad \mathcal{L}_{\vec{v}} A^i_a(s) = \dot{A}^i_a = \frac{\partial A^i_a}{\partial b}\dot{b}(s) + \frac{\partial A^i_a}{\partial c}\dot{c}(s) \quad \mathcal{L}_{\vec{v}} E^a_i(s) = \dot{E}^a_i = \frac{\partial E^a_i}{\partial p_b}\dot{p}_b(s) + \frac{\partial E^a_i}{\partial p_c}\dot{p}_c(s)$$

$$\Rightarrow \quad \vec{v} = \xi_\phi \vec{\phi} + (\xi_x + \kappa_x x)\vec{x}$$; only κ_x affects flow in phase space, and can be set to 1 w.l.o.g.

$$\Rightarrow \quad \boxed{\dot{b} = 0, \qquad \dot{p}_b = p_b, \qquad \dot{c} = c, \qquad \dot{p}_c = 0}$$

2. Even though non-canonical, can also be extended to quantum theory! Is not the focus of this talk — see Bornhoeft, Dias, and Engle (2024).

Residual Diffeomorphisms

5. **Contrast with passive equivalent**

(a) Up until now I have been presenting **active residual diffeos**: Flows on phase space.

(b) Can also speak of 'passive residual diffeos':

i. Rescaling of fiducial cell, or

ii. Rescaling of coordinates or fiducial triad/connection on the cell .

(c) Fiducial structures are the foundation on which the definitions of the model are built: They are part of the framework defining the model. Passive diffeos are thus flows in the 'space of frameworks' for the model. **One has the free choice of which structures the diffeomorphisms act on, as well as how the structures (such as the cell) are used to define the phase space variables.**

(d) Active residual diffeos are more conceptually clear: **They are flows on phase space within a single framework**. No wiggle room.

(e) **From now on in this talk** (as before this slide), **'residual diffeo' will always refer to 'active residual diffeo'.**

Residual Diffeomorphisms

PRESERVATION OF BOHR+ RES.-DIFFEO-COV.:

Most general family of Quantum Hamiltonians

(Bornhoeft, Dias, Engle (2024))

1. **Heuristically**

(a) Preservation of Bohr $\Rightarrow$ $\boxed{H_{\text{eff}} = \sum_{k=1}^{M} f_k(p_b, p_c) e^{\delta_b^k b} e^{\delta_c^k c} \text{ with } \delta_b^k, \delta_c^k \text{ functions of } p_b, p_c \text{ only}}$.

(b) Covariance under Residual-Diffeos: Arguments of exponentials must be *invariant*:

$$0 = \frac{d}{dt}(\delta_b b) = \dot{\delta}_b b = \left(\frac{\partial \delta_b}{\partial p_b}\dot{p}_b + \frac{\partial \delta_b}{\partial p_c}\dot{p}_c\right) b = \frac{\partial \delta_b}{\partial p_b} p_b b \qquad \Rightarrow \qquad \frac{\partial \delta_b}{\partial p_b} = 0 \qquad \Rightarrow \qquad \boxed{\delta_b^k = A_k(p_c)}$$

$$\begin{aligned} 0 &= \frac{d}{dt}(\delta_c c) = \dot{\delta}_c c + \delta_c \dot{c} = \left(\frac{\partial \delta_c}{\partial p_b}\dot{p}_b + \frac{\partial \delta_c}{\partial p_c}\dot{p}_c\right) c + \delta_c c \\ &= \frac{\partial \delta_c}{\partial p_b} p_b c + \delta_c c = \left(\frac{\partial \delta_c}{\partial p_b}|p_b| + \delta_c(\text{sgn} p_b)\right)(\text{sgn} p_b) c \\ &= \frac{\partial}{\partial p_b}(\delta_c |p_b|)(\text{sgn} p_b) c \qquad \Rightarrow \qquad \frac{\partial}{\partial p_b}(\delta_c |p_b|) = 0 \qquad \Rightarrow \qquad \boxed{\delta_c^k = \frac{B_k(p_c)}{|p_b|}} \end{aligned}$$

Bohr + Residual-Diffeo-Covariance

2. **General result** (Bornhoeft, Dias, Engle (2024))

(a) Imposing **preservation of Bohr, covariance under residual diffeos**, covariance under discrete automorphisms (parities), and that $A_k(p_c), B_k(p_c)$ be even ("metric loop assumption") we get:

$$H_{\text{eff}} = |p_b|^{n+1} a_0 \text{sgn}(p_b p_c) \; + \; |p_b|^{n+1} \sum_{k=1}^{M} \Bigg(a_k \text{sgn}(p_b p_c) \cos(A_k b) \cos\left(B_k \frac{c}{|p_b|}\right) + b_k \text{sgn}(p_b) \cos(A_k b) \sin\left(B_k \frac{c}{|p_b|}\right) + c_k \text{sgn}(p_c) \sin(A_k b) \cos\left(B_k \frac{c}{|p_b|}\right) + d_k \sin(A_k b) \sin\left(B_k \frac{c}{|p_b|}\right) \Bigg)$$

with $a_k, b_k, c_k, d_k, A_k, B_k$ (arbitrary) even functions of p_c alone.

(b) Natural quantum notion of residual diffeos also fixes ordering ambiguity of underlying operator.

Bohr + Residual-Diffeo-Covariance

3. **KEY POINT:**

(a) **At horizon**, $p_b \to 0$, so $\delta_c^k = \frac{A_k(p_c)}{|p_b|} \to \infty$ as well.

(b) But, for large mass BH's, space-time curvature is low at horizon, so H_{eff} should approach H_{cl}. But this is only possible if $\delta_b^k \to 0$ and $\delta_c^k \to 0$.

(c) CONTRADICTION: Necessarily incorrect classical limit.

Bohr + Residual-Diffeo-Covariance

CORRECT CLASSICAL LIMIT

Only model matching classical theory at low curvatures is Ashtekar-Olmedo-Singh (2018, 2024).

Correct Classical Limit

1. **AOS solves the problem of δ_c diverging at the horizon** quite directly: They require δ_b, δ_c to be **Dirac observables**, thus independent of time — in this case, independent of r —**thus preventing divergence at** $r = 2m$.

2. **Consequence of being Dirac observables: δ_b and δ_c can no longer be pure momentum.**

   (a) thus $e^{i\delta_b b}$, $e^{i\delta_c c}$ no longer have well-defined operator analogues on $\mathcal{H}^2_{\text{Bohr}}$ — **No underlying 'loop quantization' in usual sense**

   (b) **Way around this**: Extend phase space so that δ_b, δ_c, together with new momenta for each, are **added degrees of freedom**. Then remove these degrees of freedom with **added first class constraints** imposing their relation to the other variables.

      i. was suggested in Ashtekar, Olmedo, Singh (2018)

      ii. was carried out in EMM framework (Elizaga Navescués, Mena Marugán, Mínguez-Sánchez (2023))

   (c) But there is no analogue of such a procedure used in full LQG. **How much can we trust this to be a model of predictions from full LQG?** ***Is the main point of LQC and related models***, as compared to non-loop quantizations of symmetry reduced models.

Correct Classical Limit

3. **Additional assumption of AOS**: $\boxed{\delta_b = \delta_b[b, p_b] \text{ and } \delta_c = \delta_c[c, p_c]}$.

   (a) Allows dynamics for (b, p_b) and (c, p_c) to decouple, allowing **exact solubility of the model**.

   (b) But at cost of **non-covariance under residual diffeos:** *Suppose b.w.o.c. that H_{eff} is also exactly covariant under residual diffeos.* Then arguments of the exponents must be *invariant*:

$$0 = \frac{d}{dt}(\delta_c c) = \dot{\delta}_c c + \delta_c \dot{c} = \left(\frac{\partial \delta_c}{\partial c}\dot{c} + \frac{\partial \delta_c}{\partial p_c}\dot{p}_c\right) c + \delta_c c = \left(\frac{\partial \delta_c}{\partial c} c + \delta_c\right) c = \frac{\partial}{\partial c}(\delta_c c)\, c$$

$$\Rightarrow \qquad \frac{\partial}{\partial c}(\delta_c c) = 0 \qquad \Rightarrow \qquad \delta_c = \frac{A(p_c)}{c}$$

   so that $e^{i\delta_c c} = e^{iA(p_c)}$, and hence H_{eff}, is independent of c, so that it equals H_{cl} in no limit $\rightarrow\leftarrow$.

   (c) **However: This assumption is motivated by mathematical convenience, not physics.**
   Question: Can we relax the above assumption and find another choice of δ_b and δ_c that are **Dirac observables *and* yield diffeo-covariant effective dynamics?**

SUMMARY,

lessons?, and next steps?

Summary

1. **Preservation of Bohr + Residual Diffeo Covariance** forces a $\overline{\mu}$-type scheme, which in turn **prevents a correct classical limit** at the horizon, where curvature is small.
2. Ashtekar-Olmedo-Singh (AOS)
   (a) achieves **correct classical limit** at the horizon,
   (b) but **neither** has an underlying constraint operator on $\mathcal{H}^2_{\text{Bohr}}$ **nor** is exactly covariant under residual diffeos.
   (c) Instead of defining $\hat{H}$ on $\mathcal{H}^2_{\text{Bohr}}$, definition on *extension* of $\mathcal{H}^2_{\text{Bohr}}$ is possible — EMM framework. **Connection to full LQG, however, is no longer clear**.

Lessons? Next steps?

1. Problem is arguably that foliation becomes null at the horizon, a problem which does not occur in the spherically symmetric case which is more general anyway. **Just forget KS and use spherically symmetric models?**

2. Perhaps get insights into the KS case by **embedding into a spherically symmetric loop quantization**?

3. Or maybe the extended phase space / extended Hilbert space approach worked out by EMM for AOS has an analogue in the full theory?

4. **Can δ_b and δ_c in EMM/AOS be modified to be exactly residual diffeo covariant?**

   Would seem important: Diffeos are the central symmetry determining GR.

THANK YOU

Questions?
Remarks?
Answers?