

Shell-crossing singularities and shock formation during gravitational collapse

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FAU² Focus Workshop on quantum black holes and the relation to asymptotic infinity

Reminder: Motivation

Black holes are one of the few places quantum gravity effects are expected to be important.

There are two main problems in black hole physics that any successful theory of quantum gravity should be able to solve:

- **Singularity**

Black hole space-times in general relativity are singular. Can quantum gravity resolve the singularity?

- **Information loss problem**

Hawking radiation is thermal. If a black hole fully evaporates, an initial pure state seems to evolve to a thermal state. Can quantum gravity somehow restore unitarity?

Black Hole Collapse

Goal: study quantum gravity effects in black holes, according to loop quantum gravity, starting from the initial collapse.

There are (at least) two good reasons to study black hole collapse:

1. **How is the singularity avoided?**

This is presumably a dynamical process, so we should study space-times where (classically) the singularity forms dynamically.

2. **The role of matter**

Classically, vacuum is often thought sufficient since matter from the collapse will eventually hit the singularity and 'disappear'.

But what if there is no singularity?

→ During collapse an inner horizon forms; this is missed in vacuum.

For these reasons, we studied the Lemaître-Tolman-Bondi (LTB) space-time: spherically symmetric, with a dust field, building on a lot of earlier work studying black holes in LQG.

Main Steps

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4. Do a loop quantization at each point along the radial lattice;
5. Extract effective dynamics from the quantum theory, and take the continuum limit.

Loop Quantization: Holonomies

A key step in the loop quantization is to calculate holonomies of the Ashtekar-Barbero connection.

Due to spherical symmetry, holonomies along an edge on a great circle of a sphere at fixed radius $x = x_0$ do not depend on the angular coordinates. The only input required is the coordinate (angle) length $\bar{\mu}$ of the holonomy,

$$h_\theta(\bar{\mu}) = \mathcal{P} \exp \int_0^{\bar{\mu}} d\theta \, b(x_0) \tau_2 = \cos\left(\frac{\bar{\mu}b}{2}\right) \mathbb{I} + 2 \sin\left(\frac{\bar{\mu}b}{2}\right) \tau_2.$$

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In LQC, the fundamental discreteness of LQG geometric operators motivates an operator for (components of) the field strength to be based on holonomies with a small but finite length of order $\sim \ell_{\text{Pl}}$ (not an infinitesimal length).

Loop Quantization: Fixing $\bar{\mu}$

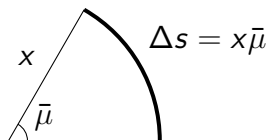
$\bar{\mu}$ is the coordinate length of the edge in the θ direction, and we want the physical length to be ℓ_{Pl} [Ashtekar, Pawłowski, Singh, 2005].

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In this case, the path for the holonomy is an arc with angle $\bar{\mu}$ of a great circle of radius x . To get the physical length $\Delta s = \ell_{\text{Pl}}$ requires

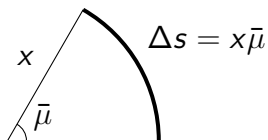
$$\bar{\mu} = \frac{\ell_{\text{Pl}}}{x}.$$

For LQC improved dynamics in spherical symmetry, see also [Boehmer, Vandersloot, 2007; Chiou, Ni, Tang, 2012; Gambini, Olmedo, Pullin, 2020].

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This procedure fails for paths that are null, but the relevant paths here are always spacelike for any x .

Effective Dynamics

In homogeneous cosmology, the quantum dynamics of sharply-peaked states is very well approximated by a set of LQC effective dynamics defined on the classical phase space that include \hbar corrections [Ashtekar,

Pawłowski, Singh, 2005; Taveras, 2008; Rovelli, WE, 2013; Bojowald, Brahma, 2015].

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Effective dynamics can also be derived for LTB space-times, it is generally expected that they will again be a good approximation for sharply-peaked states so long as we don't probe Planck-length scales

[Zhang, 2021].

Planck curvature scales are ok: the Kretschmann invariant

$$K \sim M^2/x^6, \text{ so Planck curvature arises at } x \sim (\ell_{\text{Pl}}^2 M)^{1/3} \gg \ell_{\text{Pl}}.$$

Effective Equations

The effective dynamics capture leading-order loop quantum gravity effects, and are generated by a Hamiltonian (density)

$$\mathcal{H}^{\text{eff}} = -\frac{1}{2G} \left[\frac{E^b}{\ell_{\text{Pl}}^2 x} \partial_x \left(x^3 \sin^2 \frac{\ell_{\text{Pl}} b}{x} \right) + \frac{x}{E^b} + \frac{E^b}{x} \right].$$

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$$ds^2 = -dt^2 + \mathbf{1} (dx + N^x dt)^2 + x^2 d\Omega^2.$$

There remains one degree of freedom b (the connection component in angular directions) that satisfies the non-linear equation of motion

$$\dot{b} + \frac{1}{2\ell_{\text{Pl}}^2 x} \partial_x \left(x^3 \sin^2 \frac{\ell_{\text{Pl}} b}{x} \right) = 0.$$

To find solutions to non-linear wave equations, it is typically necessary to allow weak solutions.

Weak Solutions

Weak solutions are not differentiable, so they cannot solve a differential equation—but they can solve an integral form of the equation of motion. For the conservation equation

$$\dot{u} + \partial_x[f(u)] = 0,$$

weak solutions $u(x, t)$ satisfy

$$\int_{x_1}^{x_2} dx u \Big|_{t=t_1}^{t=t_2} + \int_{t_1}^{t_2} dt f(u) \Big|_{x=x_1}^{x=x_2} = 0,$$

for all x_1, x_2, t_1, t_2 .

When the weak solution is discontinuous, the discontinuity is called a shock wave.

Weak Solutions in General Relativity

Examples of weak solutions in general relativity are thin shell solutions obtained using Israel's junction conditions [Israel, 1966], and the Dray-'t Hooft shock wave [Dray, 't Hooft, 1985].

It has also been argued that weak solutions should be considered for the LTB space-time in general relativity [Nolan, 2003; Lasky, Lun, Burston, 2006].

We will allow for weak solutions in the LQC effective dynamics for LTB space-times.

- Analytical methods are useful for simple configurations.
- Otherwise, numerics are typically necessary, for example the standard Godunov algorithm.

Method of Characteristics

The method of characteristics can be used to solve a wave equation that is linear in derivatives. For

$$\dot{u} + v(u, x) \partial_x u = 0,$$

introduce parametrized curves $t(s)$ and $x(s)$ in the $t - x$ plane, then

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This is a change of coordinates from (x, t) to (X, s) , where X labels the characteristic curves along which u is constant: X is a comoving coordinate.

When Characteristics Fail

The method of characteristics works up until characteristic curves cross.

Suppose at t_0 there are two points x_1, x_2 such that $u(x_1, t_0) \neq u(x_2, t_0)$. If the two characteristic curves passing through (x_1, t_0) and (x_2, t_0) later intersect at (x_3, t_{int}) , the method of characteristics predicts that $u(x_3, t_{int}) = u(x_1, t_0)$ and also that $u(x_3, t_{int}) = u(x_2, t_0)$, which is a contradiction.

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It can be verified that at this point the Jacobian for the coordinate transformation from $(x, t) \rightarrow (X, s)$ vanishes: the coordinate transformation is not valid. (For the previous example, the condition $J \neq 0$ is $\partial_x x \neq 0$.)

At such a point, it is necessary to return to (x, t) coordinates and look for weak solutions. The crossing of characteristics signals the formation of a shock—a discontinuity in u .

Comoving Coordinates for LTB

From the generalized Painlevé-Gullstrand coordinates for LTB, comoving coordinates can be introduced most directly by looking at the dust energy density

$$\rho = \frac{1}{8\pi G x^2} \partial_x \left(\frac{x^3}{\ell_{\text{Pl}}^2} \sin^2 \frac{\ell_{\text{Pl}} b}{x} \right),$$

which motivates the definition of the gravitational mass

$$m(x) = 4\pi \int_0^x d\tilde{x} \tilde{x}^2 \rho(\tilde{x}).$$

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The equation of motion for m is

$$\dot{m} + \frac{x}{\ell_{\text{Pl}}} \sin \frac{\ell_{\text{Pl}} b}{x} \cos \frac{\ell_{\text{Pl}} b}{x} \partial_x m = 0,$$

so characteristic curves for this PDE satisfy

$$\frac{dx}{ds} = \frac{x}{\ell_{\text{Pl}}} \sin \frac{\ell_{\text{Pl}} b}{x} \cos \frac{\ell_{\text{Pl}} b}{x}.$$

Lemaître Coordinates

The new coordinates are given by $X = x_0$, and the metric in these coordinates is

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The equation for m has trivialized to $\partial_t m = 0$, and the only truly dynamical equation is

$$\frac{dx}{dt} = \frac{x}{\ell_{\text{Pl}}} \sin \frac{\ell_{\text{Pl}} b}{x} \cos \frac{\ell_{\text{Pl}} b}{x},$$

which can be rewritten in a more familiar LQC-like form

$$\frac{1}{x^2} \cdot \left(\frac{dx}{dt} \right)^2 = \frac{2Gm(x)}{x^3} \left(1 - \frac{2Gm(x)\ell_{\text{Pl}}^2}{x^3} \right).$$

This can also be derived by working in Lemaître coordinates from the start [Giesel, Liu, Rullit, Singh, Weigl, 2023].

Shell-Crossings

In the language of general relativity, the crossing of characteristics is a shell-crossing.

A shell-crossing can signal the failure of a system of coordinates, or a true physical singularity. If it is only a coordinate singularity, then we should just use different coordinates—this is the case for

Oppenheimer-Snyder collapse [Fazzini, Rovelli, Soltani, 2023; Giesel, Liu, Singh, Weigl, 2023].

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In terms of the comoving coordinates,

$$\rho = \frac{\partial_X m}{4\pi X^2 \partial_X X}.$$

A shell-crossing happens when $\partial_X X = 0$.

⇒ This is a shell-crossing singularity when in addition $\partial_X m \neq 0$ so that ρ diverges (curvature scalars diverge too in this case).

A Simple Example

Consider the piecewise-linear initial data

$$\rho = \begin{cases} \rho_o & \text{for } X < X_1, \\ \rho_o \left(1 - \frac{X-X_1}{X_2-X_1}\right) & \text{for } X_1 < X < X_2, \\ 0 & \text{for } X > X_2. \end{cases}$$

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Focusing on $X_1 < X < X_2$, clearly $\rho(X) \neq 0$ and so $\partial_X m \neq 0$. If for any t the condition $\partial_X x = 0$ holds for $X_1 < X < X_2$, then there is a shell-crossing singularity.

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This is a direct calculation, one simply needs to solve the ODE for $x(t)$ with the initial condition $x(t_0) = X_0$.

Result: There arises a shell-crossing singularity for any $\rho_o > 0$, and any $0 < X_1 < X_2$.

Shell-Crossing Singularities are Generic

Theorem: [Fazzini, Husain, WE, 2024]

In the marginally bound case, for all initial conditions for the dust field such that the energy density $\rho \geq 0$ is continuous, of compact support, and such that $\int_0^a dx x^2 \rho > 0$ for some a , a shell-crossing singularity will form.

- This happens at the latest $\frac{2}{3}t_{PI}$ after the bounce,
- The occurrence of a shell-crossing singularity signals the formation of a shock,
- A shell-crossing singularity is a weak singularity, and it is possible to evolve past it in terms of a weak solution [Husain, Kelly, Santacruz, WE, 2022].

Weak Singularities in LQC

A shell-crossing singularity is a weak singularity: although curvature invariants diverge, nearby test particles are neither crushed together nor pushed infinitely far apart.

It has already been shown for homogeneous and isotropic LQC that weak singularities are not resolved [Singh, 2009]; it is not surprising that weak singularities persist in other spacetimes as well.

⇒ It will be necessary to allow for weak solutions, and in particular shock waves, in effective models for quantum gravity.

Coarse-Graining and Shocks

When studying collective phenomena, whether gases or fluids composed of many molecules, or regions of spacetime composed of many quanta of geometry as predicted by LQG, it is typically necessary to perform some averaging or coarse-graining procedure.

Coarse-graining presupposes the presence of many constituents, and the same averaging procedure will not necessarily give the same result for neighbouring regions: there can be discontinuities in coarse-grained quantities.

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1. Shocks in quantum gravity are not surprising if spacetime is constituted of many quanta of geometry.
2. The existence of shocks in classical GR [Hellaby, Lake, 1985; Nolan, 2003; Lasky, Lun, Burston, 2007] in fact suggests that spacetime may emerge from a coarse-graining procedure.

The Physics of Shocks in Gravity

1. Weak solutions are not unique: additional input is required to select a preferred solution.

⇒ This can be done by selecting a preferred field variable, say (A_a^i, E_i^a) .

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This requires input from the microscopic theory: this is an opportunity for quantum gravity.

2. Can we trust the effective equations when describing shocks? This requires a more detailed analysis, but the (integrated) equations of fluid dynamics remain valid in the presence of a shock—information about quantum mechanics or the molecular constituents of the fluid is not needed.

The same could be true for quantum gravity.

Summary

- ⇒ Non-marginally bound solutions: similar results [Cipriani, Fazzini, WE, 2024].
- Partial convergence between different approaches to the effective LQG treatment of LTB spacetimes;
 - Crushing singularities are resolved by a non-singular bounce;
 - Weak shell-crossing singularities persist and are guaranteed for $\rho(t_0)$ that is continuous and of compact support;
 - Shocks can be expected to arise for any theory of quantum gravity that predicts spacetime emerges through a coarse-graining of many microscopic degrees of freedom;
 - Shocks provide a window onto quantum gravity: an extra input needed to select the correct weak solution.
 - This input determines the dynamics of shocks [V. Husain's talk].

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Thank you for your attention!