Axial perturbations in Hybrid Loop Quantum Cosmology

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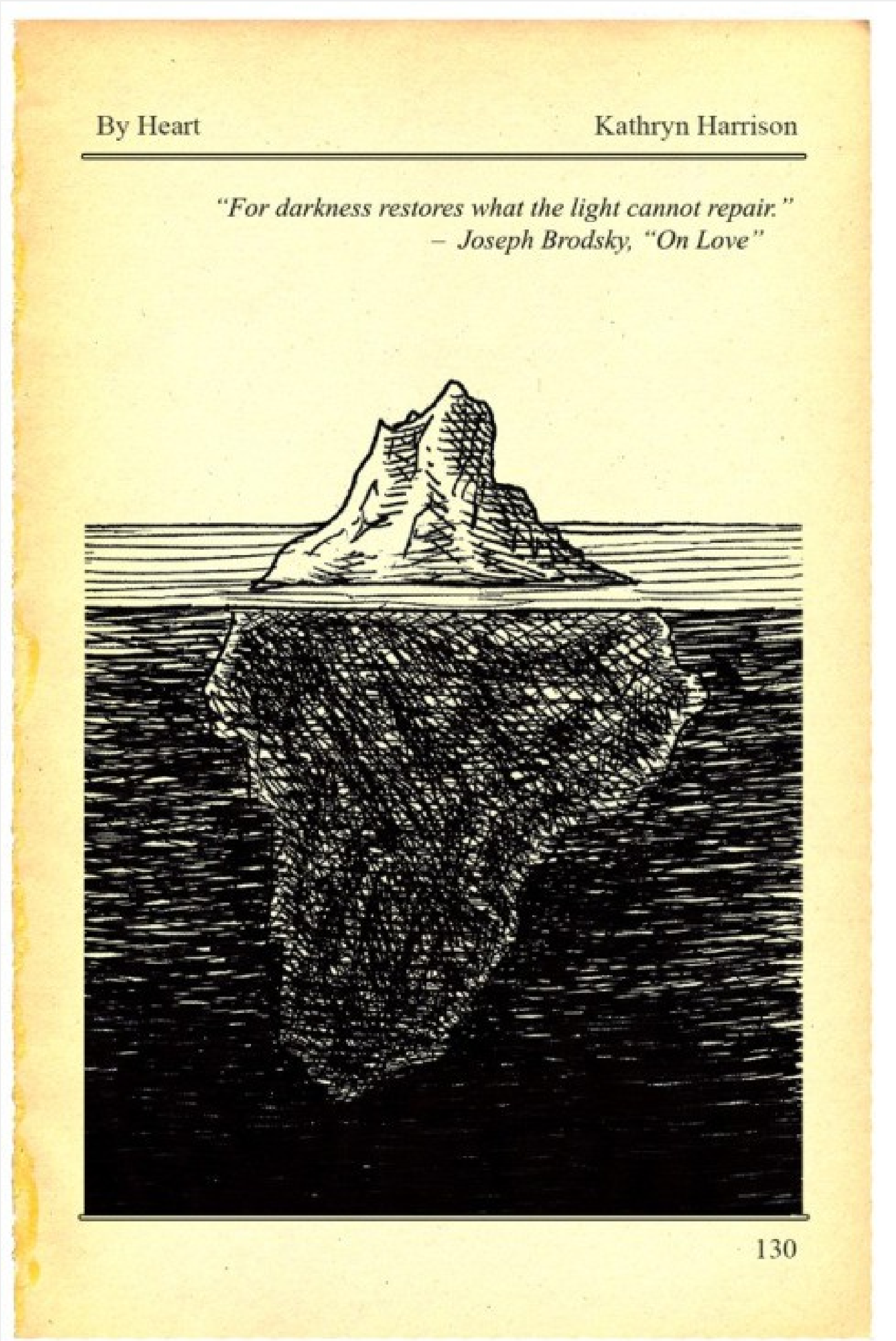
Introduction

- Many efforts have been made to apply LQG to **black hole** spacetimes.
- The interest has been revitalized by Ashtekar, Olmedo, and Singh (AOS), who introduced a model for nonrotating black holes.
- The **interior** can be described as a Kantowski-Sachs (KS) spacetime.
- To go beyond these simple scenarios, we consider perturbation theory.
- We truncate our perturbations at **second order in the action**.
- Physical perturbations correspond to perturbative gauge invariants.
- We will construct a Hamiltonian formulation for this perturbed system and proceed to its hybrid quantization within LQC.



By Heart — Kathryn Harrison

"For darkness restores what the light cannot repair."
– Joseph Brodsky, "On Love"

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The background

- Using triad variables, the metric in the interior region is of KS form:

$$ds^2 = -\frac{V^2 \underline{N}^2(\tau)}{4\pi^2} d\tau^2 + \frac{p_b^2(\tau)}{|p_c(\tau)|} dx^2 + |p_c(\tau)| \left(d\theta^2 + \sin^2\theta \, d\phi^2\right).$$

- The spatial sections have a volume $V = 4\pi p_b \sqrt{|p_c|}.$

- The geometry has **two canonical pairs** of degrees of freedom, with connection variables such that $\{b, p_b\} = \gamma, \ \{c, p_c\} = 2\gamma.$

- We include a (homogeneous) scalar field: $\{\Phi, \Pi_\Phi\} = 1.$

- This KS background is subject ONLY to the **Hamiltonian** constraint

$$\underline{N} H_{KS} = -\underline{N}\left(\Omega_b^2 + 2\Omega_b \Omega_c + p_b^2 - 4\Pi_\Phi^2\right), \quad \Omega_j = \frac{j\, p_j}{\gamma}, \quad j = b, c.$$

Perturbations

- We consider compact sections with the topology of $S^1 \times S^2$. Then, zero-modes are isolated and can be treated exactly.

- We expand our perturbations in products of REAL Fourier modes and spherical harmonics.

- Spherical harmonics split in polar and axial depending on their behavior under parity.

- A polar harmonic of eigenvalue $-l(l+1)$ for the Laplacian on S^2 has parity eigenvalue equal to $(-1)^l$. Scalar harmonics Y_l^m are polar.

- We use a real Regge-Wheeler-Zerilli basis of harmonics.

- Using capital Latin letters for S^2-indices, we can decompose any symmetric tensor as

$$T_{ab}\, dx^a\, dx^b = T_{xx}\, dx^2 + 2\mathrm{T}_{xA}\, dx\, dx^A + T_{AB}\, dx^A\, dx^B .$$

Perturbations

- For scalars on S^2, we have $\zeta(\theta,\phi)=\sum \zeta_l^m Y_l^m$.
- For covectors, $w_A(\theta,\phi)=\sum\left(W_l^m Z_l^m{}_A+w_l^m X_l^m{}_A\right)$,

  where we include polar and axial contributions.

  Using the metric γ_{AB} on S^2 and its covariant derivative, we have

  $$Z_l^m{}_A=Y_l^m{}_{:A},\quad X_l^m{}_A=\epsilon_{AB}\gamma^{BC}Y_l^m{}_{:C},\quad l\geq 1.$$

  which are orthonormalized to $l(l+1)$.

- Finally, for tensors

  $$T_{AB}(\theta,\phi)=\sum \tilde{T}_l^m\gamma_{AB}Y_l^m+\sum\left(T_l^m Z_l^m{}_{AB}+t_l^m X_l^m{}_{AB}\right),$$

  with $X_l^m{}_{AB}=\frac{1}{2}\left(X_l^m{}_{A:B}+X_l^m{}_{B:A}\right)$, $\quad Z_l^m{}_{AB}=Y_l^m{}_{:AB}+\frac{l(l+1)}{2}\gamma_{AB}Y_l^m$, $\quad l\geq 2$.

  These tensor harmonics are orthonormalized to $l(l+1)(l+2)/2$.

Perturbations

- We use real spherical harmonics to avoid "reality conditions" on the expansion coefficients,

$$Y_l^m \to \left\{ Y_l^m ,\ m=0 ;\ \ \frac{(-1)^m}{\sqrt{2}} \left(Y_l^m + Y_l^{m*} \right),\ m>0 ;\ \ \frac{(-1)^m}{i\sqrt{2}} \left(Y_l^{|m|} - Y_l^{|m|*} \right),\ m<0 \right\}.$$

- Similarly, for the Fourier expansion on S^1, we employ the real modes

$$Q_{n,\lambda} \to \left\{ Q_0 = 1 ;\ \ Q_{n,+} = \sqrt{2} \cos \omega_n x ,\ \ Q_{n,-} = \sqrt{2} \sin \omega_n x ,\ \ \omega_n = 2\pi n ,\ n \geq 1 \right\}.$$

Note that the derivative ∂_x changes the value of λ for $n \geq 1$.

- For simplicity, we will restrict ourselves to AXIAL perturbations with $l \geq 2$. The study of polar perturbations can be carried out along similar lines.

- There are no scalar axial perturbations. And we will see that axial vector pertubations are pure gauge in our system.

Axial perturbations

- Calling $\{\nu\}=\{n,\lambda,l,m\}$, we can expand the pertubations of the spatial metric, its momentum, and the shift vector as

$$\Delta h_{ab}\,dx^a\,dx^b=-2\sum h_1^{\nu}(t)\,X^m_{l\;A}(\theta,\phi)\,Q_{n,\lambda}(x)\,dx\,dx^A+\sum h_2^{\nu}(t)\,X^m_{l\;AB}(\theta,\phi)\,Q_{n,\lambda}(x)\,dx^A\,dx^B,$$

$$\Delta\left[\frac{p_{ab}}{\sqrt{h}}dx^a\,dx^b\right]=-\frac{4\pi\,p_b^2}{V}\sum\frac{p_1^{\nu}(t)}{l(l+1)}X^m_{l\;A}(\theta,\phi)\,Q_{n,\lambda}(x)\,dx\,dx^A$$
$$+\frac{8\pi\,p_c^2}{V}\sum\frac{p_2^{\nu}(t)}{l(l+1)(l+2)}X^m_{l\;AB}(\theta,\phi)\,Q_{n,\lambda}(x)\,dx^A\,dx^B,$$

$$N_a\,dx^a=-16\pi\sum h_0^{\nu}(t)\,X^m_{l\;A}(\theta,\phi)\,Q_{n,\lambda}(x)\,dx^A.$$

- At second order, the contribution of the perturbations to the action is

$$\frac{1}{16\pi}\int dt\,\sum\left(\dot{h}_1^{\nu}\,p_1^{\nu}+\dot{h}_2^{\nu}\,p_2^{\nu}-h_0^{\nu}\,C_{\nu}^{ax}-\underline{N}\,H_{\nu}^{ax}\right).$$

Perturbative diff. constraints

Hamiltonian constraint

Gauge invariants

- Considering the KS background as fixed, we can perform a linear canonical transformation in the perturbations so that they are described by gauge invariant canonical pairs, and by the perturbative constraints and variables canonically conjugated to them:

$$\{h_1^\nu, p_1^\nu, h_2^\nu, p_2^\nu\} \rightarrow \left\{Q_1^\nu, P_1^\nu, Q_2^\nu, P_2^\nu = -\frac{1}{2} C_\nu^{ax}\right\},$$

  with generating function

$$F^\nu = h_1^\nu P_1^\nu + h_2^\nu P_2^\nu - \frac{\lambda \omega_n}{2} h_2^\nu P_1^\nu + \frac{l(l+1)(l+2)}{4 p_c^2} (\Omega_b + \Omega_c)(h_2^\nu)^2$$
$$- \frac{21(l+1)}{p_b^2} \Omega_b \left(\frac{\omega_n^2}{4} (h_2^\nu)^2 + \lambda \omega_n h_1^\nu h_2^\nu \right).$$

- The perturbative term in the Hamiltonian is changed by the "time" variation of this generating function, given by its Poisson bracket with the background Hamiltonian H_{KS}.

Hamiltonian term

- With a suitable redefinition of Lagrange multipliers, the perturbative contribution to the action can be written as

$$\int dt \left\{ (\underline{N}-\underline{\tilde{N}})\, H_{KS} + \frac{1}{16\pi}\sum \left(\dot{Q}_1^{\nu} P_1^{\nu} + \dot{Q}_2^{\nu} P_2^{\nu}\right) + \frac{1}{8\pi}\sum \tilde{h}_0^{\nu} P_2^{\nu} - \underline{\tilde{N}} \sum \underline{\tilde{H}}_{\nu}^{ax} \right\},$$

where the new lapse includes quadratic perturbative terms and

$$\underline{\tilde{H}}_{\nu}^{ax} = \frac{p_b^2 \,(P_1^{\nu})^2}{l(l+1)} + l\,\frac{(l+1)}{p_b^2}\left[8\Omega_b^2 + 8\,\Omega_b\,\Omega_c + 4\,p_b^2 + (l+2)(l-1)\,p_b^2\right](Q_1^{\nu})^2$$

$$+\frac{\omega_n^2\, p_c^2}{l(l+1)(l+2)}\left[P_1^{\nu} - \frac{4\,l\,(l+1)}{p_b^2}\,\Omega_b\, Q_1^{\nu}\right]^2 - 4\,\Omega_b\, Q_1^{\nu} P_1^{\nu}.$$

- We can now eliminate the cross-terms in the perturbative contribution to the Hamiltonian and scale the momenta P_1^{ν} by performing a suitable background- and mode-dependent linear canonical transformation:

$$\left\{Q_1^{\nu}, P_1^{\nu}\right\} \rightarrow \left\{\tilde{Q}_1^{\nu}, \tilde{P}_1^{\nu}\right\}.$$

- Since the new transformation is again background-dependent, the perturbative term in the Hamiltonian changes by the variation of the generating function. In terms of the new variables, we obtain

$$\tilde{H}^{ax}_{\nu} = (\tilde{P}^{\nu}_{1})^2 + \left[(l+2)(l-1)\, p_b^2 + \omega_n^2\, p_c^2 \right] (\tilde{Q}^{\nu}_{1})^2$$

$$+ \frac{(l+2)(l-1)\, p_b^2}{(l+2)(l-1)\, p_b^2 + \omega_n^2\, p_c^2} \left[p_b^2 - \frac{(l+2)(l-1)\, p_b^2 - 2\omega_n^2\, p_c^2}{(l+2)(l-1)\, p_b^2 + \omega_n^2\, p_c^2} (\Omega_b - \Omega_c)^2 \right] (\tilde{Q}^{\nu}_{1})^2 .$$

- The resemblance with the Hamiltonian of a scalar field in KS suggests a new canonical transformation such that the limit of high frequencies is well identified. We define

$$k^2 = (l+2)(l-1) + \omega_n^2, \qquad \hat{l} = \frac{\sqrt{(l+2)(l-1)}}{k}, \qquad b_{\hat{l}}^2 = \hat{l}^2\, p_b^2 + \frac{\omega_n^2}{k^2}\, p_c^2 .$$

Role of square wavenumber

Role of component of unitary vector

Hamiltonian term

- Using then the generating function $\boldsymbol{F}_{\nu}=\sqrt{b_{\hat{l}}}\,\tilde{Q}_1^{\nu}\boldsymbol{P}_1^{\nu}-\frac{\{b_{\hat{l}},H_{KS}\}}{4\mathrm{b}_{\hat{l}}}(\tilde{Q}_1^{\nu})^2,$ and taking into account its background dependence, we obtain the following perturbative contribution to the Hamiltonian:

$$\boldsymbol{H}_{\nu}^{ax}=b_{\hat{l}}\left[(\boldsymbol{P}_1^{\nu})^2+(k^2+s_{\hat{l}})(\boldsymbol{Q}_1^{\nu})^2\right],\qquad b_{\hat{l}}^2=\hat{l}^2p_b^2+\frac{\omega_n^2}{k^2}p_c^2,$$

with the background-dependent mass

$$s_{\hat{l}}=\frac{4}{b_{\hat{l}}^2}\left[p_b^2+\Omega_b^2\right]-\frac{\hat{l}^2}{b_{\hat{l}}^4}p_b^2\left[4(\Omega_b-\Omega_c)^2+p_b^2+2(\Omega_b^2-\Omega_c^2)\right]+2\frac{\hat{l}^4}{b_{\hat{l}}^6}p_b^4(\Omega_b-\Omega_c)^2.$$

- Remarkably, the canonical transformation performed on the perturbative variables can be completed on the combined system formed by the perturbations and the background. As a consequence, the background variables are modified by quadratic perturbative terms.

Total system

- In practice, the action truncated perturbatively at second order can be expressed as the background term plus the quadratic perturbative contribution, evaluated at the modified background zero-modes. Denoting these modified variables with the same symbols as before,

$$\int dt \left(\dot{\Phi}\Pi_{\Phi} - \frac{1}{2\gamma}\dot{p}_c c - \frac{1}{\gamma}\dot{p}_b b + \frac{1}{16\pi}\sum \dot{\boldsymbol{Q}}_1^{\nu}\boldsymbol{P}_1^{\nu} + \frac{1}{16\pi}\sum \dot{Q}_2^{\nu} P_2^{\nu} \right)$$
$$+\int dt\, \frac{1}{8\pi}\sum \tilde{h}_0^{\nu} P_2^{\nu} - \int dt\ \tilde{N}\left\{ H_{KS} + \sum \left(b_{\hat{l}}\left[(\boldsymbol{P}_1^{\nu})^2 + (k^2 + s_{\hat{l}})(\boldsymbol{Q}_1^{\nu})^2 \right] \right) \right\}.$$

This system is canonical, with physical degrees of freedom contained in the background and the gauge invariant perturbations.

These degrees of freedom are subject only to one GLOBAL constraint, which is the homogeneous Hamiltonian constraint corrected with a quadratic perturbative contribution.

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Hybrid quantization

- It is simple to achieve a hybrid quantization of this total system, combining a quantum representation for the background and a Fock representation for the perturbations.

- For instance, we can adopt a loop representation for the background geometry constructed on an extended phase space which contains two polymerization parameters, δ_b and δ_c.

- It is convenient to scale the triad variables by these parameters, introducing $\tilde{p}_j = p_j/\delta_j, \quad j=b,c.$ Calling $N_{2\delta_j} = e^{i\delta_j j},$ we define

$$\hat{N}_{\delta_j}|\tilde{\mu}_c\rangle = |\tilde{\mu}_c + 1\rangle, \quad \hat{\tilde{p}}_b|\tilde{\mu}_b\rangle = \frac{\gamma\tilde{\mu}_b}{2}|\tilde{\mu}_b\rangle, \quad \hat{\tilde{p}}_c|\tilde{\mu}_c\rangle = \gamma\tilde{\mu}_c|\tilde{\mu}_c\rangle.$$

and, with them (and using a MMO prescription),

$$\hat{\Omega}_j = \frac{1}{4i\gamma}\left|\hat{\tilde{p}}_j\right|^{1/2}\left[\left(\hat{N}_{2\delta_j} - \hat{N}_{-2\delta_j}\right)\widehat{sign}(\tilde{p}_j) + \widehat{sign}(\tilde{p}_j)\left(\hat{N}_{2\delta_j} - \hat{N}_{-2\delta_j}\right)\right]\left|\hat{\tilde{p}}_j\right|^{1/2}.$$

Hybrid quantization

- In $H^{kin}_{LQC}\otimes L^2(\mathbb{R}^3, d\delta_b d\delta_c d\Phi)$, with $\hat{\delta}_j$ and $\hat{\Phi}$ acting by multiplication and $\hat{\Pi}_\Phi=-i\partial_\Phi$, we adopt the basis $|\tilde{\mu}_b,\tilde{\mu}_c,\delta_b,\delta_c,\Phi\rangle$ and define

$$\hat{H}_{KS}=-\left[\hat{\Omega}_b^2+2\hat{\Omega}_b\hat{\Omega}_c+\hat{\delta}_b^2\hat{\tilde{p}}_b^2+4\partial_\Phi^2\right].$$

- We can then superselect the background geometry and restrict their support to semilattices of two units of separation, with point closest to the origin at $\pm\epsilon$, e.g. ${}^{(2)}H^+_{\epsilon_b}\otimes{}^{(2)}H^+_{\epsilon_c}$.

- We can also quantize the two background factors appearing in the Hamiltonian of the perturbations, namely the time factor $\hat{b}_{\hat{l}}$ and the mass $\hat{s}_{\hat{l}}$, using a symmetric algebraic factor ordering in the $\tilde{p}_j$'s.

Fock quantization

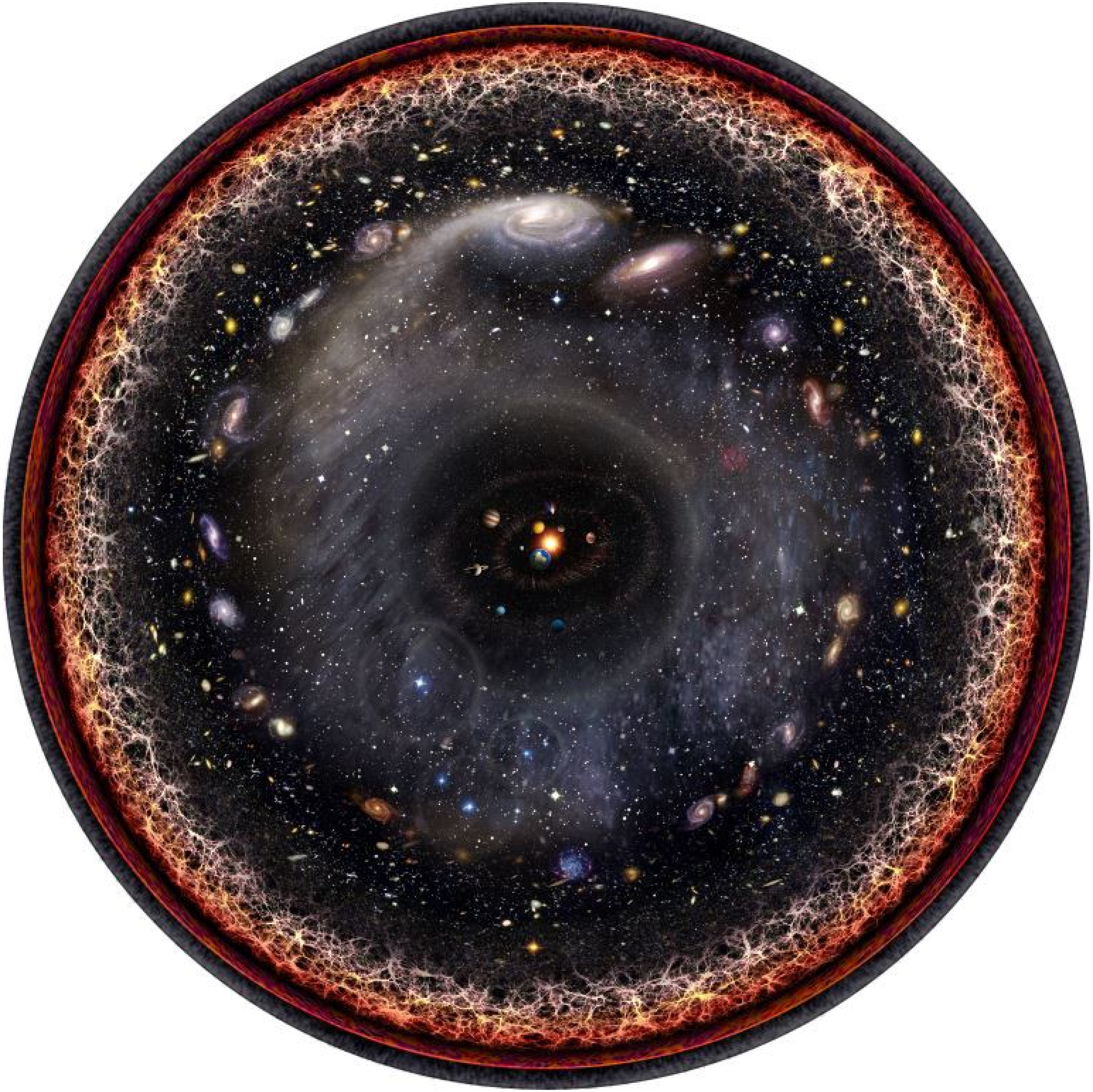
- Finally, we adopt a Fock quantization of the perturbations.

- We may allow for quite general Fock representations, with annihilation and creation variables obtained as background-dependent linear combinations of Q_1^{ν} and P_1^{ν}, which respect the symmetries of the axial perturbations dynamics.

- We further impose that the Heisenberg dynamics of these perturbations be unitarily implementable.

- Then, all acceptable Fock quantizations are **unitarily equivalent**.

- This family contains the “massless” Fock representation.



Conclusions

- We have considered perturbations around a KS spacetime, describing the interior of a nonrotating black hole.
- We have expanded the perturbations in spherical harmonics and Fourier modes. We have restricted our study to axial perturbations.
- We have truncated the action at second perturbative order and constructed a Hamiltonian formulation.
- We have performed a canonical transformation that leads to a system composed of zero-modes describing the background, perturbative gauge invariants, and perturbative constraints and their momenta.
- There is just a nontrivial, global Hamiltonian constraint on the system.
- It is the background Hamiltonian corrected with the contribution of the perturbative gauge invariants.

And beyond...

- The resulting system can be quantized combining a representation for the background and a Fock representation for the perturbations.

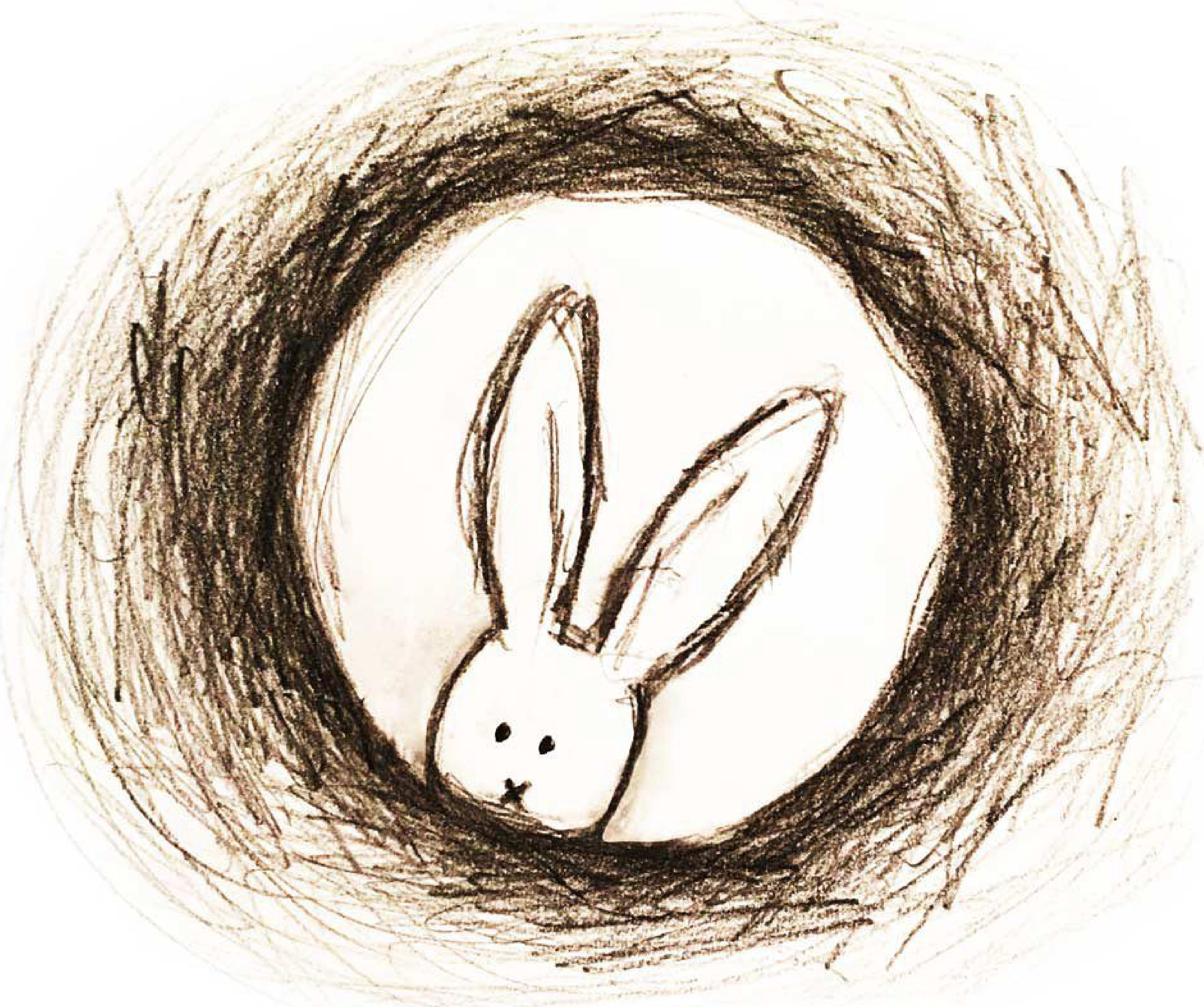
- We have adopted a loop representation for the background geometry.

- Annihilation and creation variables can be formed by background-dependent linear combinations of the perturbative gauge invariants.

- This Fock quantization is essentially unique if we require a unitary implementation of the associated Heisenberg dynamics.

- Modes with $l=1$ are pure gauge except if $n=0$, when we obtain three degrees of freedom.

- In future research, we will derive the corrected perturbation equations and discuss the relation between modes in the interior and exterior.




INTO THE BLACKNESS WE GO, BUT CAN WE EVER TRULY COME OUT?