# Classical theory of the optimal filtering in context of radio astronomy

Radio2024

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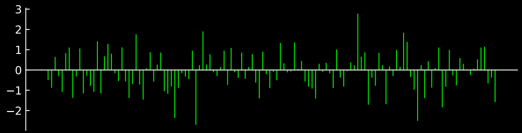
# Single telescope stream


The Effelsberg telescope




- time series type of data (baseband data)
- observations = signal + noise
- signals are weak → optimal filtering

Optimal = maximizes the detection efficiency

# Optimal filtering

- Appears naturally from Bayesian detection as likelihood ratio
- Versatile method enabling various adaptations
- In the simplest form known as the matched filter

filter/detector

expected signal

$$\Lambda = \Lambda_0 \operatorname{Re}\left\{ \int_{(T)} r(t)\, s^*(t)\, \mathrm{d}t \right\} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$$

observation

threshold

- Very practical: radar, sonar, telecommunication systems, GPS, *etc.*

# Optimal filtering — advanced form

By using the Karhunen–Loève expansion:

$$\text{Gaussian signal} - s(t) = m(t) + \sum_n s_n \phi_n(t) \qquad \phi_n(t) - \text{orthogonal functions}$$

$$\Lambda = l_R + l_D \underset{H_0}{\overset{H_1}{\gtrless}} \gamma : \quad \begin{cases} l_R = \dfrac{1}{N_0} \displaystyle\sum_n \dfrac{\lambda_n}{\lambda_n + N_0} \left| \int_{(T)} r(t)\phi_n^*(t)\,\mathrm{d}t \right|^2 \\ l_D = 2 \displaystyle\sum_n \dfrac{1}{\lambda_n + N_0} \mathrm{Re} \left\{ \left[ \int_{(T)} r(t)\phi_n^*(t)\,\mathrm{d}t \right] \left[ \int_{(T)} m(t)\phi_n^*(t)\,\mathrm{d}t \right]^* \right\} \end{cases}$$

- Threshold $\gamma$ is determined with Bayes or Neyman-Pearson criterion (fixed false-detection rate)
- Unknown parameters can be incorporated in the detector
- Detection should happen on-line $\rightarrow$ spectrogram (time-frequency distribution)

# Spectrogram and its properties

Short-time Fourier transform:

$$\text{spectrogram} - |S_t(\omega)|^2 = \left|\mathcal{F}\{s(\tau)\,\underbrace{h(\tau - t)}_{\text{window}}\}\right|^2 = \left|\int e^{-i\omega\tau} s(\tau)\,\underbrace{h(\tau - t)}_{\text{window}}\,d\tau\right|^2$$

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Properties:

- total energy conservation: $E_{\mathrm{SP}} = \int\int |S_t(\omega)|^2\,\mathrm{d}t\,\mathrm{d}\omega = \int \underbrace{|s(t)|^2}_{\text{signal energy}}\,\mathrm{d}t \times \int |h(t)|^2\,\mathrm{d}t$

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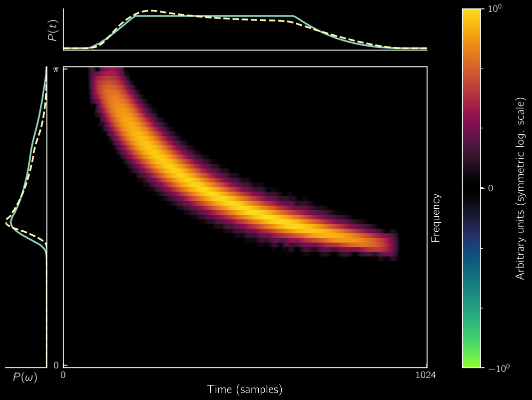
Marginals are not conserved $\rightarrow$ energy is scrambled in the t-f plane

# Spectrogram — illustrative example (fully known signal)

- Easy to understand and interpret
- Both marginals are conserved
- Not suitable for optimal detector (spectrogram correlation is not optimal by construction)



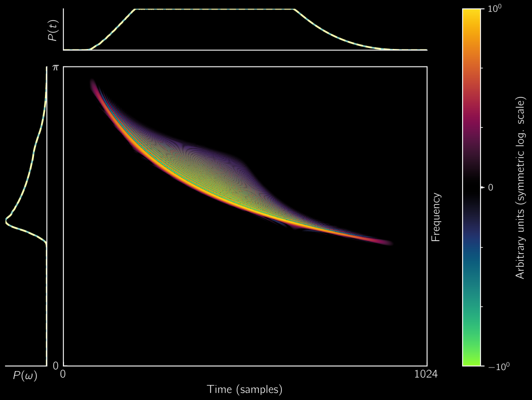
# Alternative transforms — Wigner-Ville distribution

Wigner-Ville distribution:

$$W_{xy}(t,f) = \int_{-\infty}^{\infty} x\left(t+\frac{\tau}{2}\right) y^*\left(t-\frac{\tau}{2}\right) \mathrm{e}^{-i2\pi f\tau}\,\mathrm{d}\tau$$

- Both marginals are conserved
- Opt. det. for the fully known signal:

$$\Lambda = \mathrm{Re}\{W_{rs}\} W_{ss}$$



# Alternative transforms — Unterberger distribution

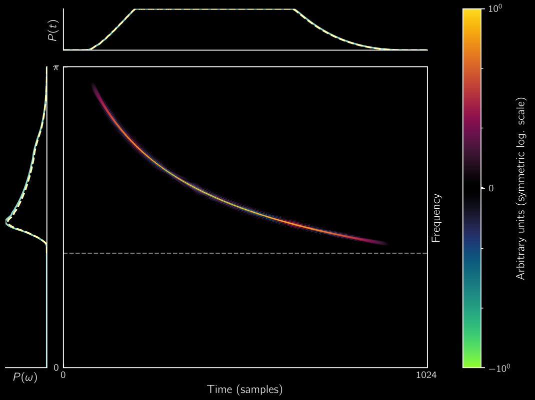
Unterberger distribution:

$$U_{xy}(t,f) = f^{3} \int_{-\infty}^{\infty} X(f\lambda(u))\, Y^{*}(f\lambda(-u)) \times \mu(u)\, \mathrm{e}^{i2\pi tf\,[\lambda(u)-\lambda(-u)]}\, \mathrm{d}u$$

$$\lambda(u) = \sqrt{\frac{1-\mathrm{e}^{u}}{\mathrm{e}^{-u}-1}}, \qquad \mu(u) \text{ — arbitrary function}$$

- Provides perfect localization
- Both marginals are conserved
- Opt. det. for the fully known signal:

$$\Lambda = \mathrm{Re}\{U^{A}_{rs}\} U^{P}_{ss}$$



# Take-home messages

- Optimal = maximizes the detection efficiency
- Optimal detector has a particular structure that can guide our choice of representations
- Spectrogram is a simple time-frequency distribution, but with a certain issues
- There are alternatives to spectrogram (Wigner-Ville distribution, Unterberger distribution, *etc*)