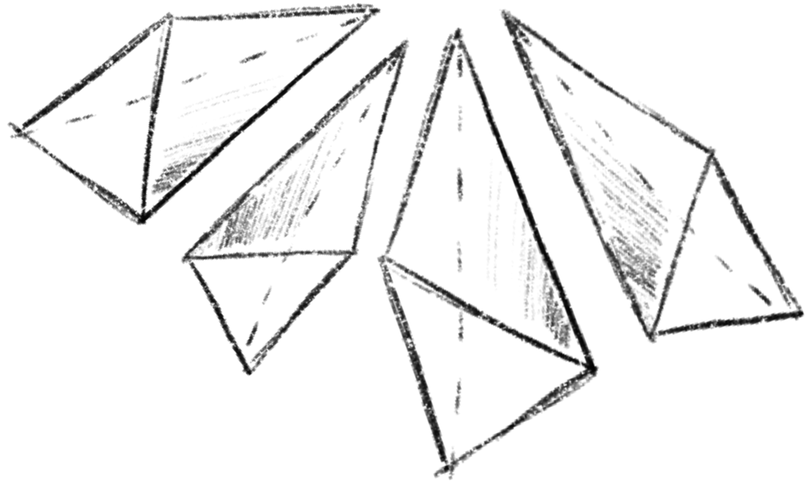
Random geometries arising from random tensor models



STRUCTURES
CLUSTER OF
EXCELLENCE

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Plan

- Discrete gravity
- Random matrices (2D quantum gravity)
- Random tensors (D>2 quantum gravity)
- The continuum limit

Discrete gravity

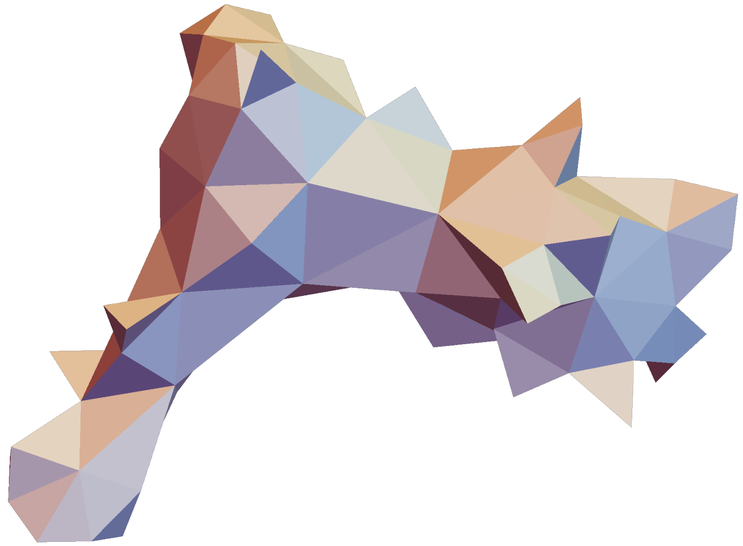
The Regge action

$$\int \mathscr{D}g\; e^{-S_{EH}(g)} \quad \rightarrow \quad \sum_G \frac{1}{|G|} e^{-S_R}$$

$$\frac{\Lambda}{8\pi G}\int d^d x\sqrt{g} \qquad \frac{\Lambda}{8\pi G}\sum_{\sigma_d} \mathrm{vol}(\sigma_d)$$

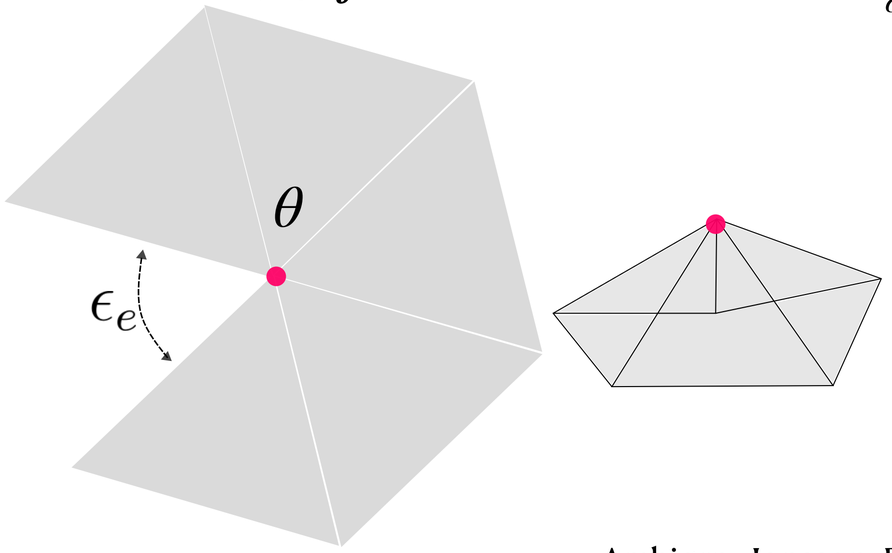
$$\frac{1}{16\pi G}\int d^d x\sqrt{g}R \qquad \frac{1}{16\pi G}\sum_{\sigma_{d-2}} \mathrm{vol}(\sigma_{d-2})\;\epsilon_{d-2}(\sigma_{d-2})$$

$$2\pi - \theta(\sigma_{d-2})$$



d dimension
σ_d d-dim. simplex

e.g. $d = 2$



Regge (1961)

Ambjørn, Jonsson, Durhuus “Quantum geometry - A statistical field theory approach”

Discrete gravity

The Regge action



$$\sum_G \frac{1}{|G|} e^{-S_R}$$

$$\frac{\Lambda}{8\pi G} \sum_{\sigma_d} \mathrm{vol}(\sigma_d) \qquad \frac{1}{16\pi G} \sum_{\sigma_{d-2}} \mathrm{vol}(\sigma_{d-2})\ \epsilon_{d-2}(\sigma_{d-2})$$

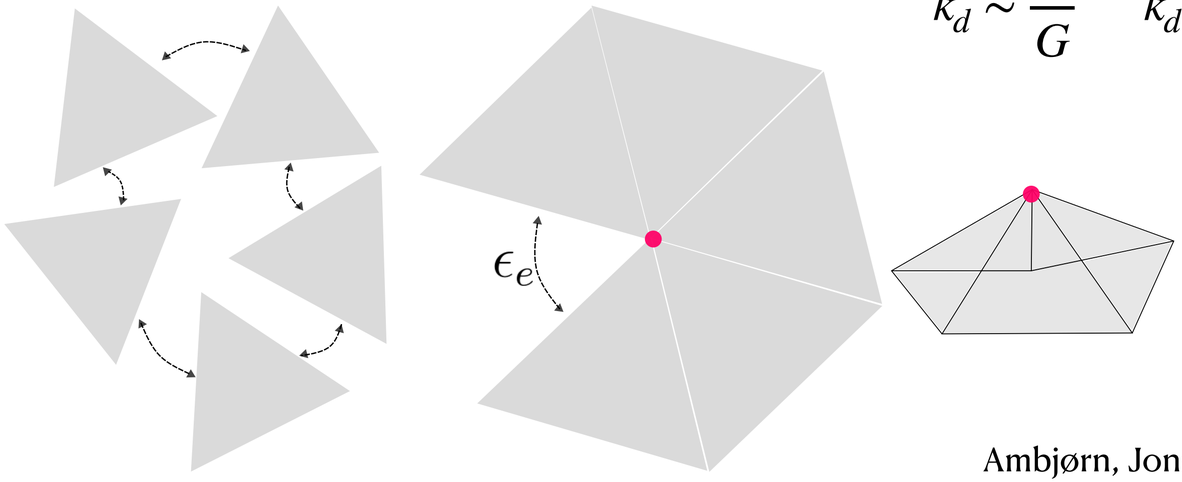
$$S_R = k_d N_d(T) - k_{d-2} N_{d-2}(T)$$

$$k_d \sim \frac{\Lambda}{G} \qquad k_{d-2} \sim \frac{1}{G}$$

T Equilateral triangulation

d dimension

e.g. $d = 2$

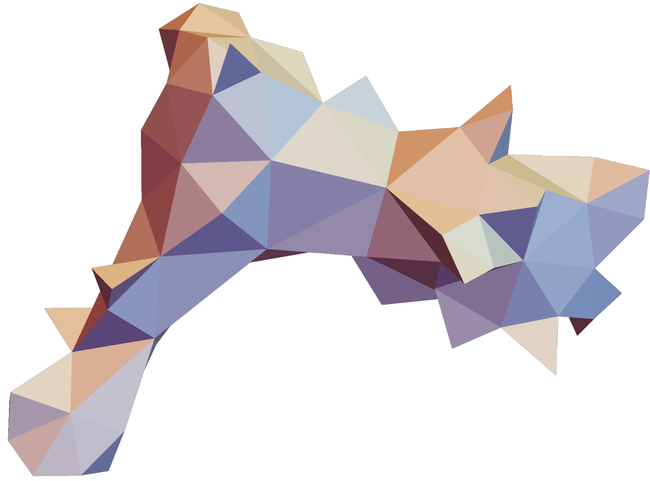


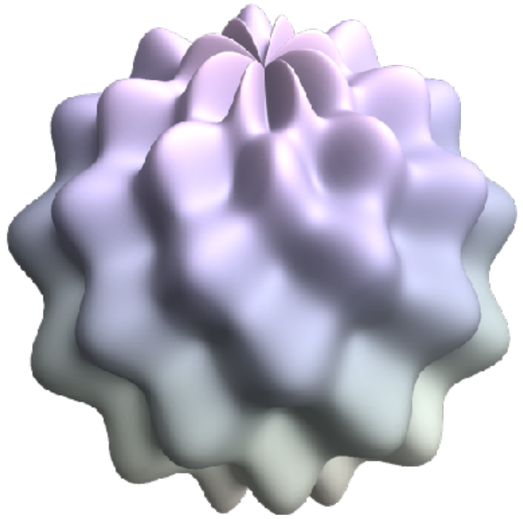
Regge (1961)

Ambjørn, Jonsson, Durhuus “Quantum geometry - A statistical field theory approach”

Discrete gravity

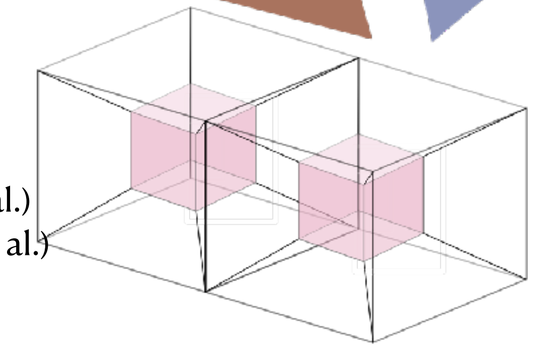
$$\int \mathscr{D}g \; e^{-S_{EH}(g)} \quad \rightarrow \quad \sum_{G} \frac{1}{|G|} e^{-S_R}$$





- Control over geometric d.o.f.
- Background independent
- Consequence in QG approac
- ...

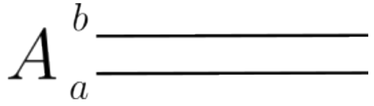
Random matrices/tensors
DO THIS!



Regge (1961)
Regge calculus (Dittrich, Bahr, Asante, Steinhaus, et. al.)
(Causal) Dynamical Triangulations (Ambjorn, Loll, Jurkiewicz, Goerlich, et al.)
Spin networks and spin foams (Baez, Rovelli, Smolin, Barret, crane, Livine, et. al.)

Random Matrices

0-dimensional field theories

- NxN Matrix A_{ab}  $a, b \in 1,\ldots,N$
- Symmetry: Hermitian $U(N)$

  Real and symmetric $O(N)$

t'Hooft, Brézin, Itzykson, Parisi, Zuber, Tutte, Di Francesco, Ginsparg, Zinn-Justin, Kazakov, Ambjorn, et. al.

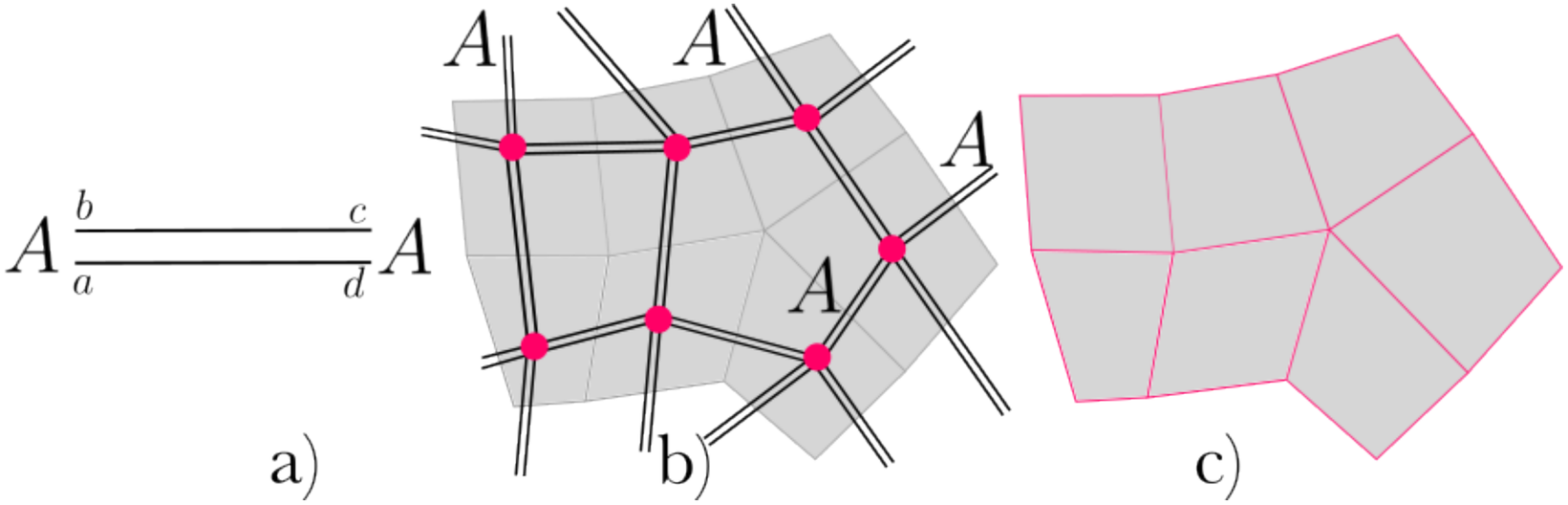
REVIEW: Eynard, "Counting surfaces" (2016)

Random Matrices

- Action $\int dA\ e^{-N\,S[A]}$ invariant under $U(N)$

e.g. $$S[A] = \frac{1}{2}\mathrm{Tr}(A^2) - \frac{\lambda}{4}\mathrm{Tr}(A^4) = \frac{1}{2}\sum_{a,b} A_{ab}A_{ba} - \frac{\lambda}{4}\sum_{a,b,c,d} A_{ab}A_{bc}A_{cd}A_{da}$$

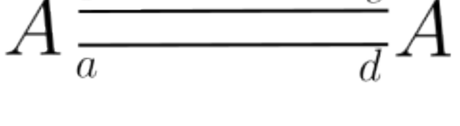
- Feynman rules $\langle A_{ab}A_{cd}\rangle = N^{-1}\delta_{ad}\delta_{bc}$ $\langle A_{ab}A_{cd}A_{ef}A_{gh}\rangle = -N\,\lambda\,\delta_{ad}\delta_{cf}\delta_{eh}\delta_{gb}$

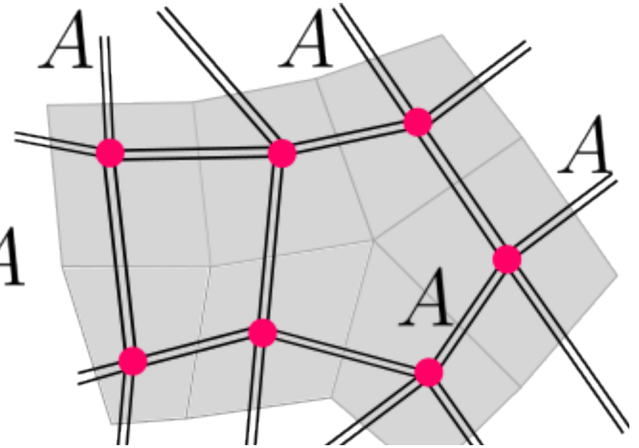


Random Matrices

- Feynman rules $\langle A_{ab} A_{cd} \rangle = N^{-1} \delta_{ad} \delta_{bc}$ $\langle A_{ab} A_{cd} A_{ef} A_{gh} \rangle = -N \lambda \, \delta_{ad} \delta_{cf} \delta_{eh} \delta_{gb}$

Edge (propagator)	N^{-1}
Vertex (interaction)	$-N \lambda$
Face	N

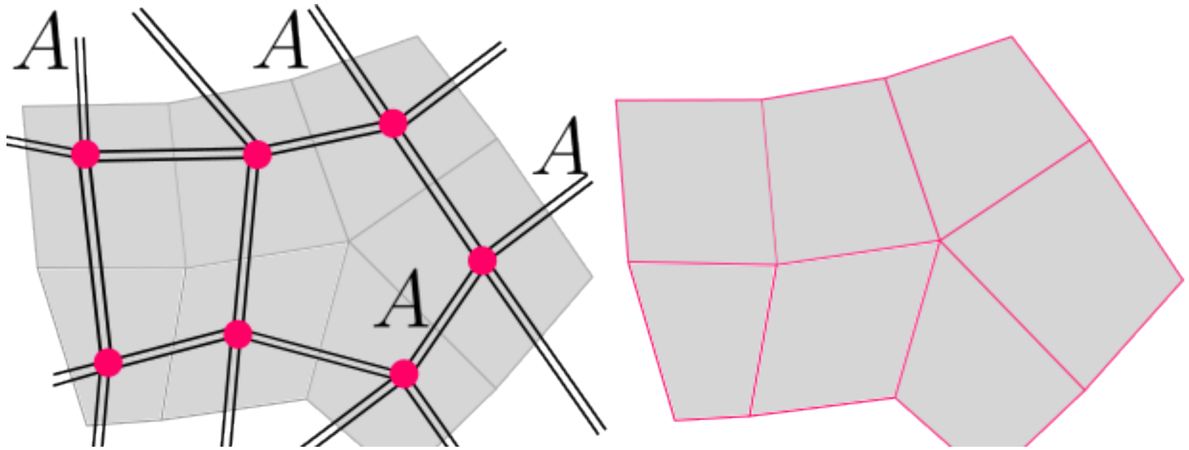




$$Z_N(\lambda) = \sum_G \frac{(-\lambda)^{v(G)}}{|G|} N^{\underbrace{V(G)+F(G)-E(G)}_{2-2g}}$$

Random Matrices

- Feynman rules $\langle A_{ab} A_{cd} \rangle = N^{-1} \delta_{ad} \delta_{bc}$ $\langle A_{ab} A_{cd} A_{ef} A_{gh} \rangle = N \, \lambda \, \delta_{ad} \delta_{cf} \delta_{eh} \delta_{gb}$



$$Z_N(\lambda) = \sum_G \frac{(-\lambda)^{v(G)}}{|G|} N^{2-2g}$$

$$\int \mathscr{D}g \; e^{\frac{1}{16\pi G} \underbrace{\int \sqrt{g} R}_{4\pi(2-2g)}} e^{-\frac{\Lambda}{8\pi G} \int \sqrt{g}}$$

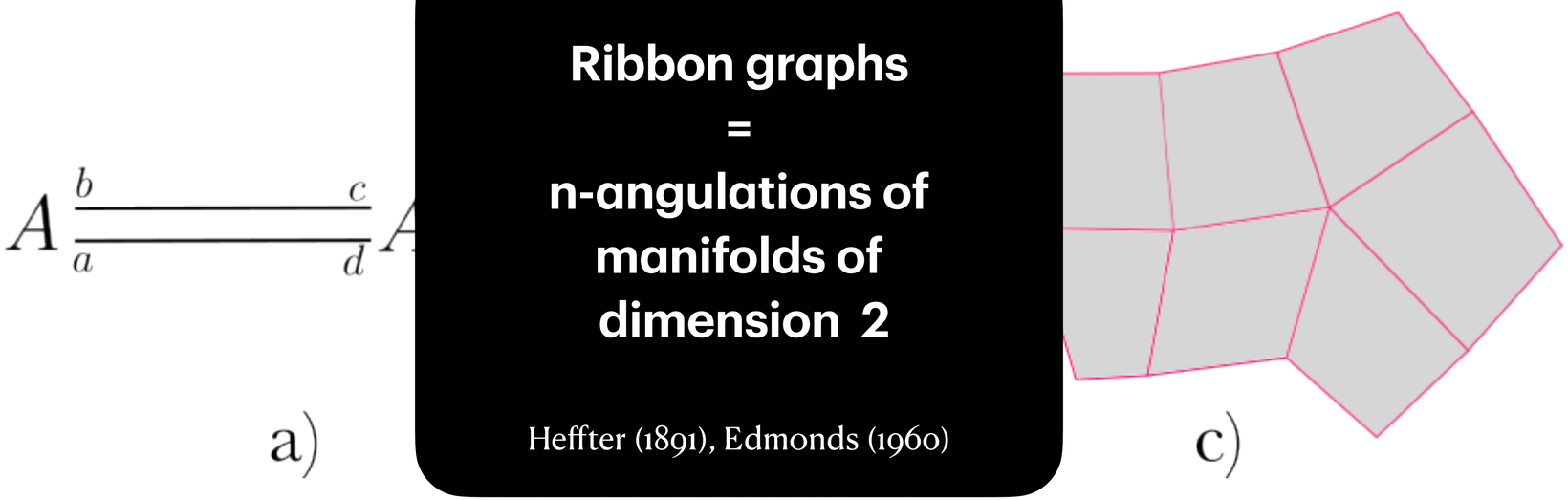
$$N \sim e^{\frac{1}{G}} \qquad \lambda \sim -e^{\frac{-\Lambda}{G}}$$

Random Matrices

EH action	Regge action	Random matrix model	Geometry/ Operator
Λ	$G\ k_2$	$G\ \ln\frac{1}{\lambda}$	Volume $\frac{\Lambda}{8\pi G}\int d^2x\sqrt{g}$
G	$\frac{1}{k_0}$	$\frac{1}{\ln N}$	Ricci scalar $\frac{1}{16\pi G}\int d^2x\sqrt{g}R$

Random Matrices

- Geometric interpretation



Ribbon graphs
=
n-angulations of manifolds of dimension 2

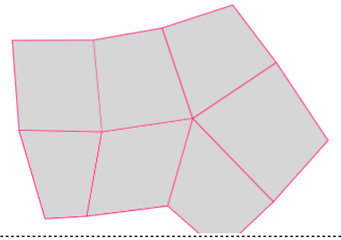
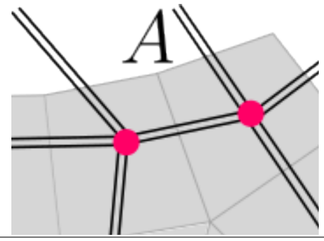
Heffter (1891), Edmonds (1960)

$$\sum_G \frac{1}{|G|} e^{-S_R}$$

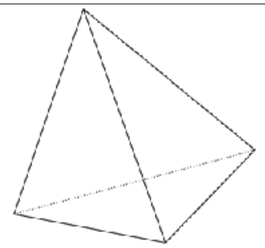
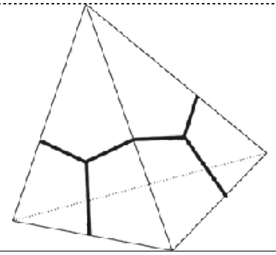
matrix

2-dim. building block + gluing rules

How do we go to D>2 ?

Dimension	2	
# of indices	2	A_{ab} 
Contractions of indices	2-dimensional building blocks	

How do we go to D>2 ?

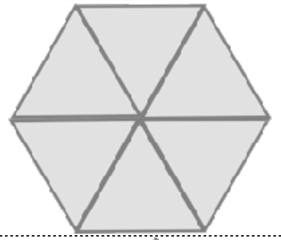
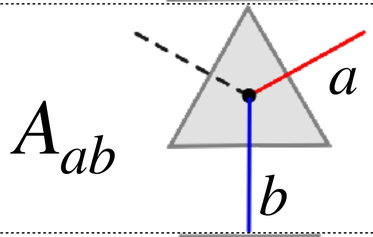
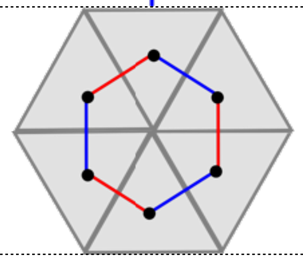
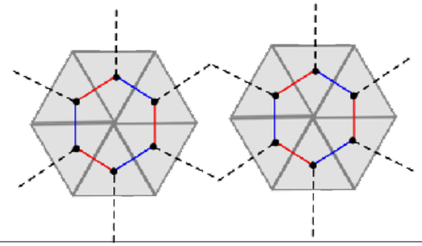
Dimension	3	
# of indices		T_{abc} 
Contractions of indices	3-dimensional building blocks	

Not enough!

Gurau “Lost in Traslation”(2018)

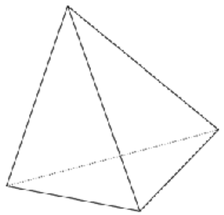
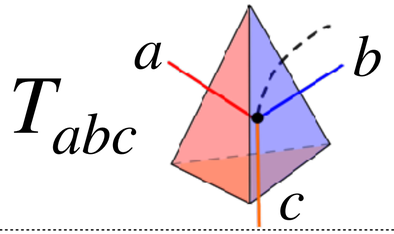
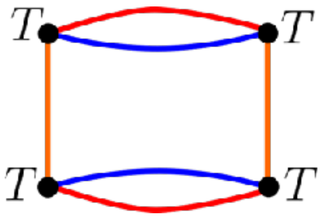
Ambjorn, Durhuus, Jonsson (1991), Sasakura (1991), Gross (1992)

How do we go to D>2 ?

Dimension	2	
# of indices	2	
Contraction of “coloured” indices	Gluing of triangles	
Contraction of “extra” index	Gluing of gluing of triangles	

3-Colored graphs

How do we go to D>2 ?

Dimension	3	
# of indices	3	
Contraction of indices	Gluing of tetrahedra	

4-Colored graphs

Random Tensors

- Tensor $T_{a_1 a_2 \ldots a_D}$

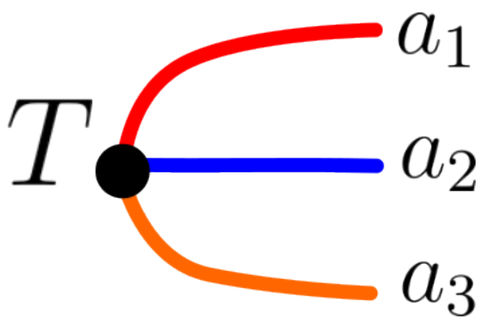
$D \in \mathbb{N}$ order

$a_i \in \{1, 2, \ldots, N\}$

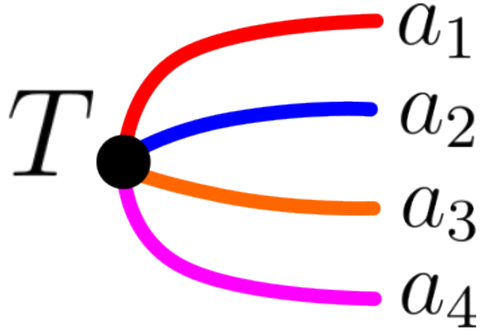
- Symmetry $O(N)^{\otimes D}$

$$T_{a_1\, a_2\, \ldots a_D} = O^{(1)}_{a_1\, b_1}\, O^{(2)}_{a_2\, b_2} \cdots O^{(D)}_{a_D\, b_D} T_{b_1\, b_2\, \ldots b_D}$$

e.g. **D=3**



D=4



Colored tensors

Gurau, Rivasseau, Oriti, Ben Geloun, Tanasa, Bonzom, Benedetti, Ryan, Carroza, Lionni, Toriumi, Riello, et. al.

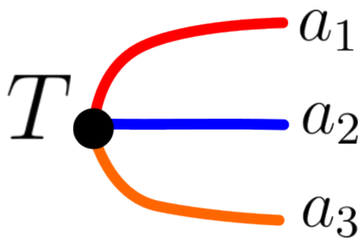
REVIEW: Lionni, "Colored discrete spaces: higher dimensional combinatorial maps and quantum gravity" (2017)

SHORT READ: Gurau "Why random tensors?" (2016)

Random Tensors

- Action $\int dT\, e^{-N\, S[T]}$ invariant under $O(N)^{\otimes D}$

e.g. **D=3**



$$S[T] = \text{Tr}\,T\,T + g_1$$

$$T_{a_1a_2a_3}T_{a_1a_2a_3}$$

$$Tr(T^2)$$

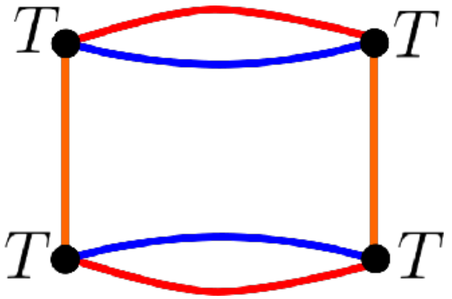
Trace invariants=bubbles



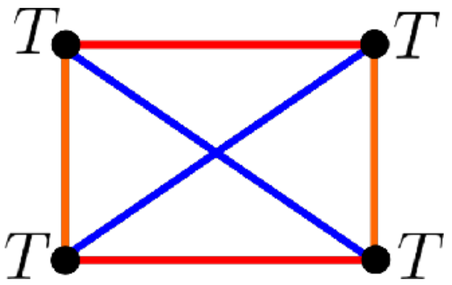


$$(Tr(T^2))^2$$

$+ g_2$

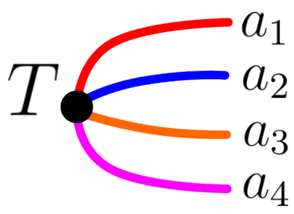


$+ g_3$

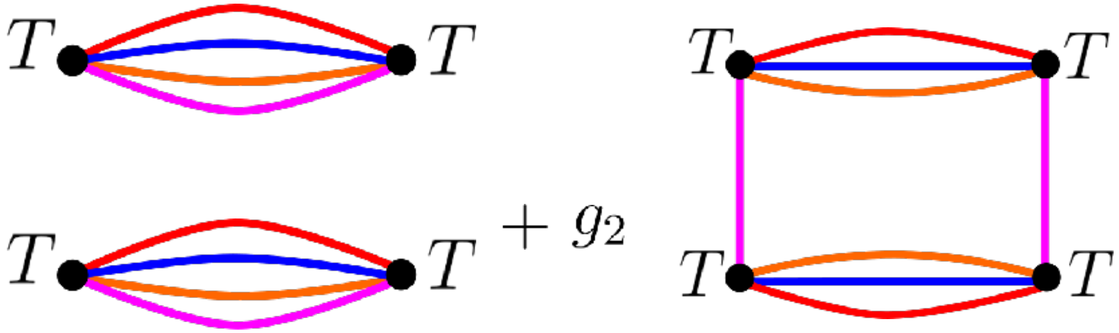


Random Tensors

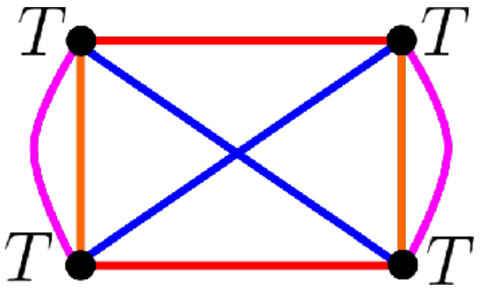
e.g. **D=4**



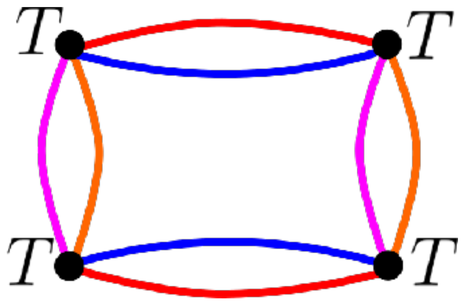
$$S[T] = T \quad T + g_1 \quad T \quad T + g_2$$



$$+ g_3$$



$$+ g_4$$



Random Tensors

- Feynman rules

e.g. D=4

$$S[T] = T \quad T + g_1 \quad T \quad T + g_2 \quad T \quad T$$

Propagator

$$T_{a_1a_2a_3a_4}\ (\delta_{a_1b_1}\delta_{a_2b_2}\delta_{a_3b_3}\delta_{a_4b_4})\ T_{b_1b_2b_3b_4}$$

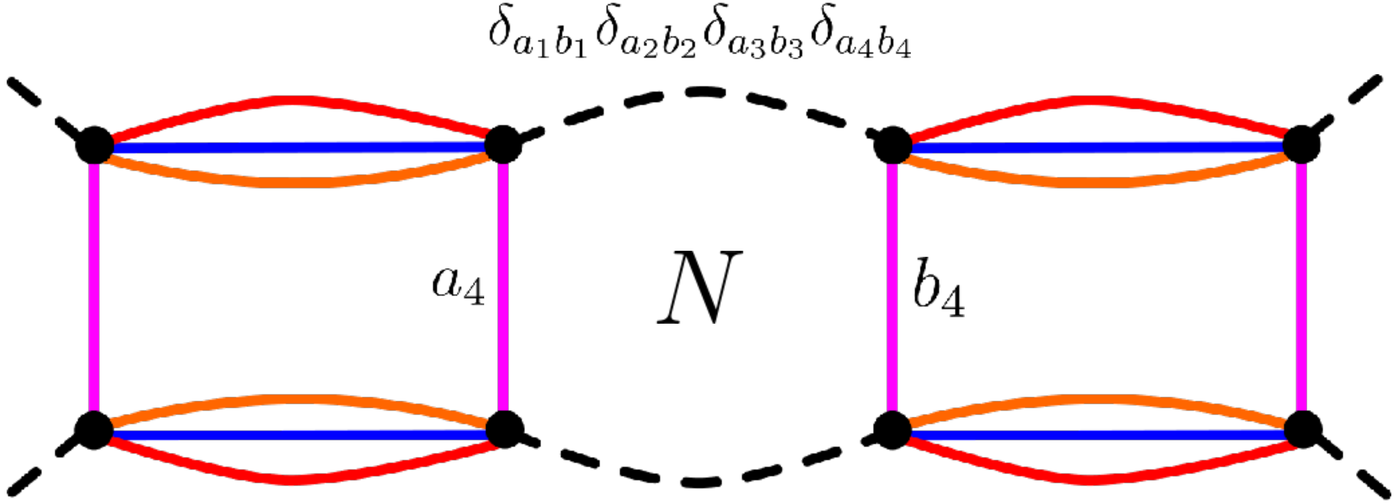
$$+ g_3 \quad T \quad T \quad T \quad T + g_4 \quad T \quad T \quad T \quad T$$

Interactions

Random Tensors

- Feynman rules

e.g. D=4 with one bubble type λ



$$Z_N(\lambda) \sim \sum_G \frac{(-\lambda)^{v(G)}}{|G|} N^{D-\omega(G)}$$

Positive number only depending on the colored graph G

Gurau (2011)

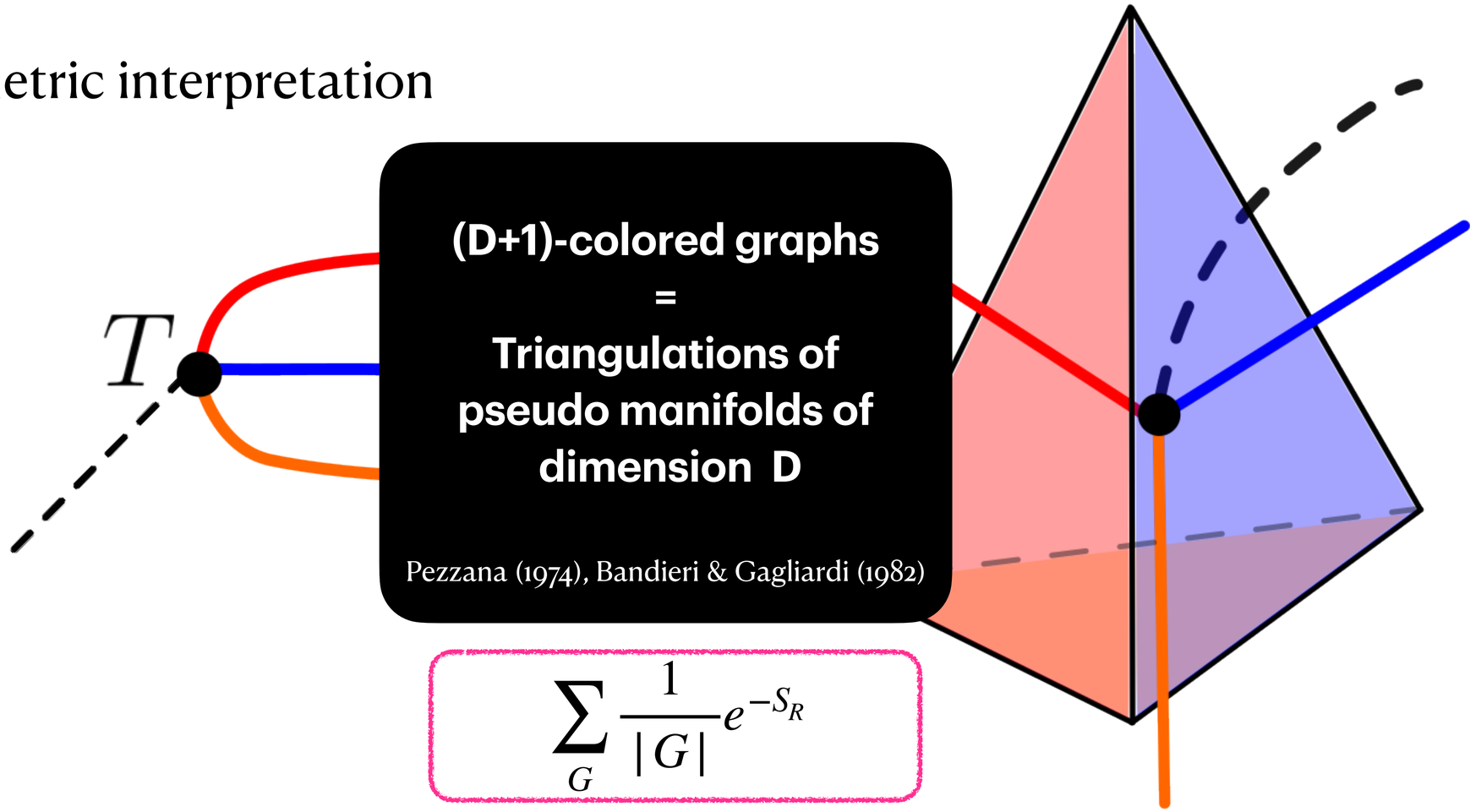
Random Tensors

EH action	Regge action	Random tensor model	Geometry/ Operator
Λ	$G\ k_d$	$G\ \ln\frac{1}{\lambda}$	Volume $\frac{\Lambda}{8\pi G}\int d^d x\sqrt{g}$
G	$\frac{1}{k_{d-2}}$	$\frac{1}{\ln N}$	Ricci scalar $\frac{1}{16\pi G}\int d^d x\sqrt{g}R$

Random Tensors

- Geometric interpretation

e.g. D=3



(D+1)-colored graphs
=
Triangulations of pseudo manifolds of dimension D

Pezzana (1974), Bandieri & Gagliardi (1982)

$$\sum_G \frac{1}{|G|} e^{-S_R}$$

D-tensor + propagator

D-dim. building block + gluing rules

The continuum limit

of building blocks $\to \infty$
size of building blocks $\to 0$

Keeping the total volume fixed



The continuum limit

# of building blocks $\to \infty$
size of building blocks $\to 0$

Keeping the total volume fixed

For random matrix and tensor models, this is:

$N \to \infty$ **Many degrees of freedom** → **Renormalization Group scale**

$N^{d-2}(\lambda - \lambda_{crit})^{\gamma}$ **fixed**

$\lambda \to \lambda_{crit}$ **Reaching a critical regime** → **Fixed point**

Brézin & Zinn-Justin (1992)
Di Francesco, Ginsparg, Zinn-Justin (1995)
Dartois, Gurau, Rivasseau (2013)
Bonzom, Gurau, Ryan, Tanasa (2014)

So what...

What can I do with this?

Continuum limit

Critical exponents

Renormalisation group analysis

Brézin & Zinn-Justin (1992)
Eichhorn & Koslowski (2013, 2017)
Pereira, Lumma, Pithis, Ben Geloun, **Castro** et. al.

Random geometry

Brownian sphere

Le Gall, Miermont (2012)
Market, Mokkadem (2023)

So what...

What can I do with this?

Discrete geometry

Construct models interesting for QG

Gauge information

Oriti, Freidel, Livine, Carroza, Ben Geloun, et. al.

Causality

Benedetti, Henson, **Castro**, Toriumi, Abranches, et. al.

So what...

What can I do with this?

Discrete geometry

Construct models interesting for QG

EH action	Regge action	Random tensor model	Geometry/ Operator
Λ	$G\ k_d$	$G\ \ln\frac{1}{\lambda}$	Volume $\frac{\Lambda}{8\pi G}\int d^dx\sqrt{g}$
G	$\frac{1}{k_{d-2}}$	$\frac{1}{\ln N}$	Ricci scalar $\frac{1}{16\pi G}\int d^dx\sqrt{g}R$
α		New coupling	$\frac{1}{16\pi G}\int d^dx\sqrt{g}R^2$

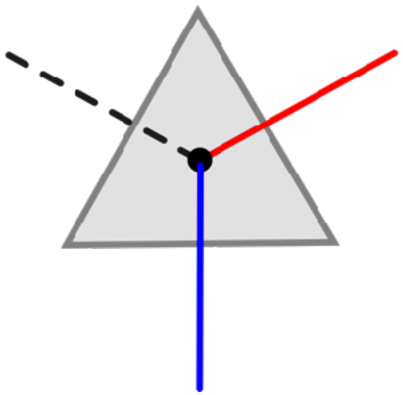
New geometric operators

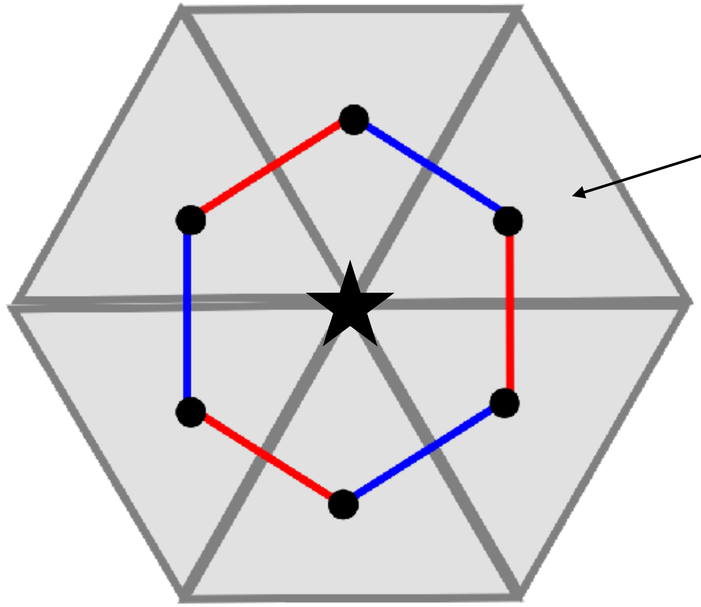
d=2: Ambjorn, Jurkiewicz, Kristjansen, Kazakov, Wynte, Szabo, Wheater,

d>2: **Castro**, Keppler (w.i.p.)

Ambjorn, Jurkiewicz, Kristjansen (1992)
Kazakov, Wynter (1995)
Szabo, Wheater (1997)

e.g. D=2 (3-colored graph, a.k.a. colored matrix model)





Length of bicolored cycles

$$K = \frac{\pi}{3}(6 - deg(\bigstar))$$

Generalizable to higher curvature invariants and D>2

Castro, Keppler (w.i.p.)