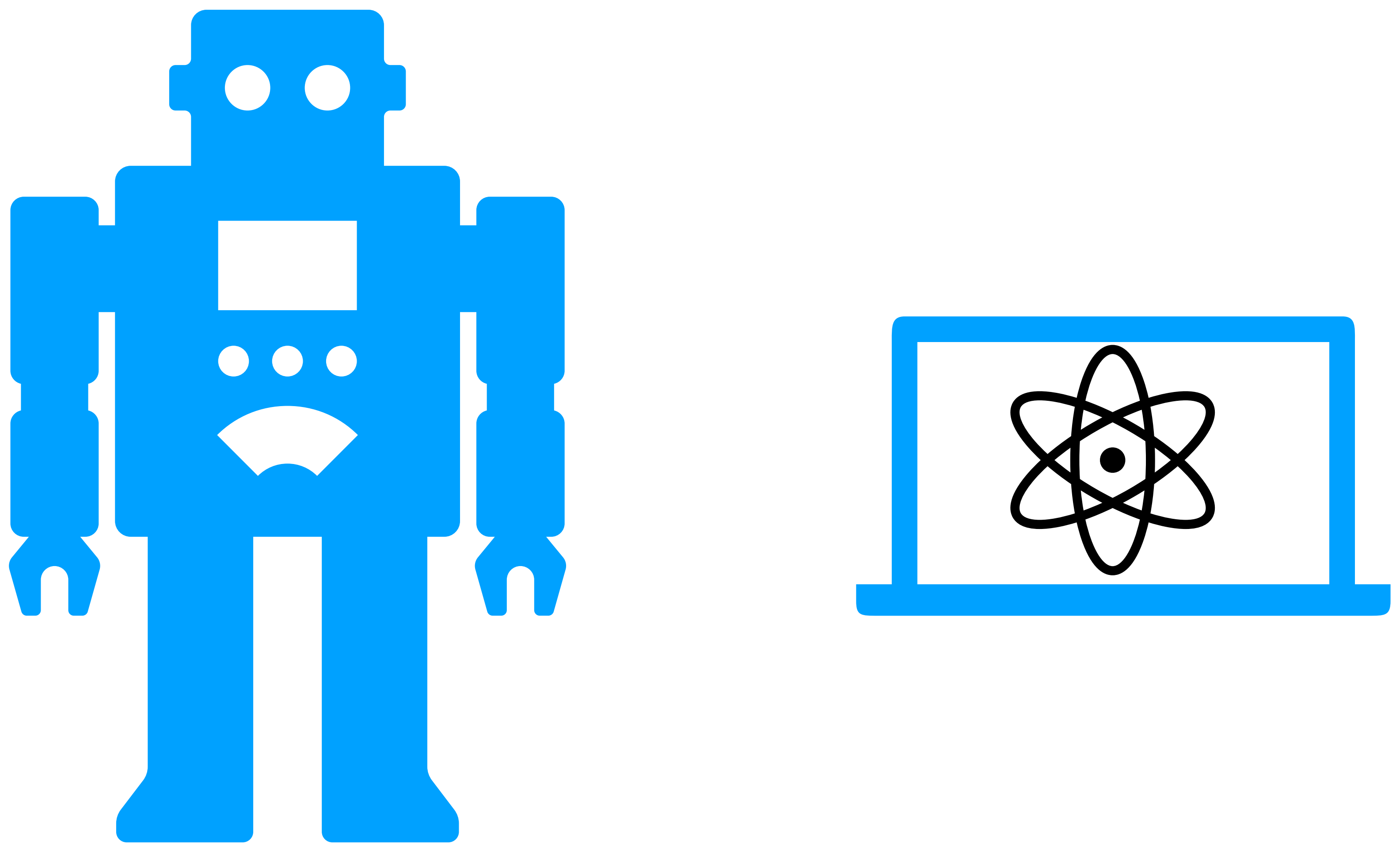
Leibniz Universität Hannover

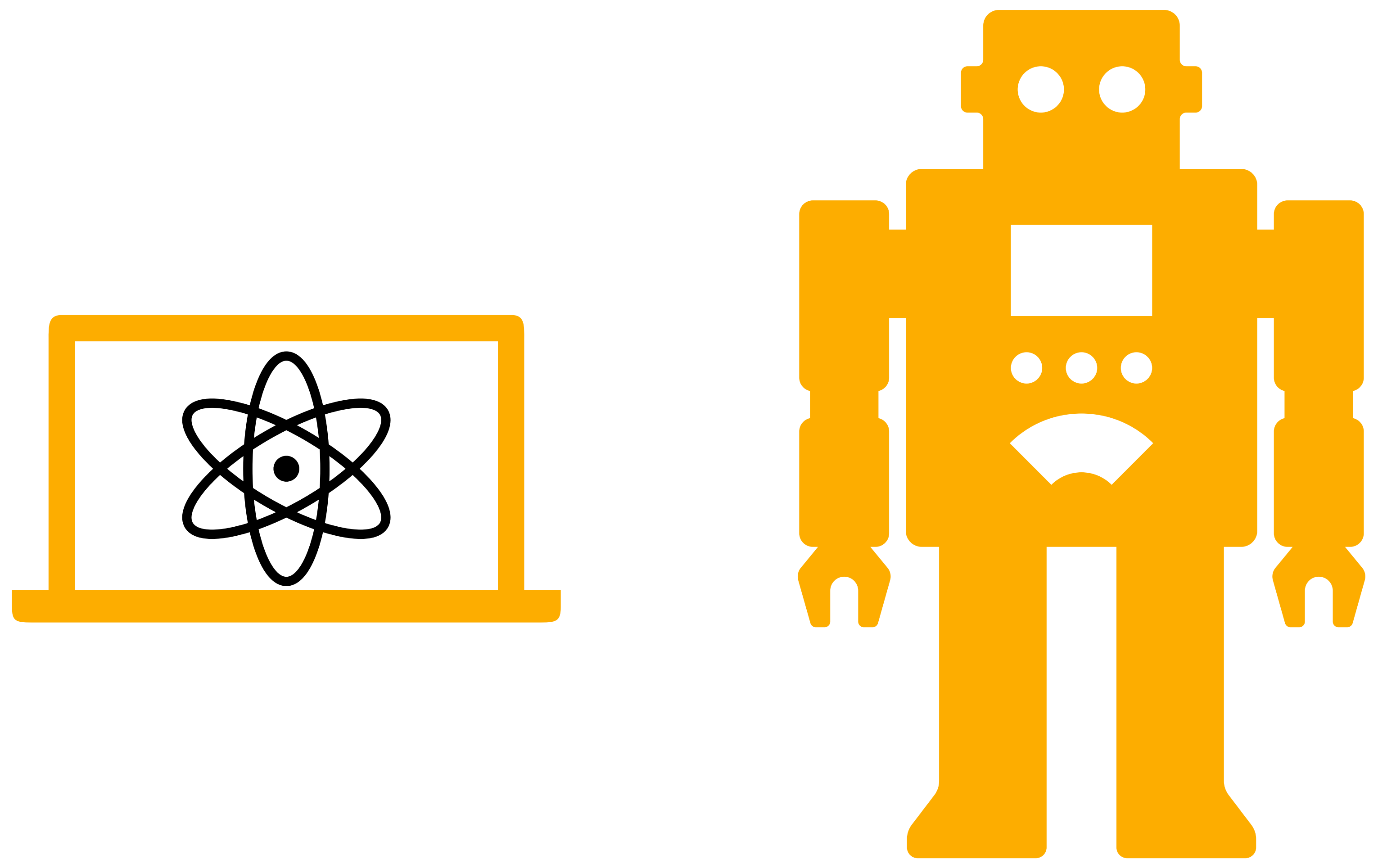
Large-scale structures of entanglement

Alexander Stottmeister

Joint work with Lauritz van Luijk, Henrik Wilming, and Reinhard F. Werner

Alice **Bob**





Entanglement

What cannot be created by **local (quantum) operations** and **classical communication** (LOCC).

Main topics for today

Main topics for today

1. A theory of embezzlement of entanglement

Main topics for today

1. **A theory of embezzlement of entanglement**

2. **Applications to many-body systems**

Connections to other talks and topics?

Connections to other talks and topics?

(a) Operator algebraic methods

Connections to other talks and topics?

(a) Operator algebraic methods

(b) Operational approach

Connections to other talks and topics?

(a) Operator algebraic methods

(b) Operational approach

(c) Model independence (to some extent)

Connections to other talks and topics?

(a) Operator algebraic methods

(b) Operational approach

(c) Model independence (to some extent)

The devil is in the details!

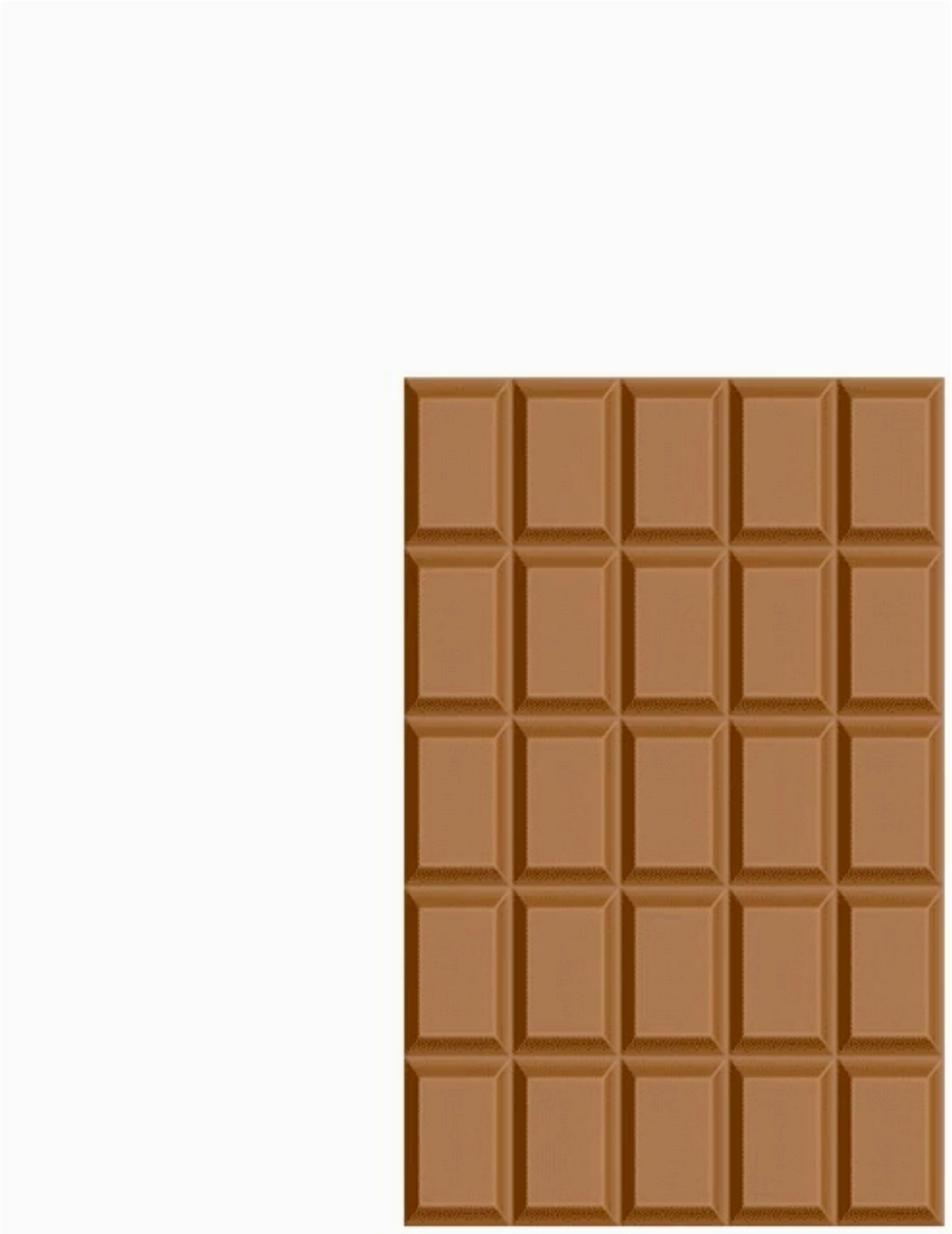
Embezzlement of entanglement

Embezzlement
Noun. /ɪmˈbez.əl.mənt/

The crime of secretly taking money that is in your care or that belongs to an organization or business you work for.

Cambridge Dictionary

Embezzlement of chocolate



user585825, How many whole pieces can be taken out in this way? (Infinite chocolate bar problem), https://math.stackexchange.com/q/2889188

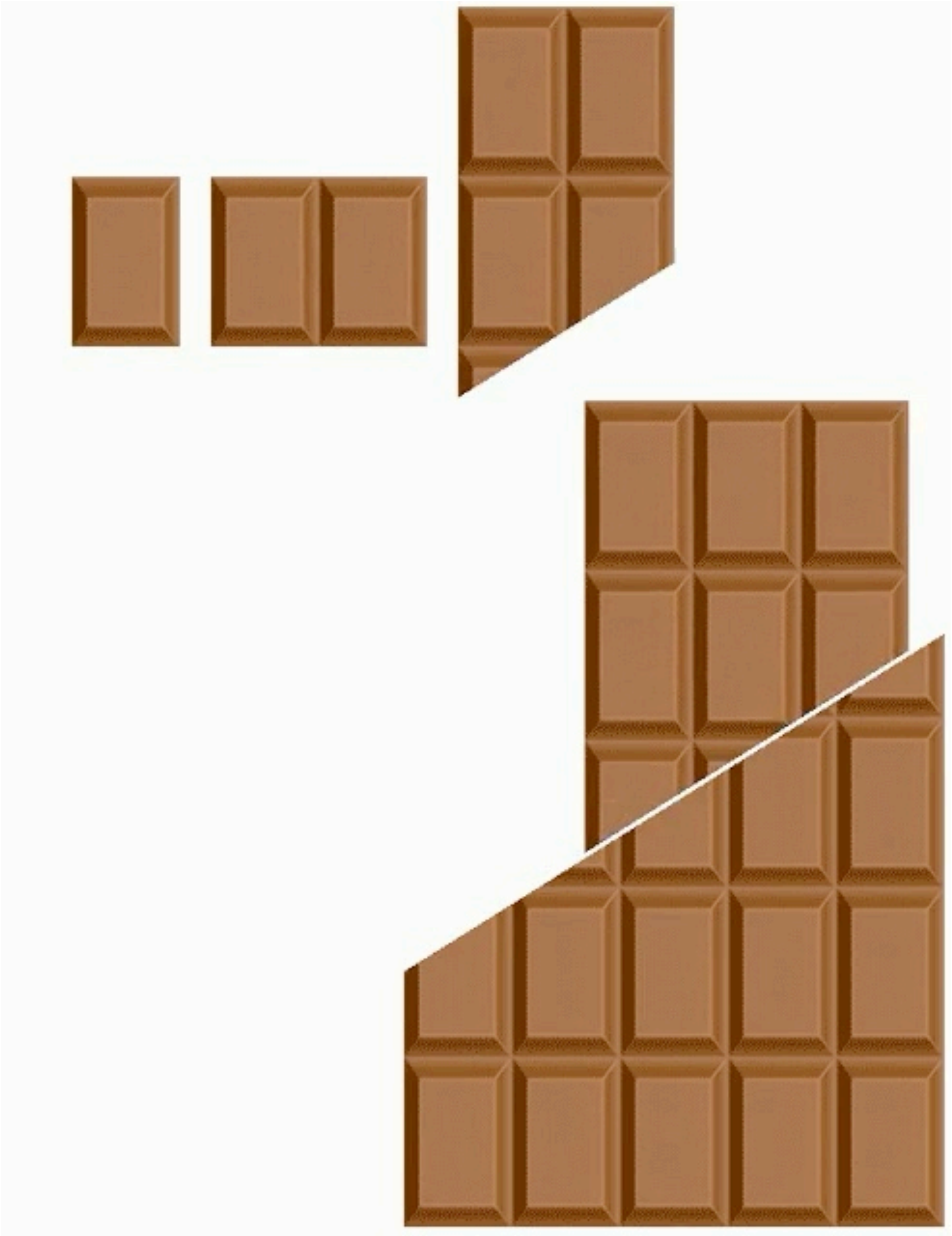
Embezzlement of chocolate

user585825, How many whole pieces can be taken out in this way? (Infinite chocolate bar problem), https://math.stackexchange.com/q/2889188

Embezzlement of chocolate

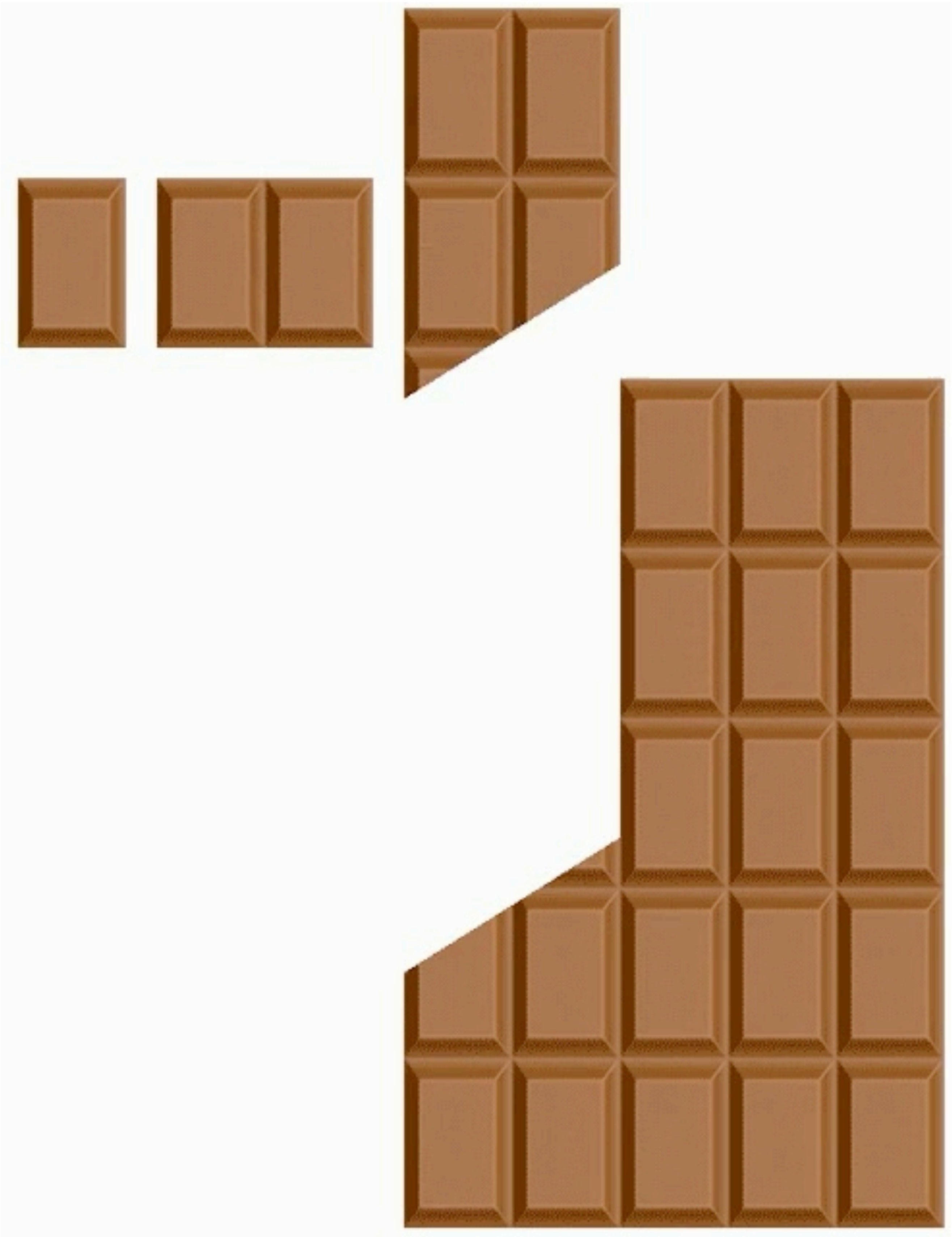
user585825, How many whole pieces can be taken out in this way? (Infinite chocolate bar problem), https://math.stackexchange.com/q/2889188

Embezzlement of chocolate



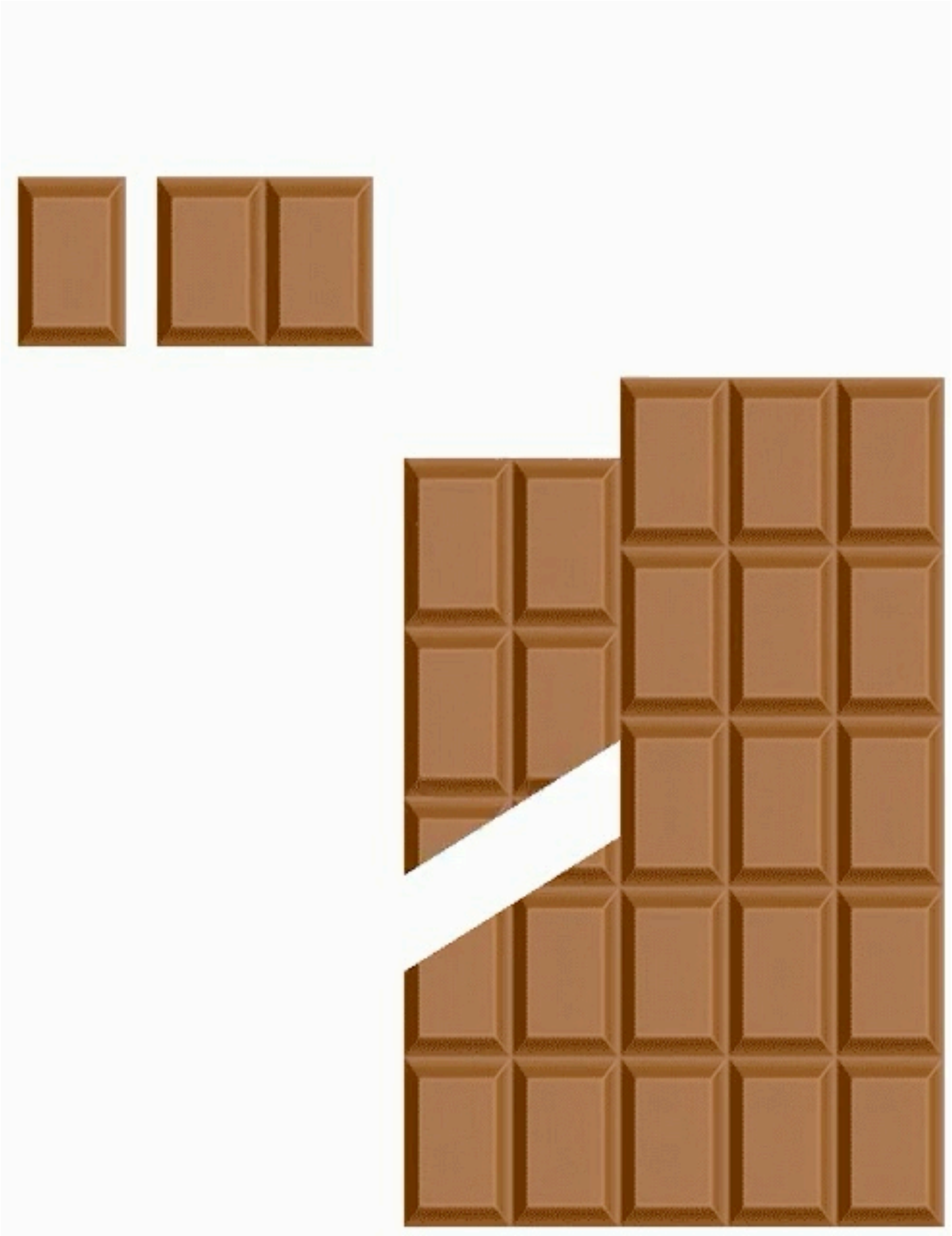
user585825, How many whole pieces can be taken out in this way? (Infinite chocolate bar problem), https://math.stackexchange.com/q/2889188

Embezzlement of chocolate



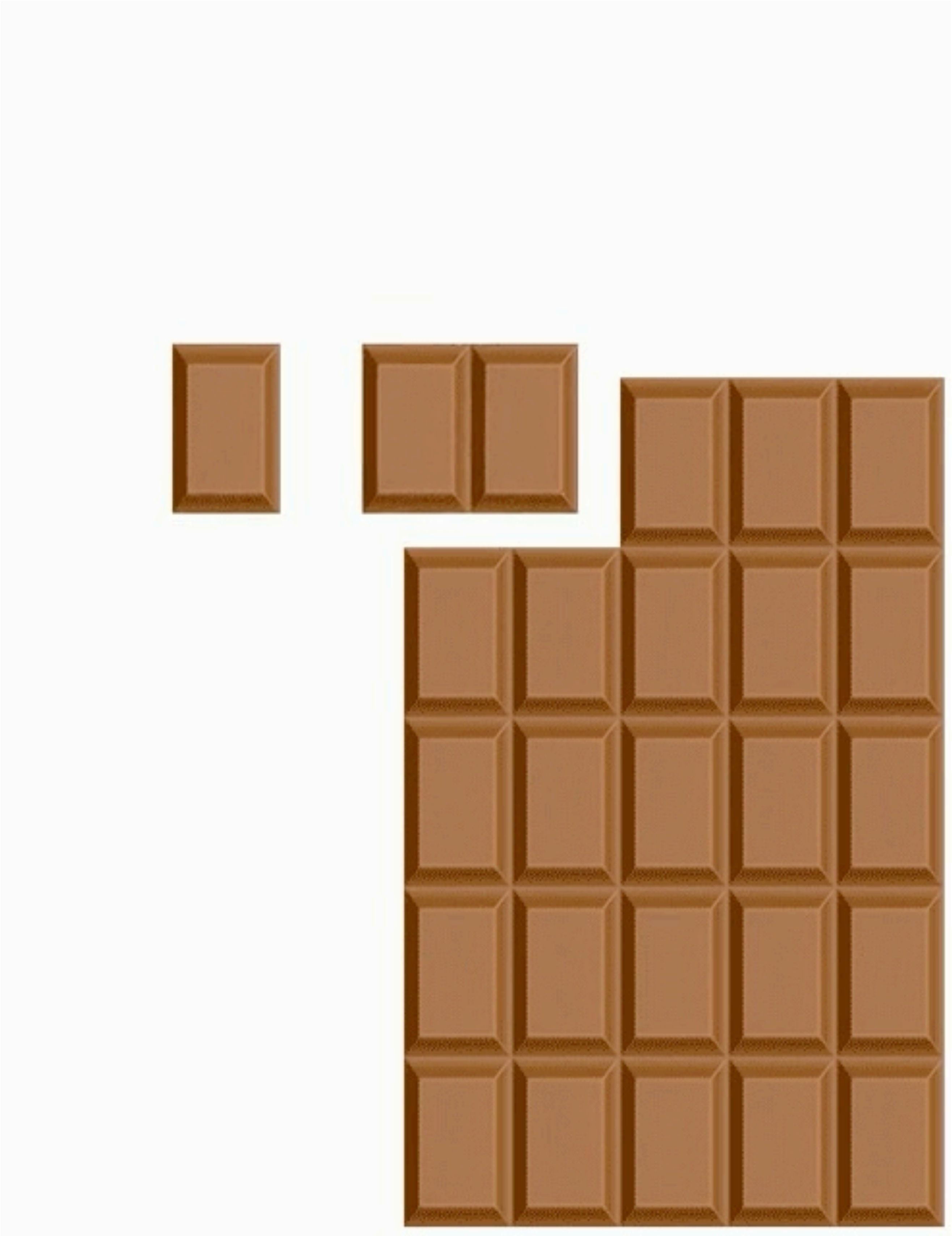
user585825, How many whole pieces can be taken out in this way? (Infinite chocolate bar problem), https://math.stackexchange.com/q/2889188

Embezzlement of chocolate



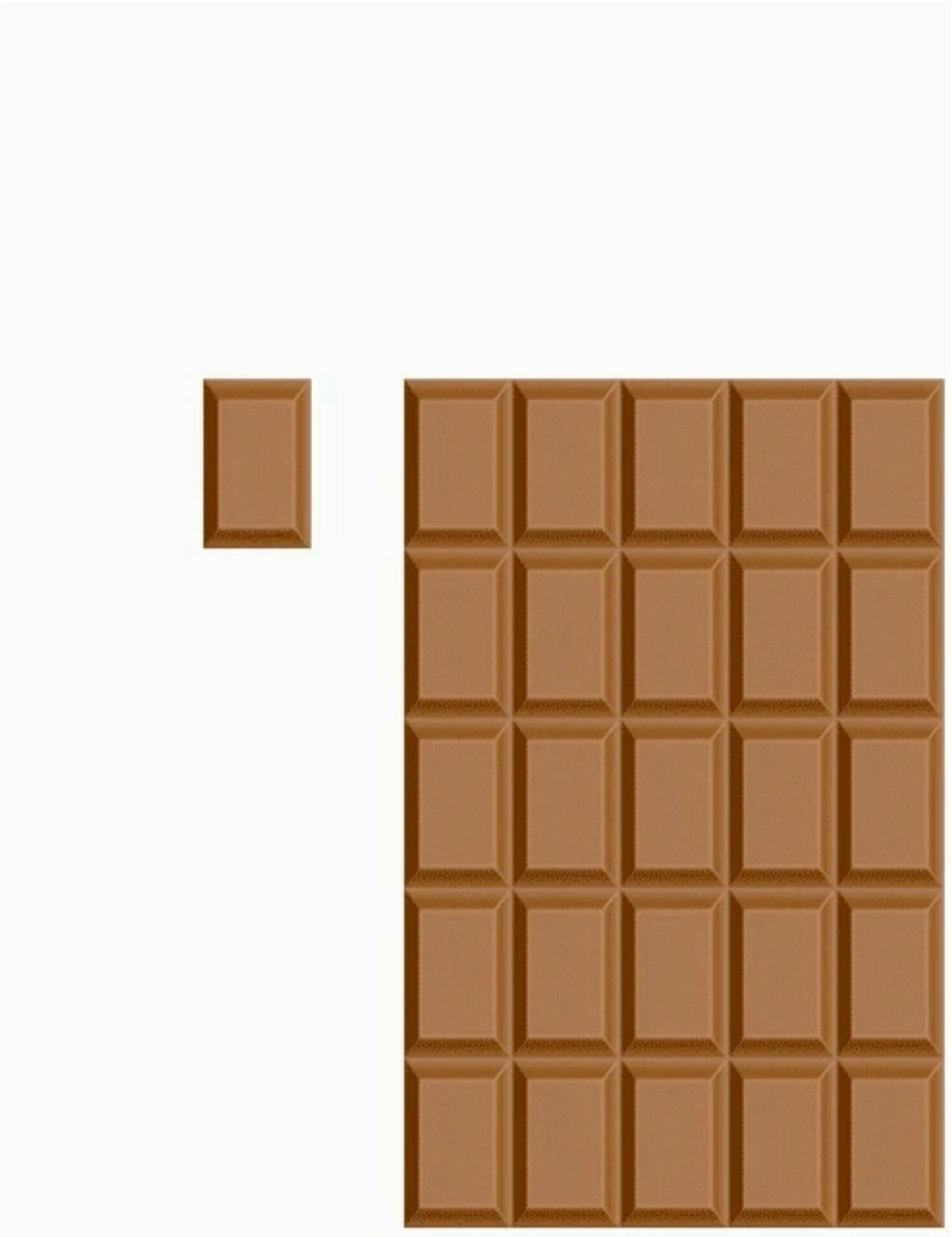
user585825, How many whole pieces can be taken out in this way? (Infinite chocolate bar problem), https://math.stackexchange.com/q/2889188

Embezzlement of chocolate



user585825, How many whole pieces can be taken out in this way? (Infinite chocolate bar problem), https://math.stackexchange.com/q/2889188

Embezzlement of chocolate

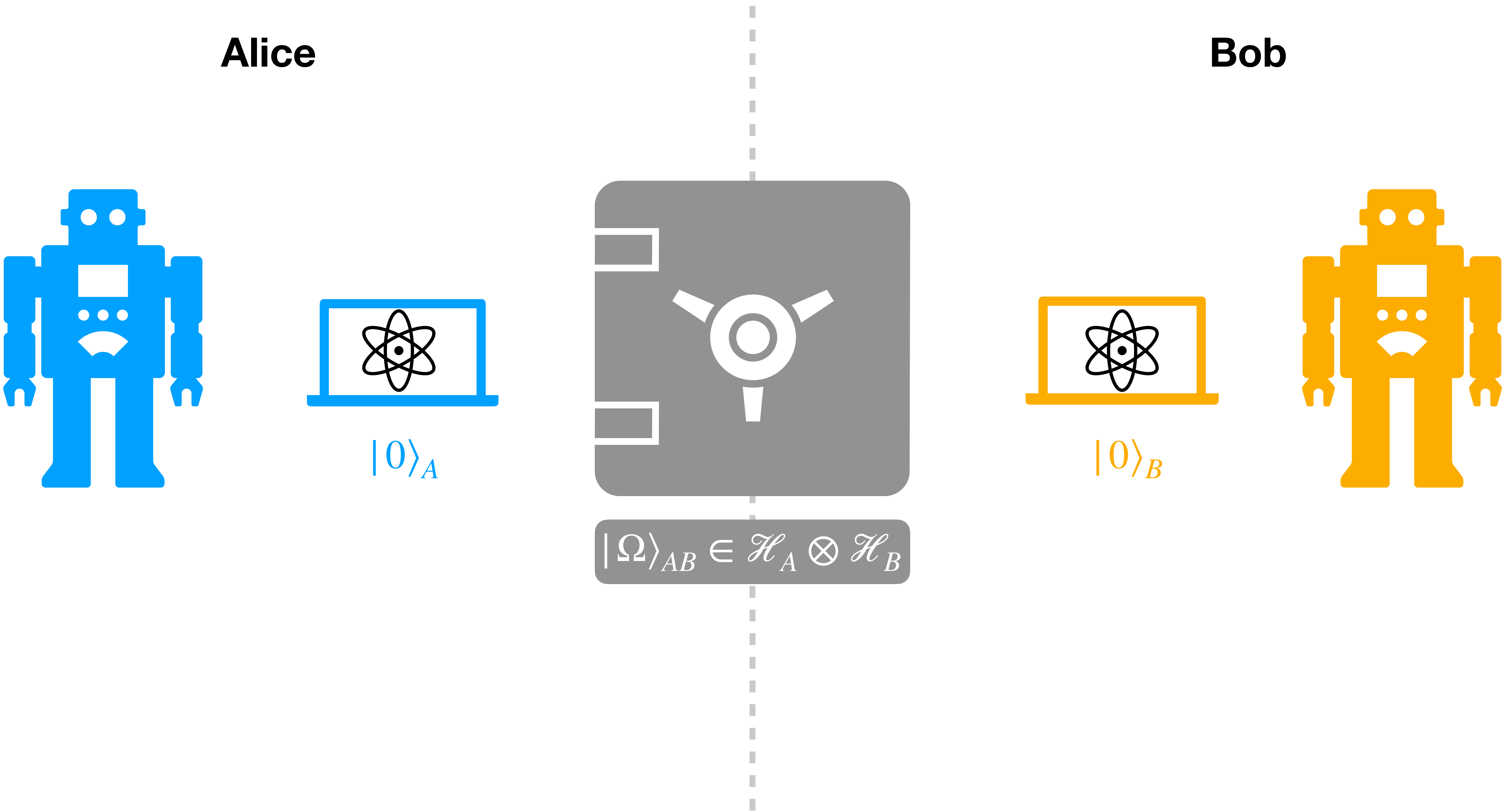


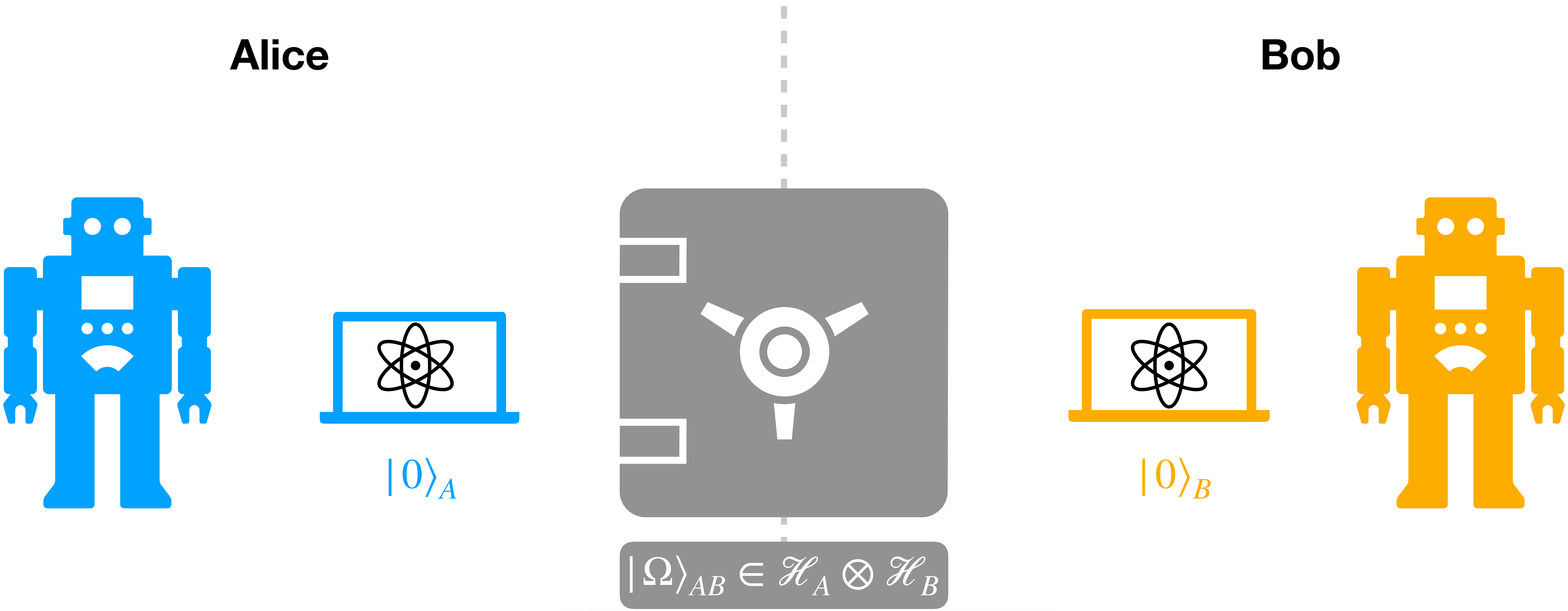
user585825, How many whole pieces can be taken out in this way? (Infinite chocolate bar problem), https://math.stackexchange.com/q/2889188

Embezzlement of entanglement
Noun. /ɪmˈbez.əl.mənt əv ɪnˈtæŋ.gəl.mənt/

The crime of secretly taking entanglement that is in your care or that belongs to an organization or business you work for.

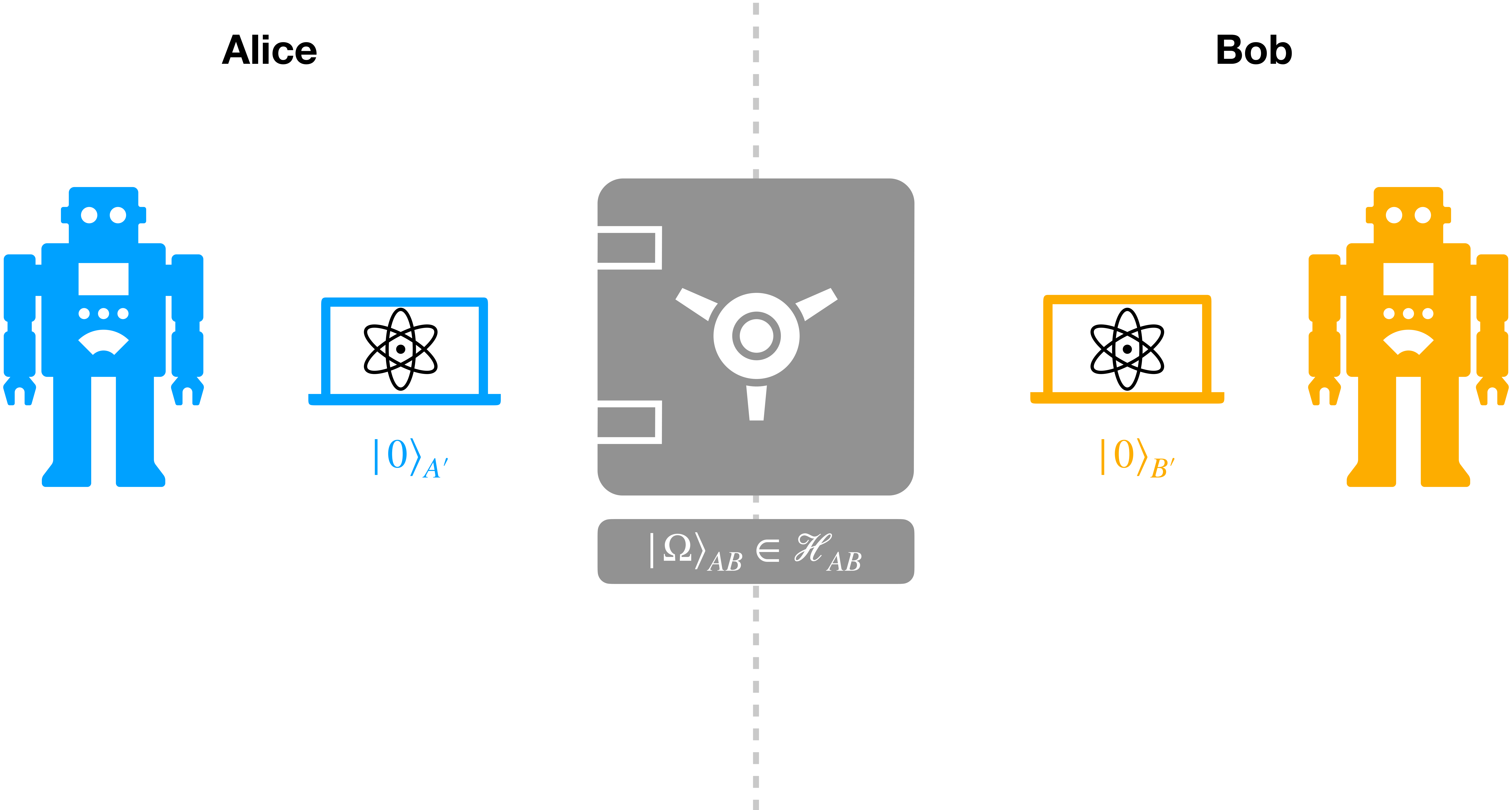
Cambridge Dictionary (next edition)

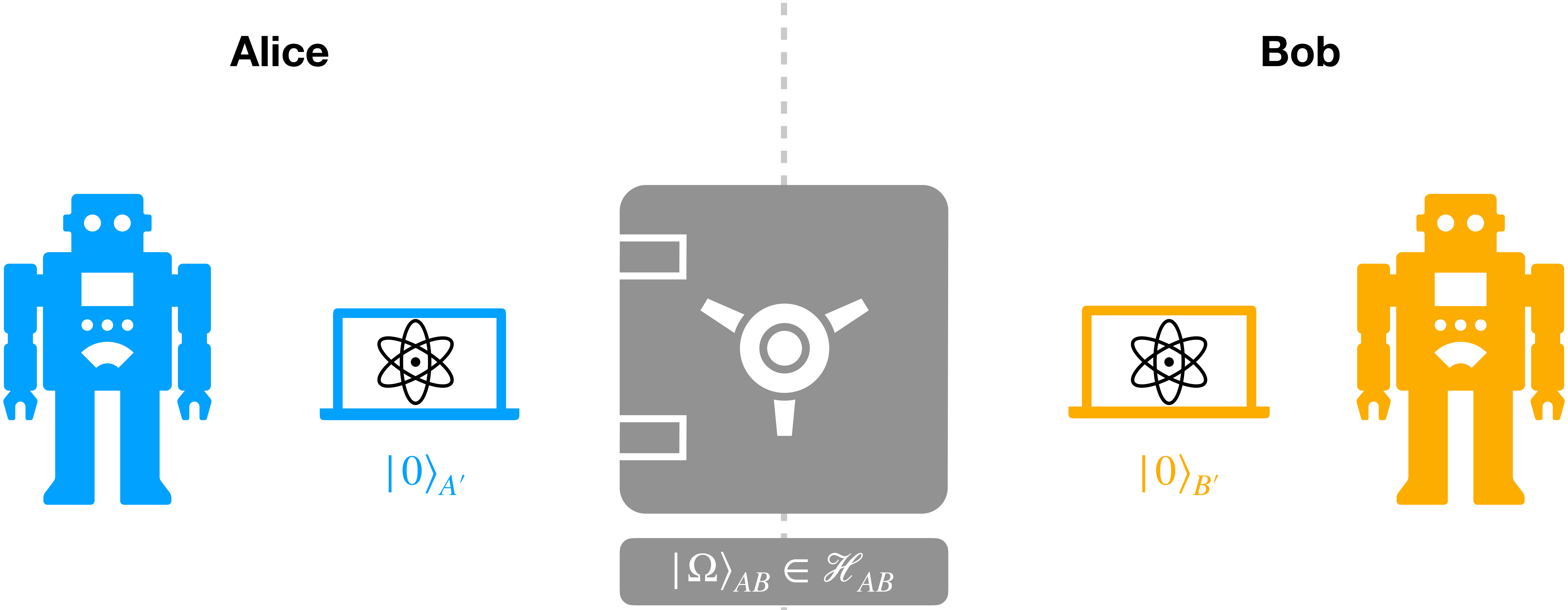

Alice
Bob
$|0\rangle_A$
$|0\rangle_B$
$|\Omega\rangle_{AB} \in \mathscr{H}_A \otimes \mathscr{H}_B$



The crime of secretly taking entanglement from […]

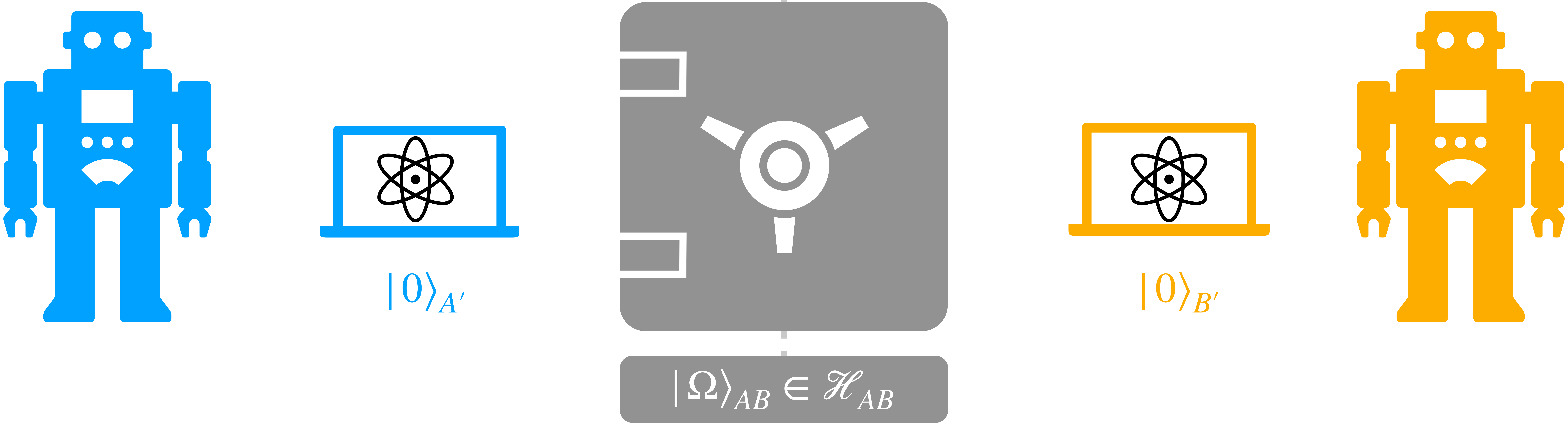
To be sure that no record of the crime exists, should be done without classical communication!


Alice
Bob
$|0\rangle_{A'}$
$|\Omega\rangle_{AB} \in \mathscr{H}_{AB}$
$|0\rangle_{B'}$



Ideally: $U_{AA'}U_{BB'}(|\Omega\rangle_{AB} \otimes |0\rangle_{A'}|0\rangle_{B'}) = |\Omega\rangle_{AB} \otimes |\Psi\rangle_{A'B'}$

Alice **Bob**



Ideally: $U_{AA'}U_{BB'}(|\Omega\rangle_{AB} \otimes |0\rangle_{A'}|0\rangle_{B'}) = |\Omega\rangle_{AB} \otimes |\Psi\rangle_{A'B'}$ **Impossible!**

R. Cleve, L. Liu, V. I. Paulsen, "Perfect Embezzlement of Entanglement", *J. Math. Phys.* 58, 2017

The van Dam — Hayden family

The family of states

$$|\Omega_n\rangle_{AB} := \frac{1}{\sqrt{H_n}} \sum_{j=1}^{n} \frac{1}{\sqrt{j}} |j\rangle_A |j\rangle_B$$

fulfills

$$\boxed{\inf_{U_{AA'},U_{BB'}} \| U_{AA'} U_{BB'} (|\Omega_n\rangle_{AB} \otimes |0\rangle_{A'} |0\rangle_{B'}) - |\Omega_n\rangle_{AB} \otimes |\Psi\rangle_{A'B'} \|^2 \leq 2 \frac{\log(d)}{\log(n)}}$$

Schmidt rank

Embezzling family

W. van Dam, P. Hayden, “Universal Entanglement Transformations without Communication”, *Physical Review A* 67, 060302, 2003

The van Dam — Hayden family

The family of states

$$|\Omega_n\rangle_{AB} := \frac{1}{\sqrt{H_n}} \sum_{j=1}^{n} \frac{1}{\sqrt{j}} |j\rangle_A |j\rangle_B$$

fulfills

$$\inf_{U_{AA'},U_{BB'}} \| U_{AA'} U_{BB'} (|\Omega_n\rangle_{AB} \otimes |0\rangle_{A'} |0\rangle_{B'}) - |\Omega_n\rangle_{AB} \otimes |\Psi\rangle_{A'B'} \|^2 \leq 2 \frac{\log(d)}{\log(n)}$$

Schmidt rank

Embezzling family

W. van Dam, P. Hayden, “Universal Entanglement Transformations without Communication”, *Physical Review A* 67, 060302, 2003

- The error is essentially optimal (Fannes-Audenaert & Powers-Størmer)
- Every embezzling family has a Schmidt spectrum scaling roughly as

$$\lambda_j \sim \frac{1}{j}$$

- The normalization factor H_n diverges with n

D. Leung, B. Wang, “Characteristics of Universal Embezzling Families”, *Physical Review A* 90, no. 4 (2014): 042331

E. Zanoni, T. Theurer, G. Gour, “Complete Characterization of Entanglement Embezzlement”, *Quantum* 8, 1368, 2024

The van Dam — Hayden family

The family of states

$$|\Omega_n\rangle_{AB} := \frac{1}{\sqrt{H_n}} \sum_{j=1}^{n} \frac{1}{\sqrt{j}} |j\rangle_A |j\rangle_B$$

fulfills

$$\boxed{\inf_{U_{AA'}, U_{BB'}} \| U_{AA'} U_{BB'} (|\Omega_n\rangle_{AB} \otimes |0\rangle_{A'} |0\rangle_{B'}) - |\Omega_n\rangle_{AB} \otimes |\Psi\rangle_{A'B'} \|^2 \leq 2 \frac{\log(d)}{\log(n)}}$$

Schmidt rank

Embezzling family

W. van Dam, P. Hayden, "Universal Entanglement Transformations without Communication", *Physical Review A* 67, 060302, 2003

- The error is essentially optimal (Fannes-Audenaert & Powers-Størmer)
- Every embezzling family has a Schmidt spectrum scaling roughly as

$$\lambda_j \sim \frac{1}{j}$$

- The normalization factor H_n diverges with n
  Cannot simply take the limit!

D. Leung, B. Wang, "Characteristics of Universal Embezzling Families", *Physical Review A* 90, no. 4 (2014): 042331

E. Zanoni, T. Theurer, G. Gour, "Complete Characterization of Entanglement Embezzlement", *Quantum* 8, 1368, 2024

Setting: Commuting operator framework

	Commuting Operator	Tensor Product
Hilbert space	$\mathscr{H}_{AB}$	$\mathscr{H}_A \otimes \mathscr{H}_B$
Quantum States	$\lvert\Omega\rangle \in \mathscr{H}_{AB}$	$\lvert\Omega\rangle \in \mathscr{H}_A \otimes \mathscr{H}_B$
Operators of Alice	$\mathscr{M}_A \subseteq \mathscr{B}(\mathscr{H}_{AB})$	$\mathscr{B}(\mathscr{H}_A) \otimes 1$
Operators of Bob	$\mathscr{M}_B \subseteq \mathscr{B}(\mathscr{H}_{AB})$	$1 \otimes \mathscr{B}(\mathscr{H}_B)$
Haag Duality	$\mathscr{M}_A = \mathscr{M}_B'$	automatic
No classical d.o.f.	$\mathscr{M}_A \cap \mathscr{M}_A' = \mathbb{C}1$	automatic

$$\mathscr{M}' := \{a \in \mathscr{B}(\mathscr{H}) \mid [a, \mathscr{M}] = 0\}$$ Commutant

- All Hilbert spaces separable
- All algebras von Neumann algebras:

$$\mathscr{M} = \mathscr{M}''$$

- Factor:

$$\mathscr{M} \cap \mathscr{M}' = \mathbb{C}1$$

Bipartite systems and embezzling states

Bipartite system: A triple $(\mathscr{H}, \mathscr{M}, \mathscr{M}')$.

Embezzling state: $|\Omega\rangle_{AB} \in \mathscr{H}$ such that for all $|\Psi\rangle_{A'B'} \in \mathbb{C}^n \otimes \mathbb{C}^n$ and all $\varepsilon > 0$ exist local unitaries such that

$$\| U_{AA'} U_{BB'} \big(|\Omega\rangle_{AB} \otimes |0\rangle_{A'} |0\rangle_{B'} \big) - |\Omega\rangle_{AB} \otimes |\Psi\rangle_{A'B'} \| < \varepsilon$$

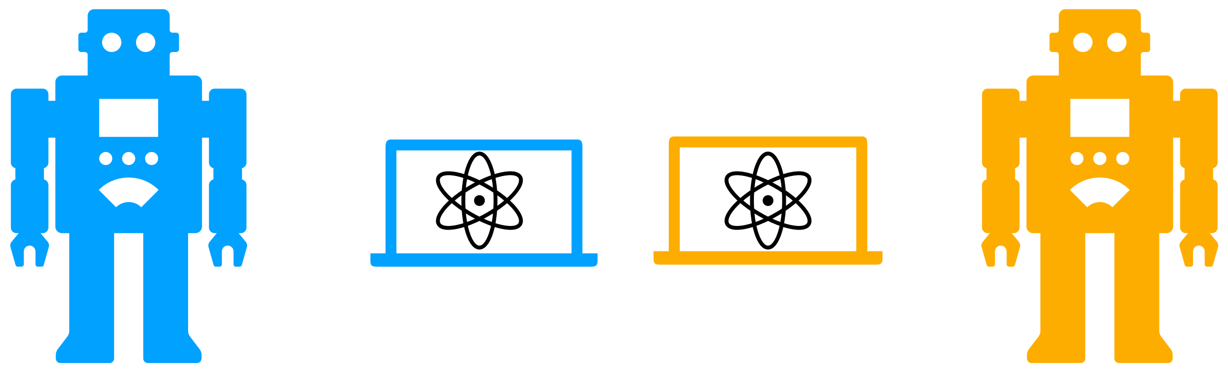
$U_{AA'} \in \mathscr{M} \otimes M_n \otimes 1, \quad U_{BB'} \in \mathscr{M}' \otimes 1 \otimes M_n$

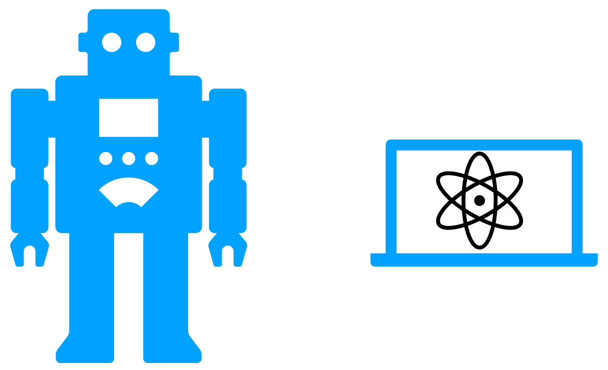
From bipartite to monopartite: LOCC

Pure state
entanglement theory

Nielsen’s theorem

Majorization theory on
reduced state





$|\Psi\rangle_{AB} \in \mathscr{H}_A \otimes \mathscr{H}_B$

partial trace

purification

$\rho_A \in \mathcal{S}(\mathscr{H}_A)$

M. A. Nielsen, “Conditions for a Class of Entanglement Transformations”, *Physical Review Letters* 83, 436-439, 1999

J. Crann et al., “State Convertibility in the von Neumann Algebra Framework”, *Commun. Math. Phys.* 378, 1123, 2022

L. van Luijk, A. S., R.F. Werner. H. Wilming, “Pure state entanglement and von Neumann algebras”, *Commun. Math. Phys.* 406, 296, 2025

From bipartite to monopartite: Embezzlement

Monopartite embezzling state: State ω on $\mathscr{M}$ such that for all $\varepsilon > 0$ and all states ψ on M_n there exists a unitary such that

$$\|U(\omega \otimes \langle 0 | \cdot | 0 \rangle)U^* - \omega \otimes \psi\| < \varepsilon$$

$U \in \mathscr{M} \otimes M_n$

From bipartite to monopartite: Embezzlement

Monopartite embezzling state: State ω on $\mathscr{M}$ such that for all $\varepsilon > 0$ and all states ψ on M_n there exists a unitary such that

$$\|U(\omega \otimes \langle 0| \cdot |0\rangle)U^* - \omega \otimes \psi\| < \varepsilon$$

$U \in \mathscr{M} \otimes M_n$

Theorem.

entangelement embezzlement $\iff$ monopartite embezzlement.

$$\|U_A U_B(|\Omega\rangle_{AB} \otimes |0\rangle_A |0\rangle_B) - |\Omega\rangle_{AB} \otimes |\Psi\rangle_{AB}\| < \varepsilon$$

$$\|U_A(\omega \otimes \langle 0| \cdot |0\rangle)U_A^* - \omega \otimes \psi_A\| < \varepsilon'$$

Quantifying embezzlement

Quality of a state to act as an embezzler in worst case:

$$\kappa(\omega) = \sup_{n\in\mathbb{N}} \sup_{\rho,\sigma} \inf_{U} \|U(\omega \otimes \rho)U^* - \omega \otimes \sigma\|$$

Quantifying embezzlement

Quality of a state to act as an embezzler in worst case:

$$\kappa(\omega) = \sup_{n\in\mathbb{N}} \sup_{\rho,\sigma} \inf_{U} \| U(\omega \otimes \rho)U^* - \omega \otimes \sigma \|$$

Minimize error over unitaries

Quantifying embezzlement

Quality of a state to act as an embezzler in worst case:

$$\kappa(\omega) = \sup_{n\in\mathbb{N}} \sup_{\rho,\sigma} \inf_{U} \| U(\omega \otimes \rho)U^* - \omega \otimes \sigma \|$$

Minimize error over unitaries

Maximize error over all n-dim. density matrices

Quantifying embezzlement

Quality of a state to act as an embezzler in worst case:

$$\kappa(\omega) = \sup_{n\in\mathbb{N}} \sup_{\rho,\sigma} \inf_{U} ||U(\omega \otimes \rho)U^* - \omega \otimes \sigma||$$

Minimize error over unitaries

Maximize error over all n-dim. density matrices

Maximize over all finite dimensions

Quantifying embezzlement

Quality of a state to act as an embezzler in worst case:

$$\kappa(\omega) = \sup_{n\in\mathbb{N}} \sup_{\rho,\sigma} \inf_{U} \| U(\omega \otimes \rho)U^* - \omega \otimes \sigma \|$$

Best worst-case error:

Worst worst-case error:

$$\kappa_{\min} = \inf_{\omega} \kappa(\omega) \quad \leq \quad \kappa(\omega) \quad \leq \quad \kappa_{\max} = \sup_{\omega} \kappa(\omega)$$

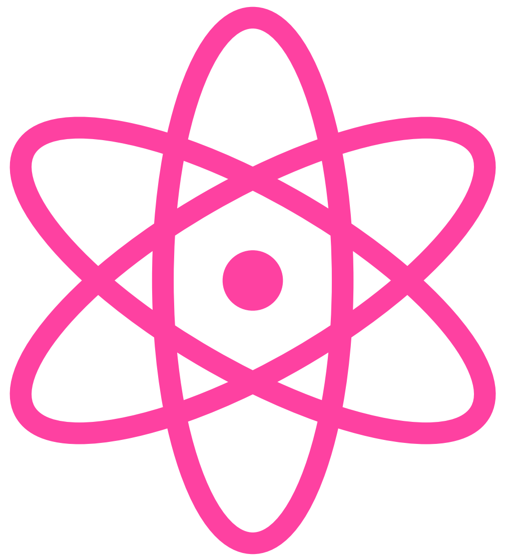
$\kappa_{\min}$ and $\kappa_{\max}$ are algebraic invariants of $\mathscr{M}$

A substitute for majorization theory

$(\mathscr{M}, \omega) \longrightarrow$ **Flow of Weights** $\longrightarrow \big((X, \hat{\sigma}_s), P_\omega\big)$

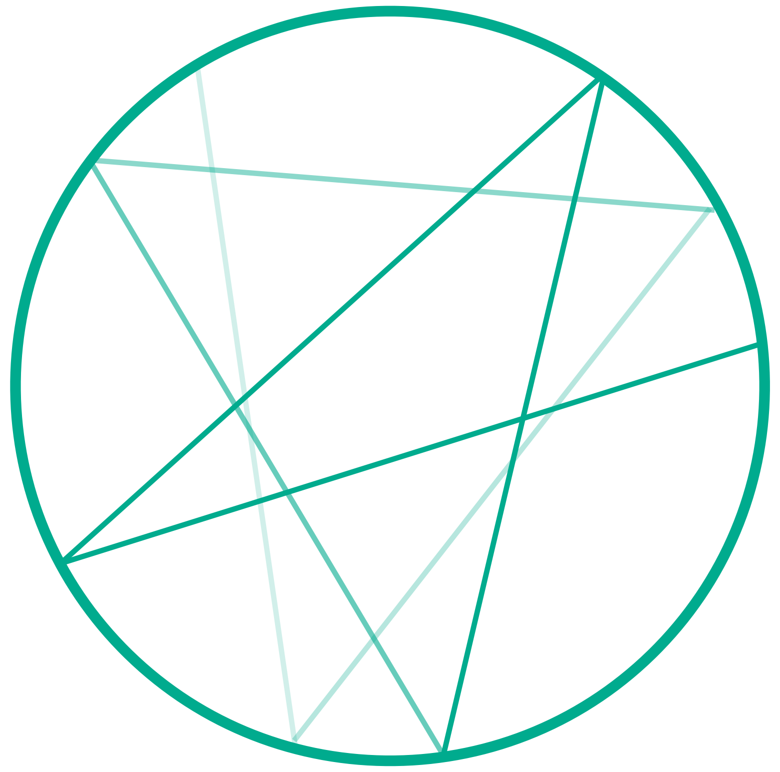
von Neumann Algebra,
state

classical dynamical system,
probability measure



A. Connes, M. Takesaki, “The Flow of Weights on Factors of Type III”, *Tohoku Mathematical Journal* 29, 473-575, 1977

U. Haagerup, E. Størmer, “Equivalence of Normal States on von Neumann Algebras and the Flow of Weights”, *Advances in Mathematics* 83, 180-262, 1990



A substitute for majorization theory

$$(\mathscr{M}, \omega)$$

von Neumann Algebra,
state

Flow of Weights

$$\mathscr{N} = \mathscr{M} \rtimes_{\sigma^\phi} \mathbb{R}, \quad \sigma_t^\phi(m) = \pi_\phi^{-1}\left(\Delta_\phi^{it}\, \pi_\phi(m)\, \Delta_\phi^{-it}\right)$$

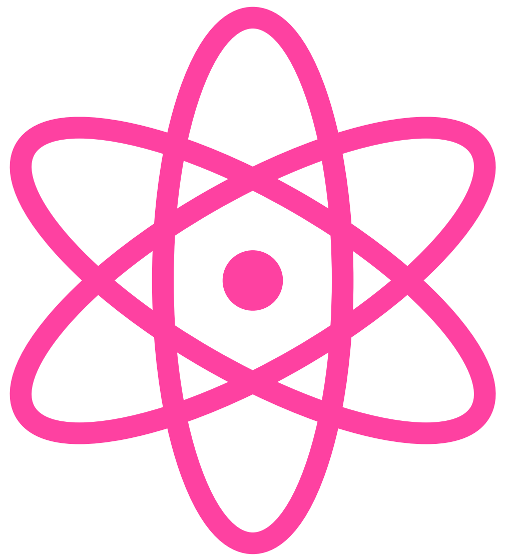
generators $\lambda(t)$ modular flow on $\mathscr{M}$

Dual action: $\theta_t(m) = m, \quad \theta_s(\lambda(t)) = e^{-ist}\lambda(t)$

$$X = Z(\mathscr{N}), \quad \hat{\sigma}_s = \theta_{s|Z(\mathscr{N})}, \quad P_\omega = \tau\left(\chi_{(1,\infty)}\left(\frac{d\tilde{\omega}}{d\tau}\right) \cdot \right)$$

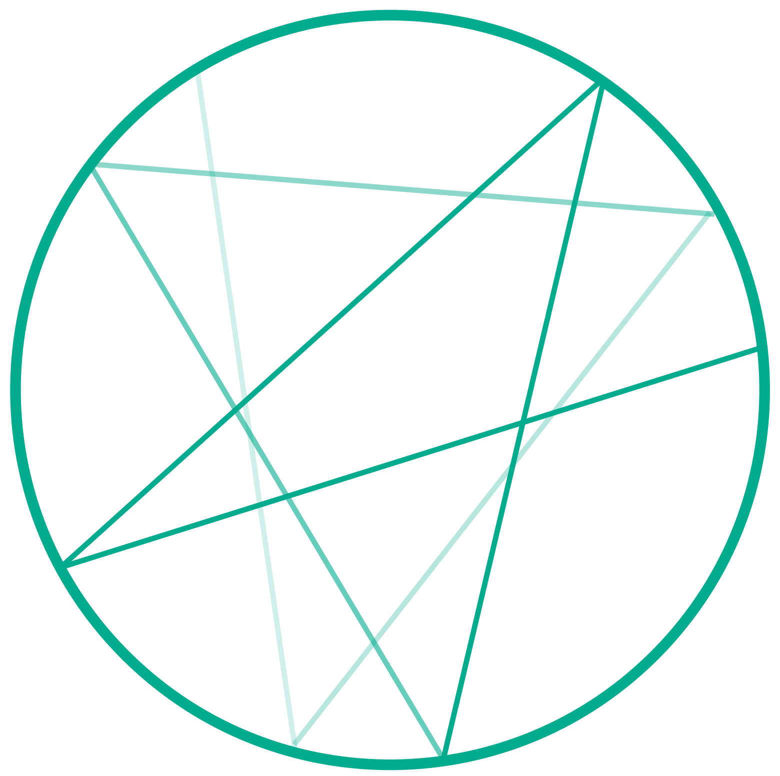
$$\left((X, \hat{\sigma}_s), P_\omega\right)$$

classical dynamical system,
probability measure



A. Connes, M. Takesaki, “The Flow of Weights on Factors of Type III”, *Tohoku Mathematical Journal* 29, 473-575, 1977

U. Haagerup, E. Størmer, “Equivalence of Normal States on von Neumann Algebras and the Flow of Weights”, *Advances in Mathematics* 83, 180-262, 1990



Spectral states of Haagerup-Størmer

ω on $\mathscr{M}$ $\longrightarrow$ P_ω state (= prob. distr.) on X

Theorem. Distance of unitary orbits = distance of spectral states:

$$\inf_{U \in \mathscr{U}(\mathscr{M})} \|U\omega U^* - \varphi\| = \|P_\omega - P_\varphi\|$$

U. Haagerup and E. Størmer. "Equivalence of normal states on von Neumann algebras and the flow of weights", *Advances in Mathematics* 83.2 (1990)

- P_ω contains full spectral information
- The set $\{P_\omega : \omega\}$ can be characterized precisely

The fundamental theorem

$$(\mathscr{M}, \omega) \longrightarrow \boxed{\textit{Flow of Weights}} \longrightarrow \big((X, \hat{\sigma}_s), P_\omega\big)$$

von Neumann Algebra,
state

classical dynamical system,
probability measure

Embezzlement $\longleftrightarrow$ Invariance under $\hat{\sigma}_s$

The fundamental theorem

$(\mathscr{M},\omega) \longrightarrow$ **Flow of Weights** $\longrightarrow \big((X,\hat{\sigma}_s),P_\omega\big)$

von Neumann Algebra, state

classical dynamical system, probability measure

Theorem. $$\kappa(\omega) = \sup_{s\in\mathbb{R}} \|\hat{\sigma}_s(P_\omega) - P_\omega\|$$

Embezzlement $\longleftrightarrow$ Invariance under $\hat{\sigma}_s$

Spectral states for $\mathscr{M} = M_n$

Flow of weights: $X = (0,\infty)$, $\sigma_s(t) = e^s t$

Spectral states for $\mathscr{M} = M_n$

Flow of weights: $X = (0,\infty),\ \sigma_s(t) = e^s t$

Density matrix

$$\rho = \sum_i p_i P_i$$

$$\longrightarrow$$

Probability density

$$dP_\rho(t) = D_\rho(t)\,dt$$

Spectral states for $\mathscr{M} = M_n$

Flow of weights: $X = (0,\infty)$, $\sigma_s(t) = e^s t$

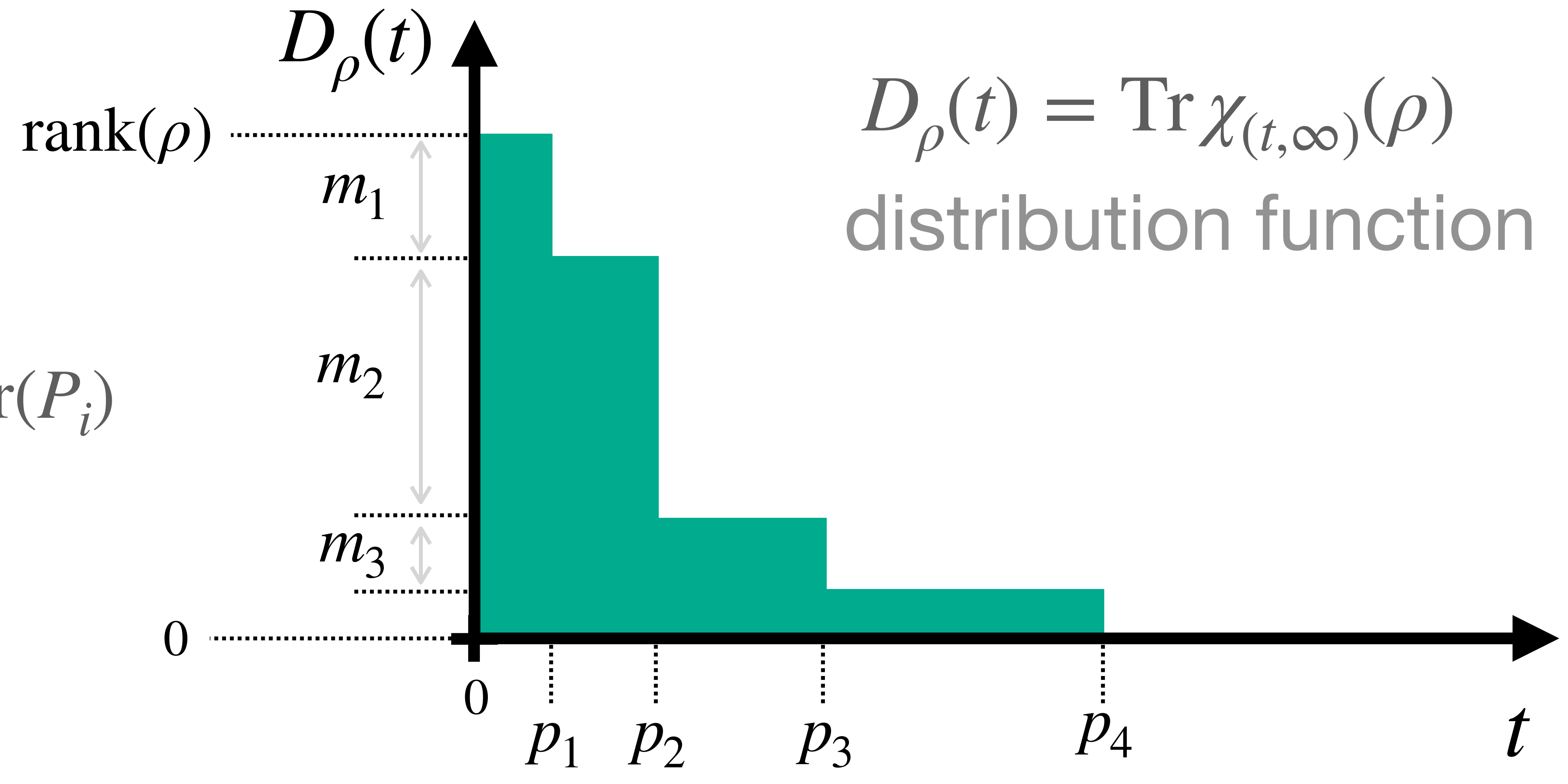
Density matrix

$$\rho = \sum_i p_i P_i$$

$\longrightarrow$

Probability density

$$dP_\rho(t) = D_\rho(t)\, dt$$

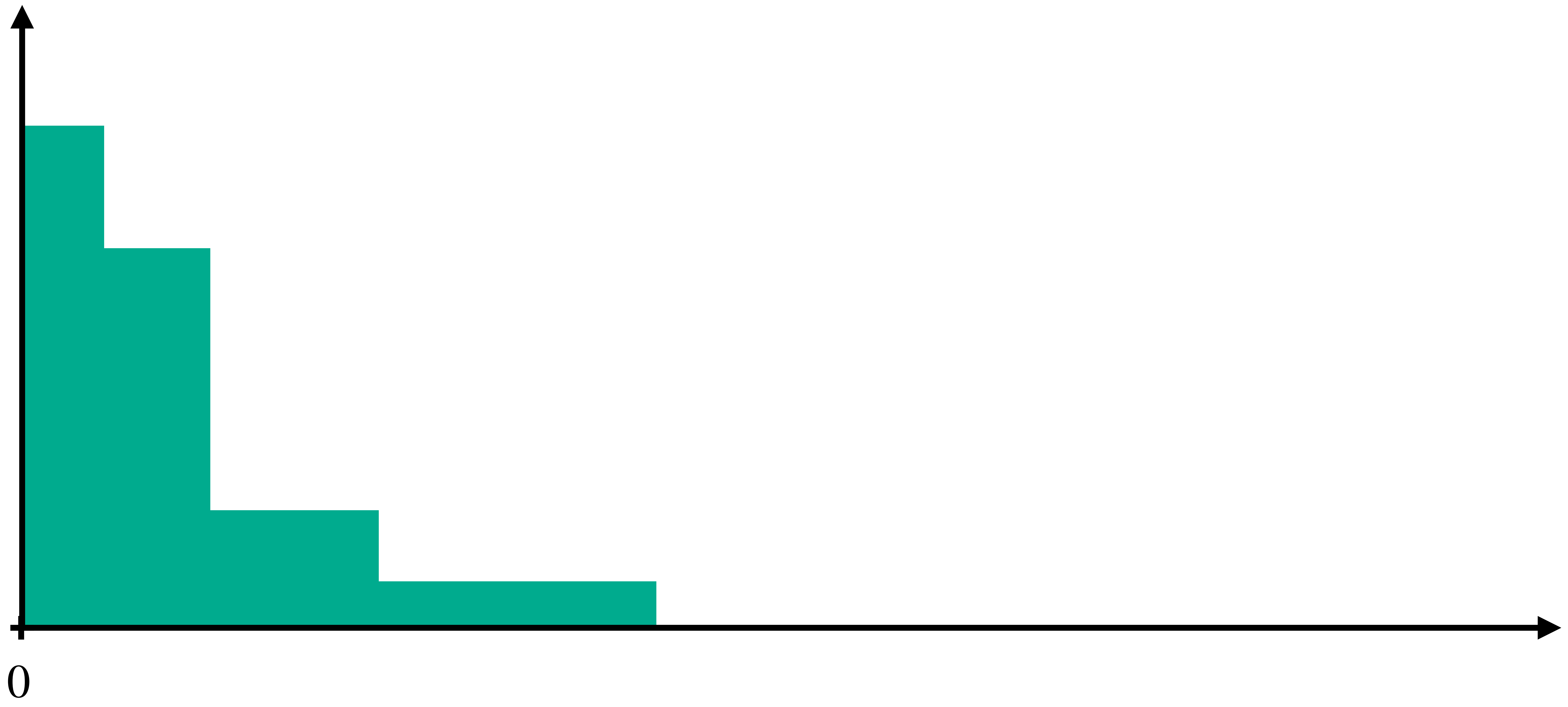


$D_\rho(t) = \mathrm{Tr}\,\chi_{(t,\infty)}(\rho)$

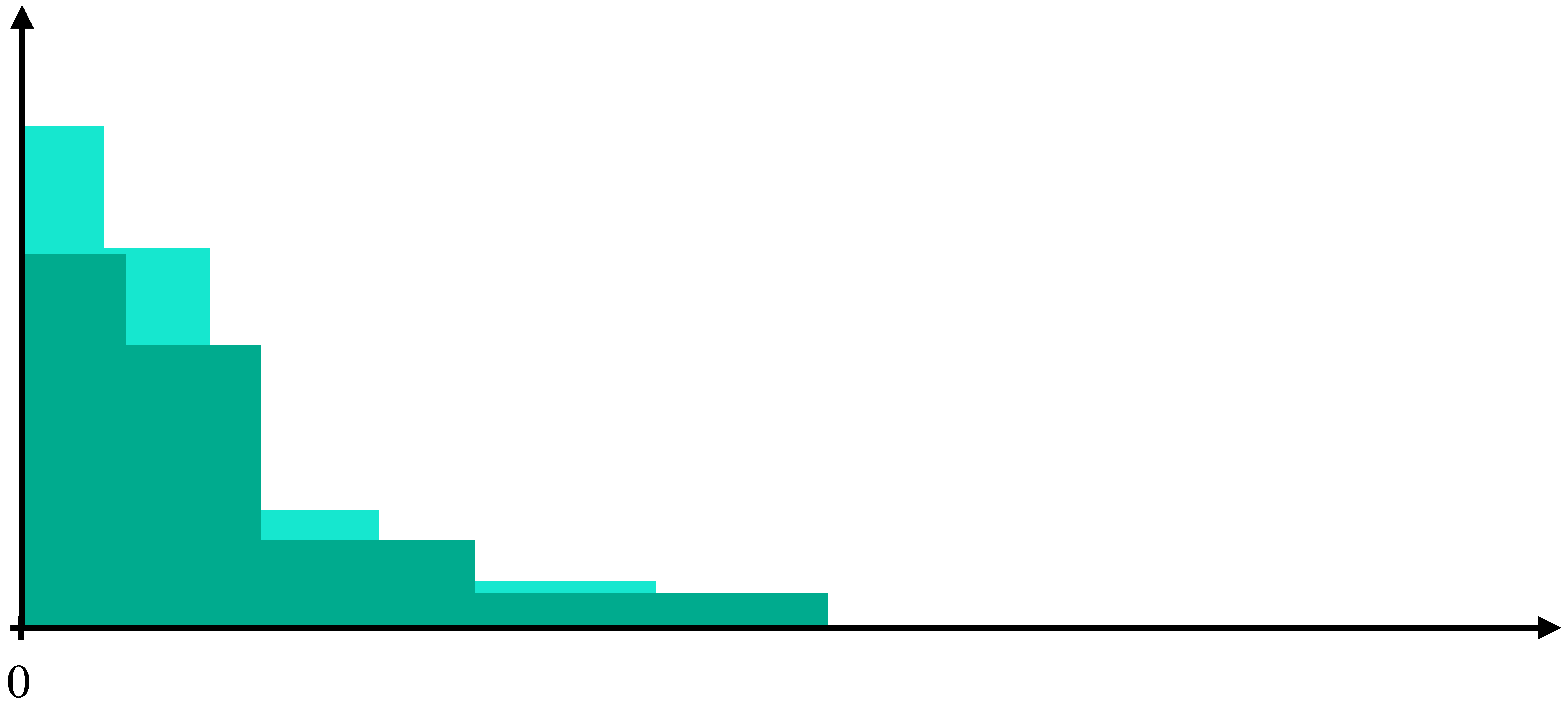
distribution function

Multiplicities $m_i = \mathrm{Tr}(P_i)$

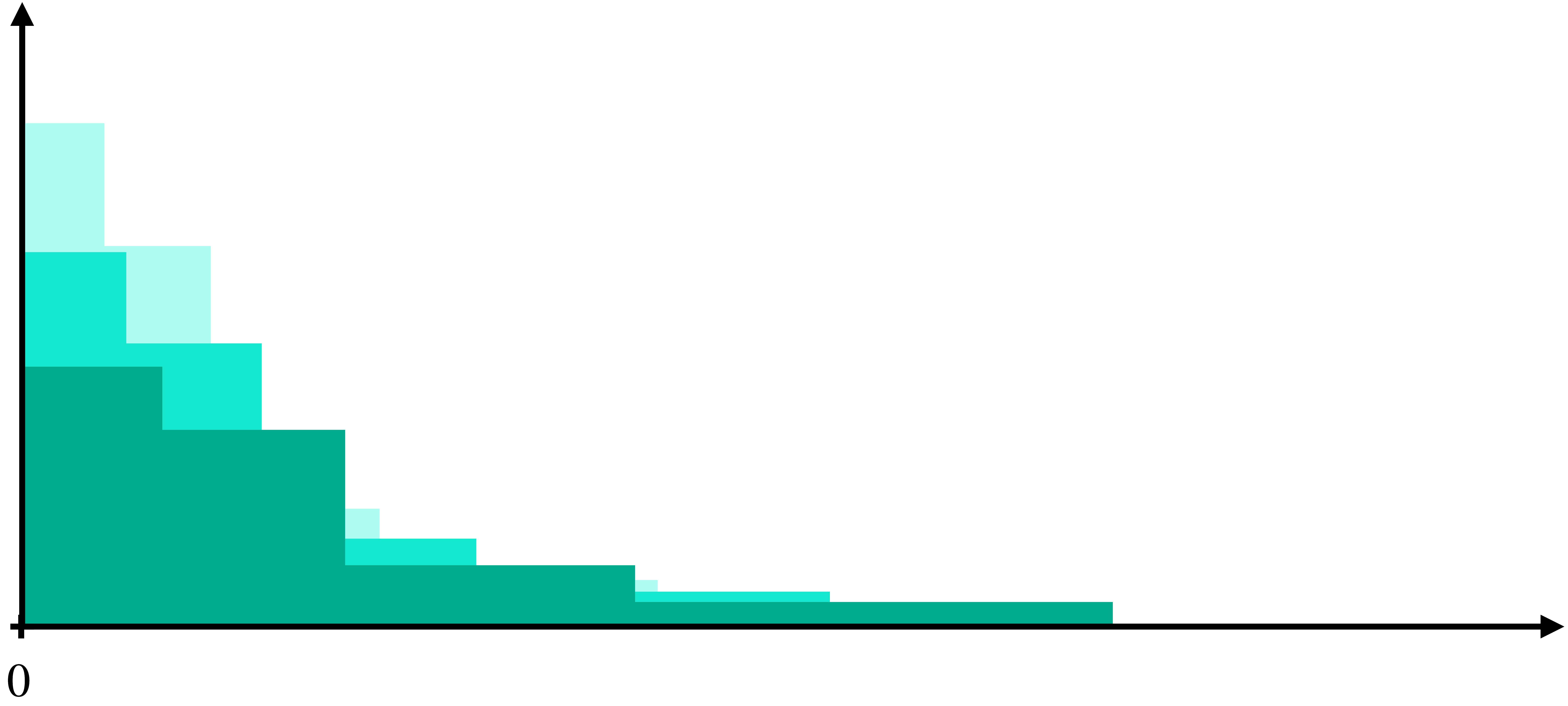
Spectral states for $\mathscr{M} = M_n$ Flow of weights: $X = (0,\infty)$, $\hat{\sigma}_s(t) = e^s \cdot t$



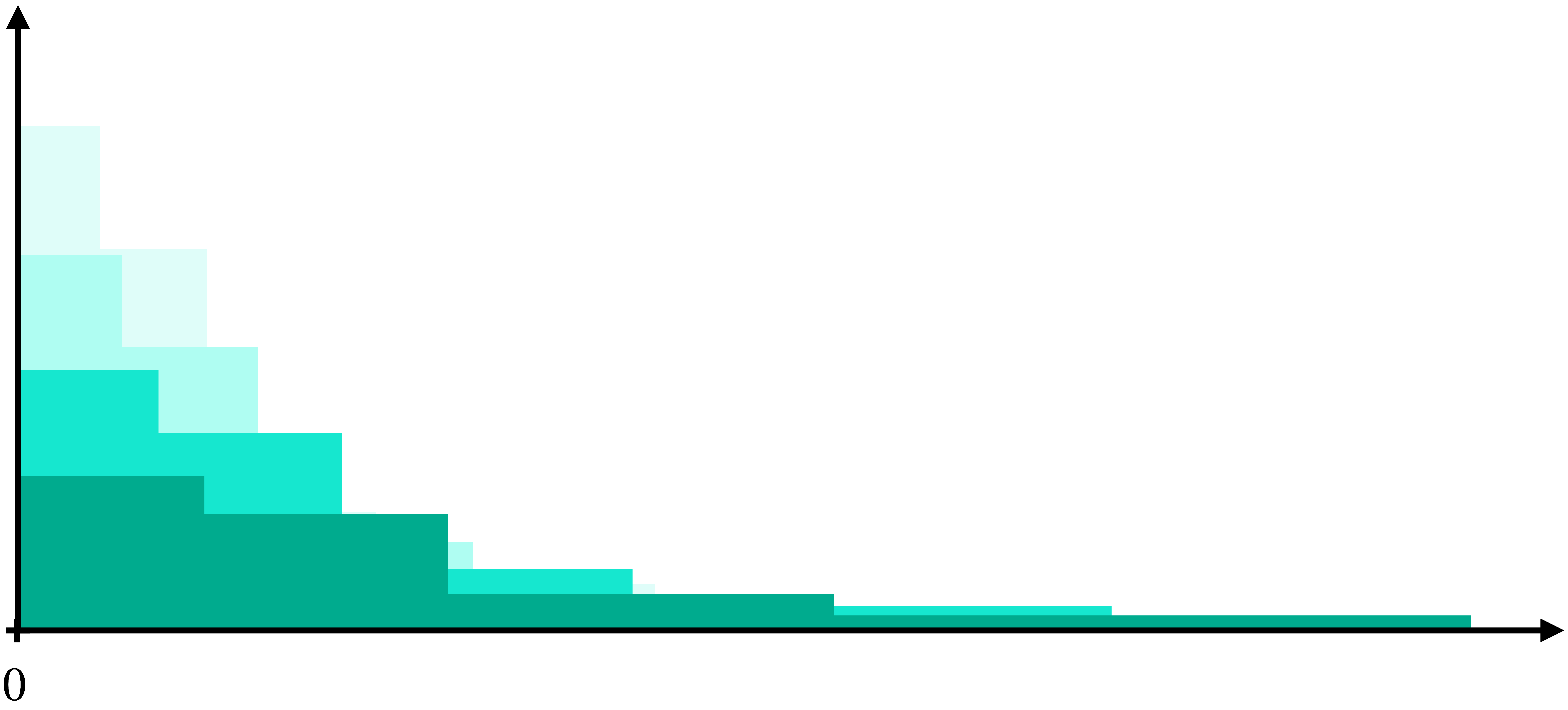
Spectral states for $\mathscr{M} = M_n$ Flow of weights: $X = (0,\infty)$, $\hat{\sigma}_s(t) = e^s \cdot t$



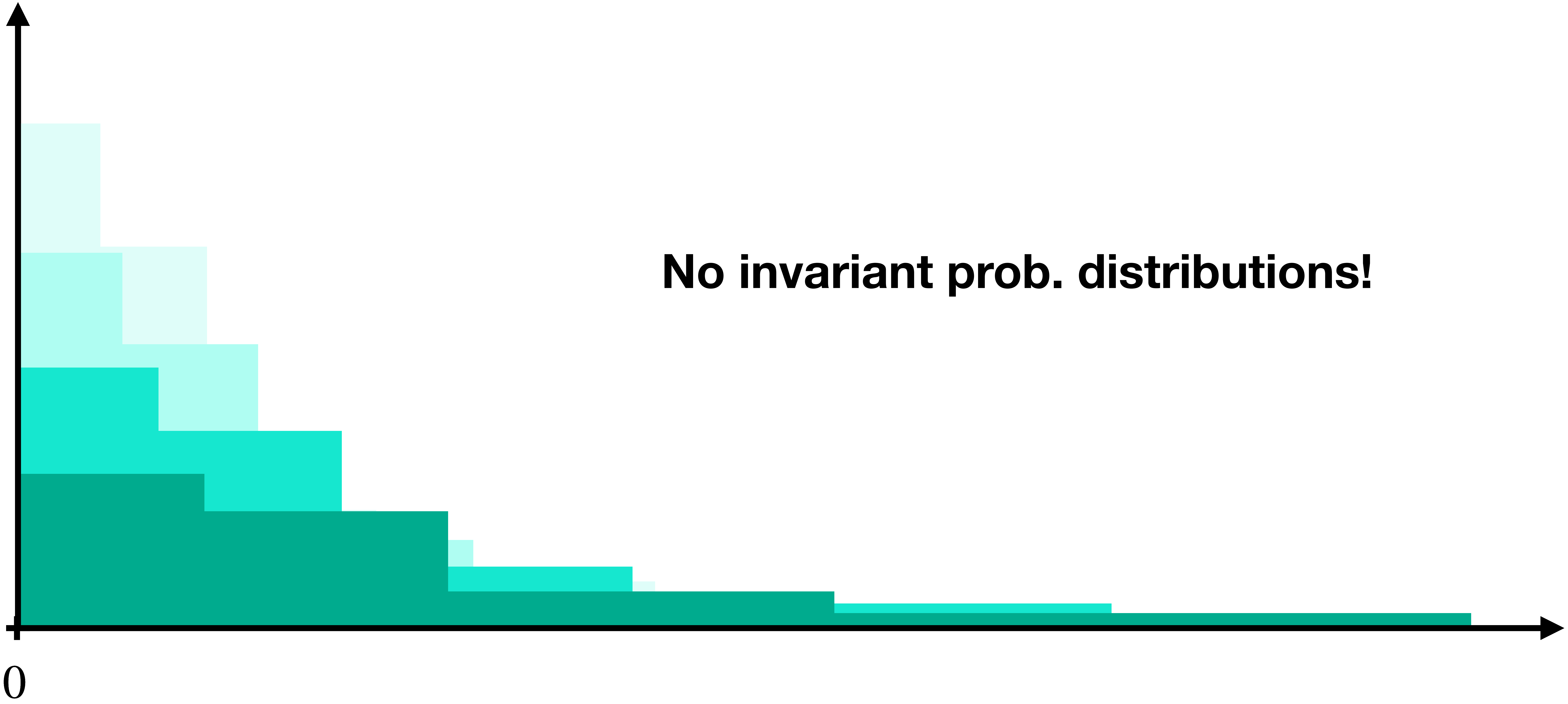
Spectral states for $\mathscr{M} = M_n$ Flow of weights: $X = (0,\infty)$, $\hat{\sigma}_s(t) = e^s \cdot t$



Spectral states for $\mathscr{M} = M_n$ Flow of weights: $X = (0,\infty)$, $\hat{\sigma}_s(t) = e^s \cdot t$

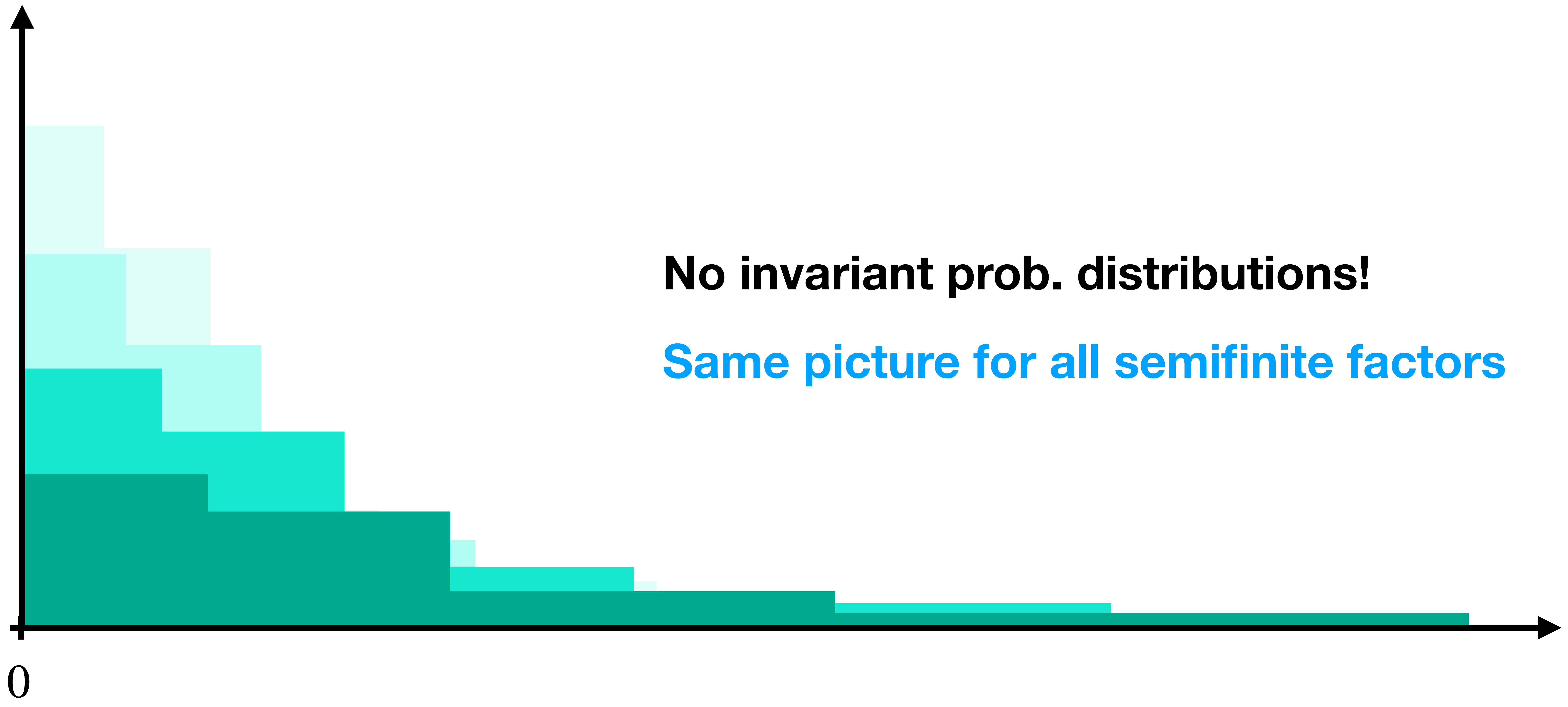


Spectral states for $\mathscr{M} = M_n$ Flow of weights: $X = (0,\infty),\ \hat{\sigma}_s(t) = e^s \cdot t$



No invariant prob. distributions!

Spectral states for $\mathscr{M} = M_n$ Flow of weights: $X = (0,\infty)$, $\hat{\sigma}_s(t) = e^s \cdot t$



No invariant prob. distributions!

Same picture for all semifinite factors

Types of Factors

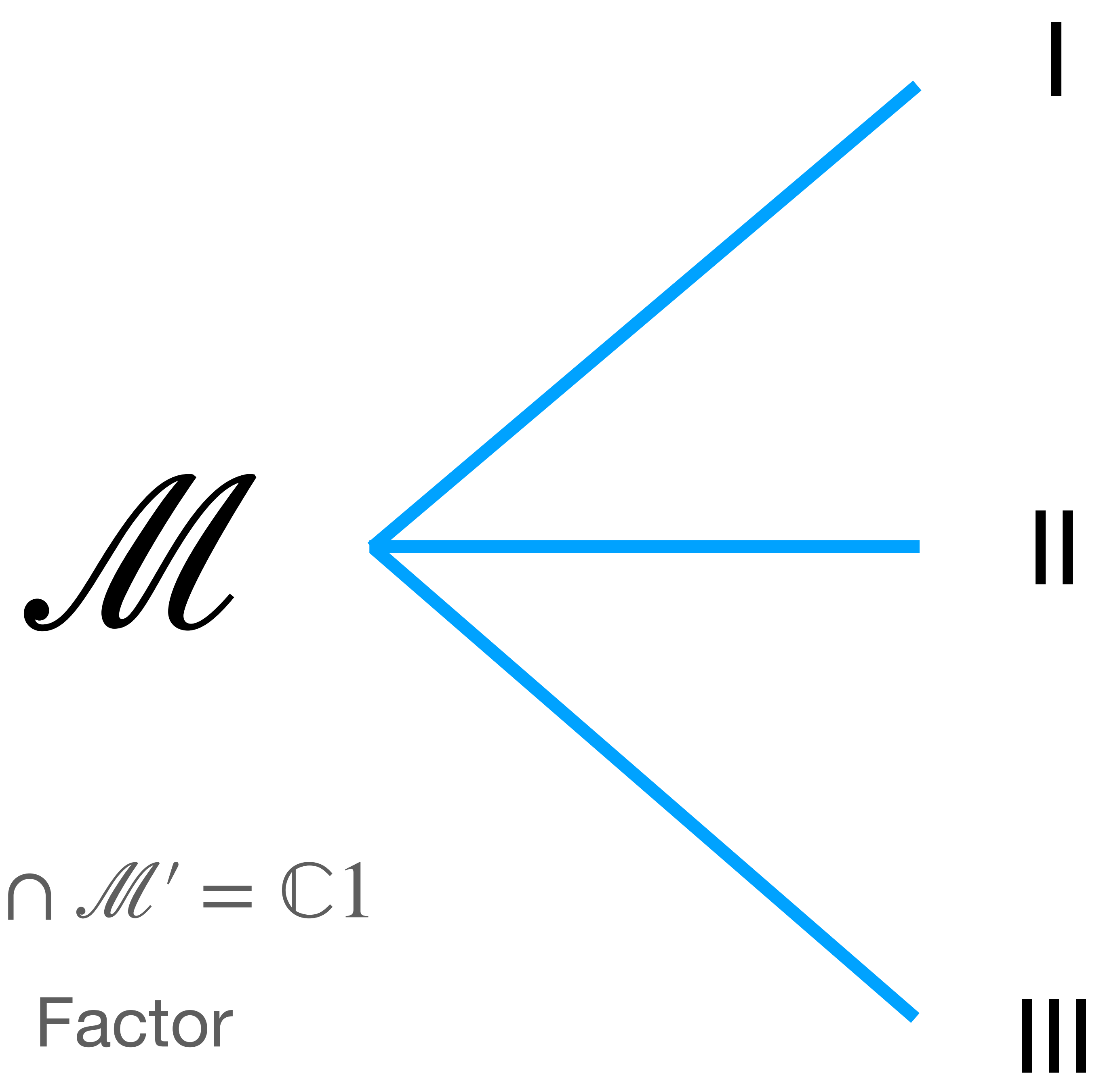
$$\mathscr{M}$$

$\mathscr{M} \cap \mathscr{M}' = \mathbb{C}1$

Factor

Types of Factors

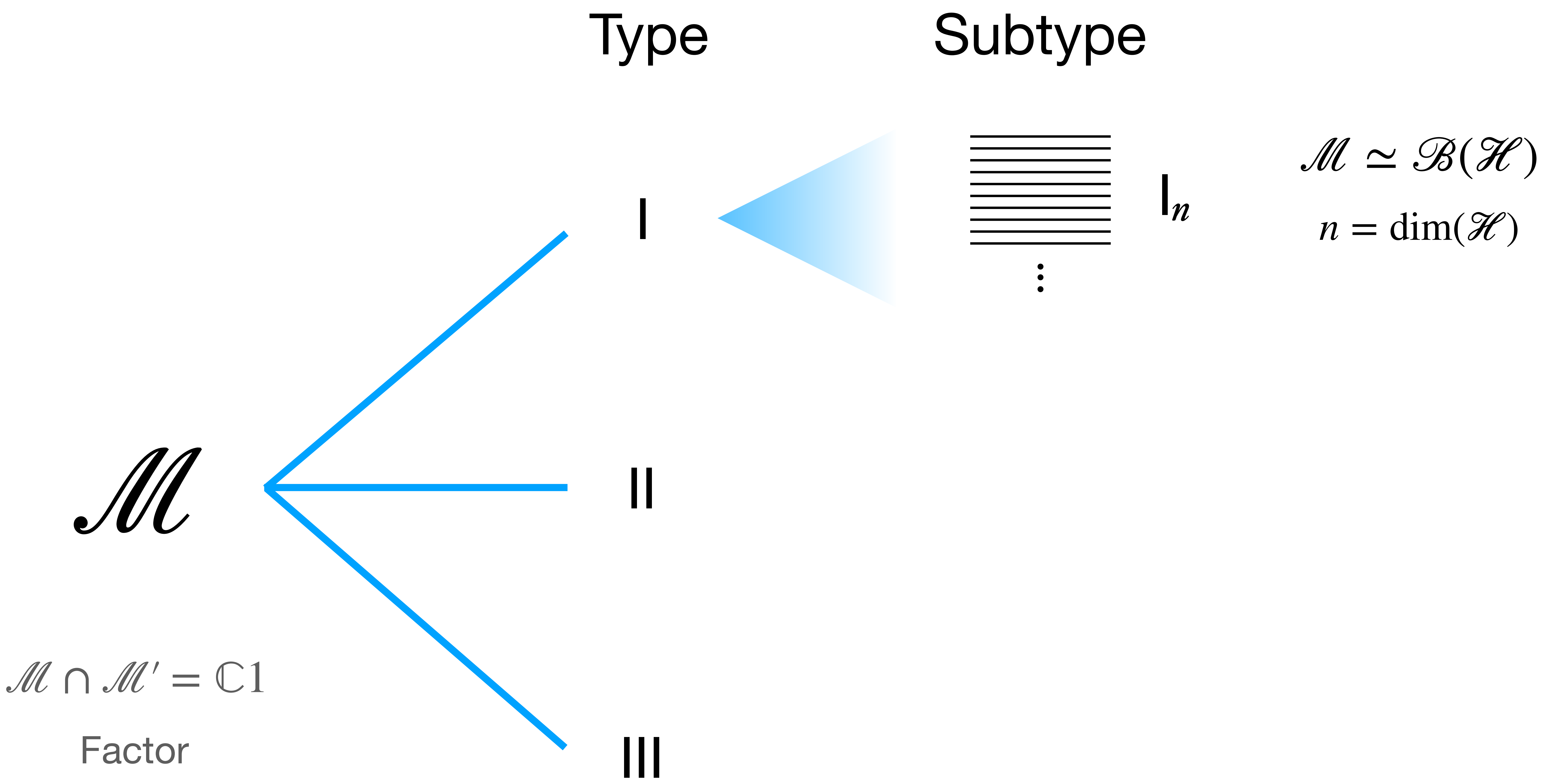
Type



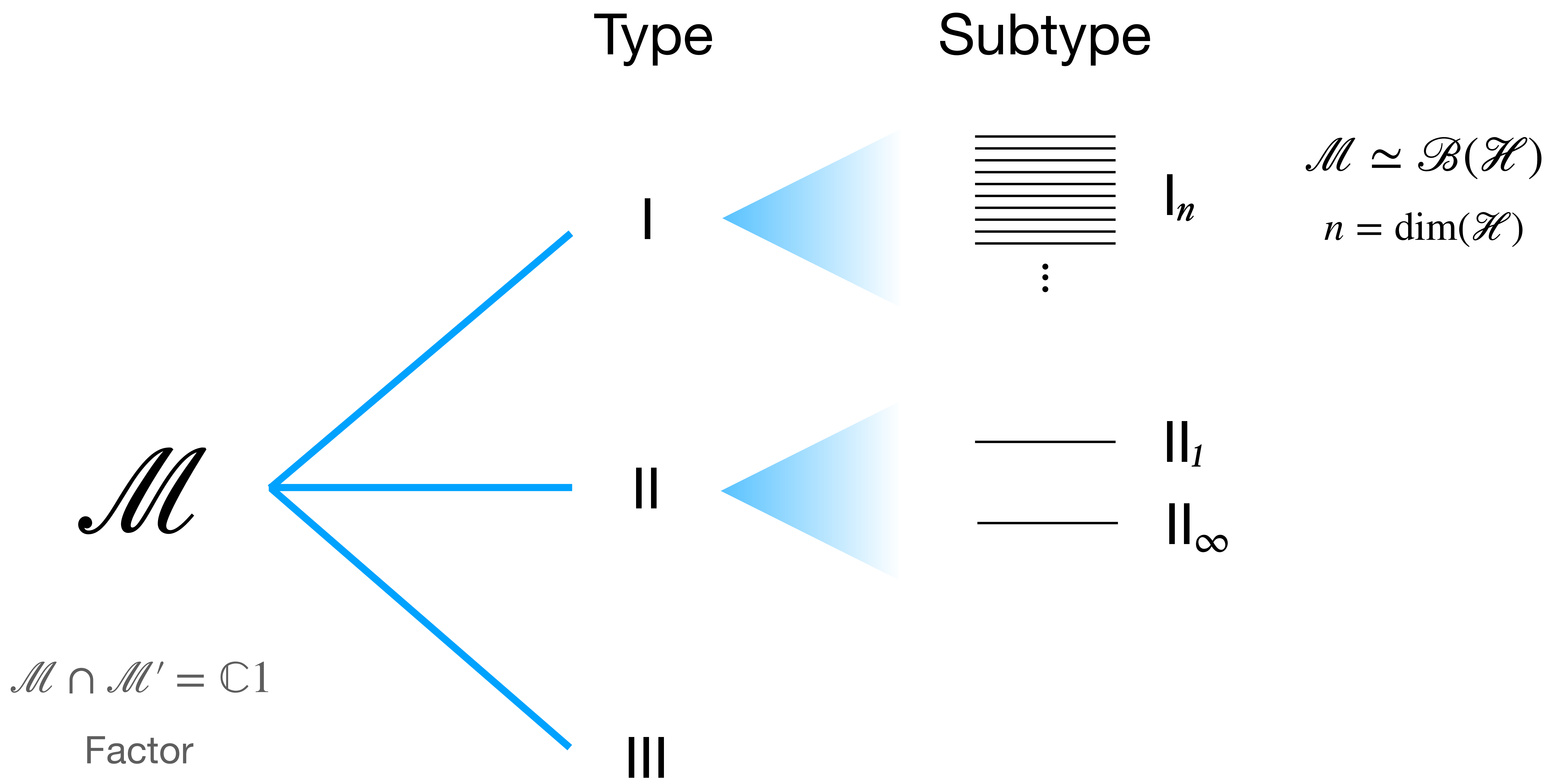
$\mathscr{M} \cap \mathscr{M}' = \mathbb{C}1$

Factor

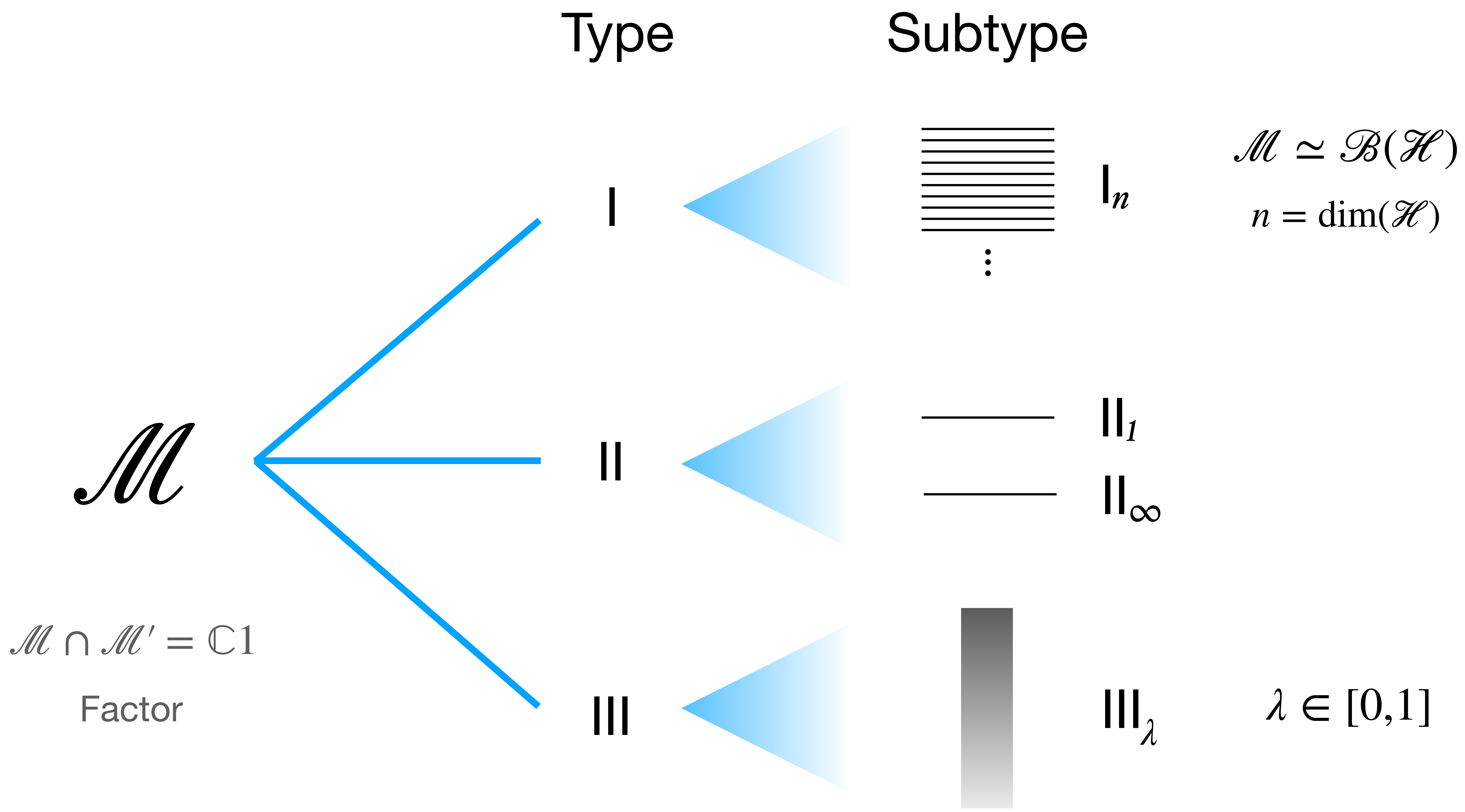
Types of Factors



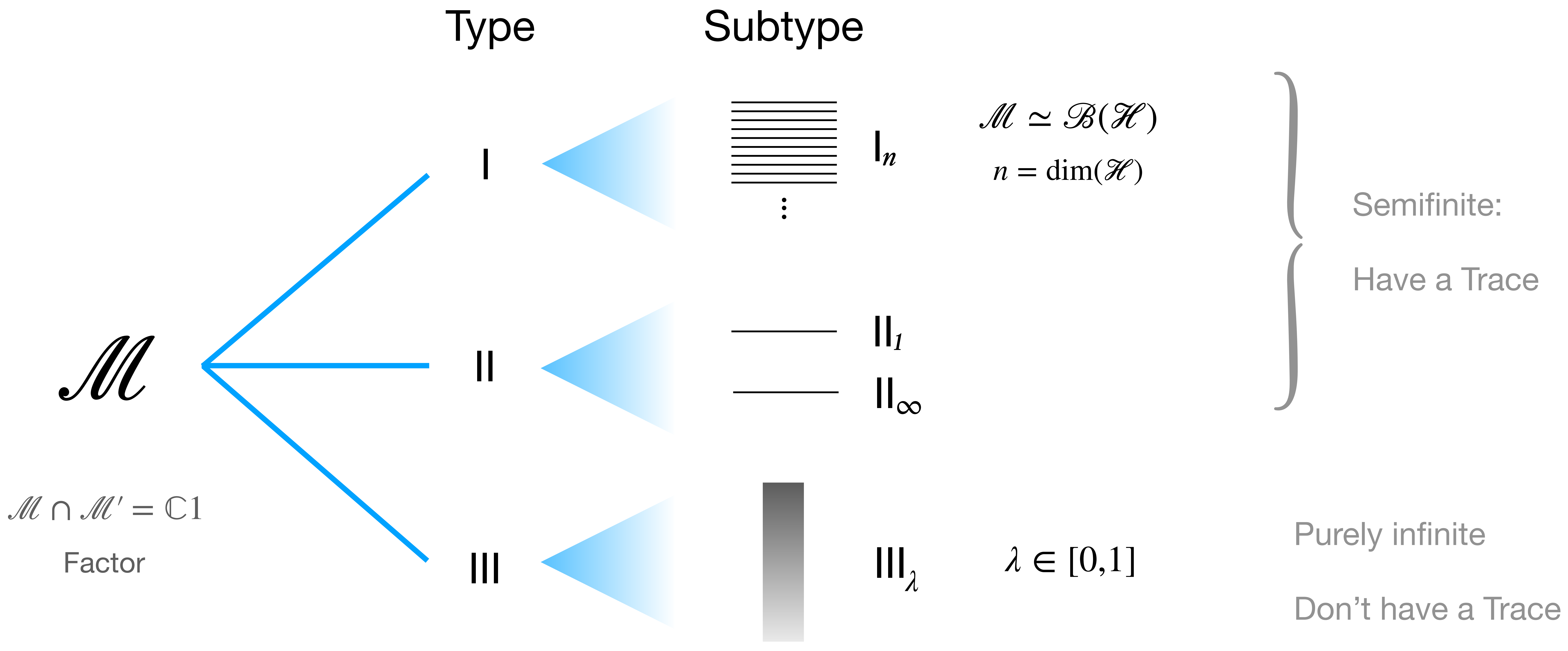
Types of Factors



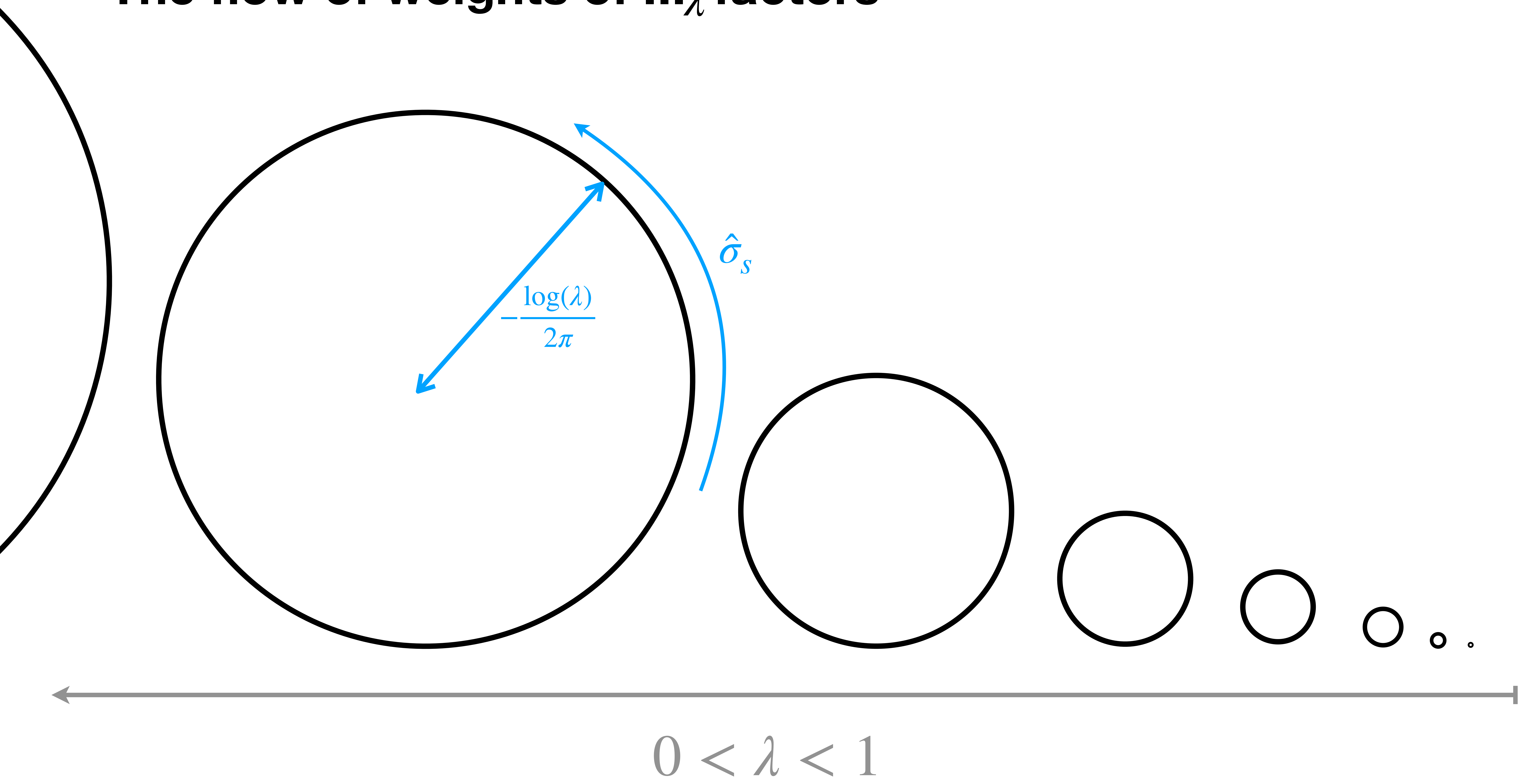
Types of Factors



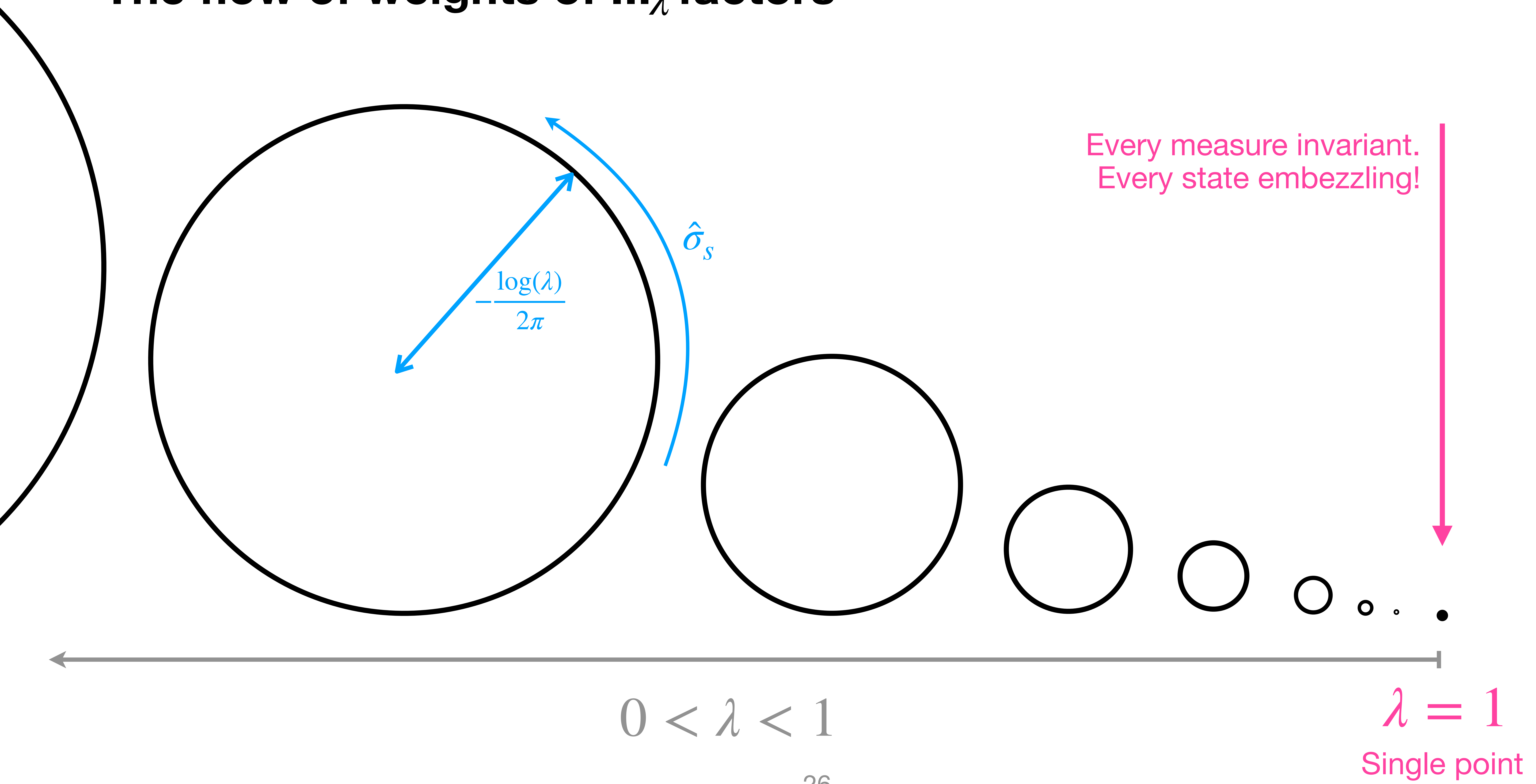
Types of Factors



The flow of weights of III_λ factors



The flow of weights of III_λ factors



Main results

Type	I	II	III		
Subtype	*	*	$\lambda = 0$	$0 < \lambda < 1$	$\lambda = 1$
κ_{min}	2	2	$\in \{0,2\}$	**0**	**0**
κ_{max}	2	2	2	$2\dfrac{1-\sqrt{\lambda}}{1+\sqrt{\lambda}}$	**0**

Main results

Type	I	II	III		
Subtype	*	*	$\lambda = 0$	$0 < \lambda < 1$	$\lambda = 1$
κ_{min}	2	2	$\in \{0,2\}$	**0**	**0**
κ_{max}	2	2	2	$2\frac{1-\sqrt{\lambda}}{1+\sqrt{\lambda}}$	**0**

No embezzling states

Main results

Type	I	II	III		
Subtype	*	*	$\lambda = 0$	$0 < \lambda < 1$	$\lambda = 1$
κ_{min}	2	2	$\in \{0,2\}$	**0**	**0**
κ_{max}	2	2	2	$2\frac{1-\sqrt{\lambda}}{1+\sqrt{\lambda}}$	**0**

$\underbrace{\text{I} \quad \text{II}}_{\text{No embezzling states}}$

No embezzling states

Some embezzling states

Main results

Type	I	II	III		
Subtype	*	*	$\lambda = 0$	$0 < \lambda < 1$	$\lambda = 1$
$\kappa_{\min}$	2	2	$\in \{0,2\}$	**0**	**0**
$\kappa_{\max}$	2	2	2	$2\frac{1-\sqrt{\lambda}}{1+\sqrt{\lambda}}$	**0**

No embezzling states

All states embezzling

Some embezzling states

Main results

Type	I	II	III		
Subtype	*	*	$\lambda = 0$	$0 < \lambda < 1$	$\lambda = 1$
κ_{min}	2	2	$\in \{0,2\}$	**0**	**0**
κ_{max}	2	2	2	$2\frac{1-\sqrt{\lambda}}{1+\sqrt{\lambda}}$	**0**

No embezzling states

All states embezzling

Some embezzling states

Sometimes embezzling states, sometimes not

Main results

Type	I	II	III		
Subtype	*	*	$\lambda = 0$	$0 < \lambda < 1$	$\lambda = 1$
κ_{min}	2	2	$\in \{0,2\}$	**0**	**0**
κ_{max}	2	2	2	$2\frac{1-\sqrt{\lambda}}{1+\sqrt{\lambda}}$	**0**

Within type III, the embezzlement quantifiers tell us the subtype!

Embezzlement reveals Connes' classification.

Types of infinite entanglement

Option A

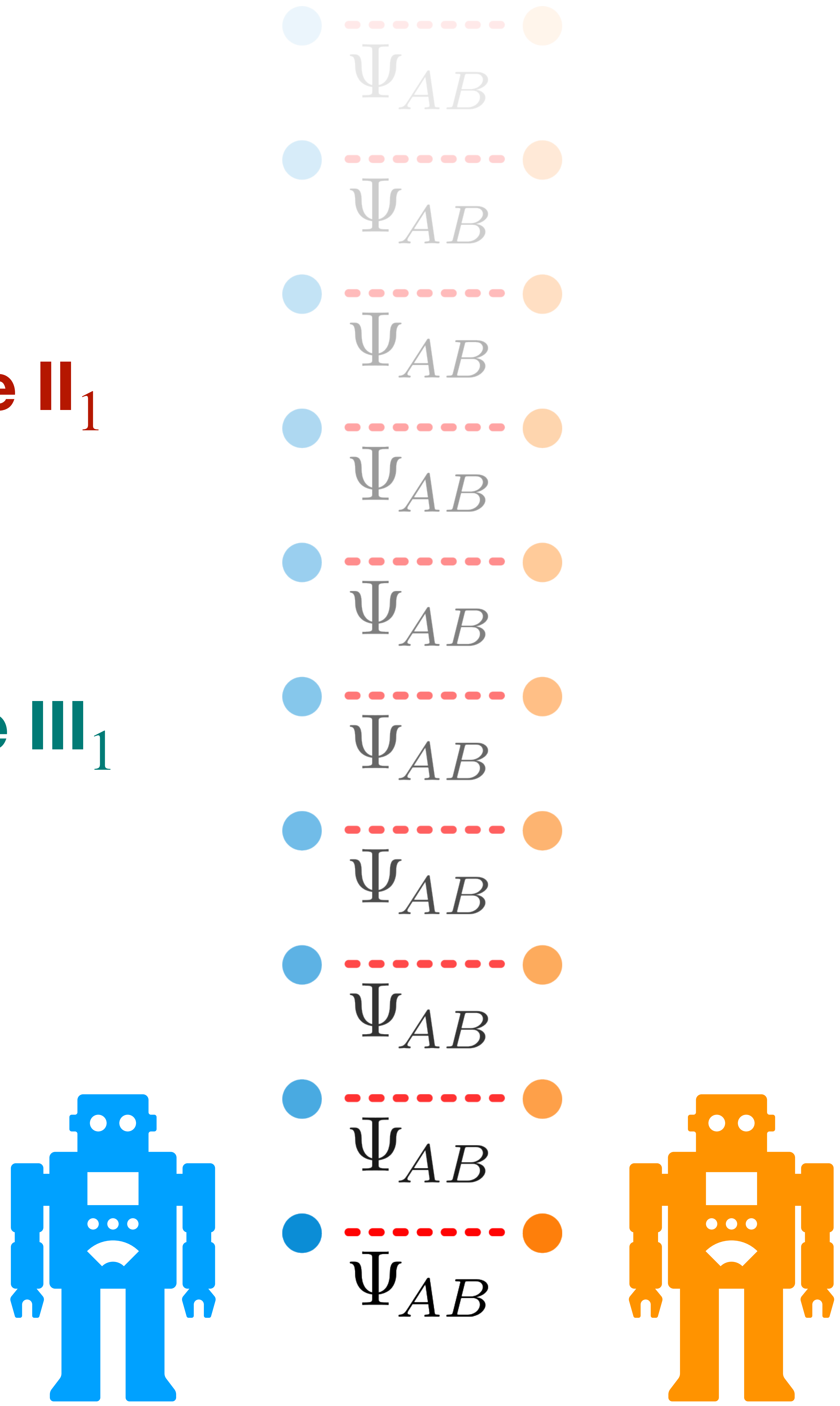
$$\Psi_{AB} \propto |1\rangle_A |1\rangle_B + |2\rangle_A |2\rangle_B + |3\rangle_A |3\rangle_B$$

Type II_1

Option B

$$\Psi_{AB} \propto |1\rangle_A |1\rangle_B + 2\,|2\rangle_A |2\rangle_B + 3\,|3\rangle_A |3\rangle_B$$

Type III_1



Types of infinite entanglement

Option A

$$\Psi_{AB} \propto |1\rangle_A |1\rangle_B + |2\rangle_A |2\rangle_B + |3\rangle_A |3\rangle_B$$

Type II_1

Option B

$$\Psi_{AB} \propto |1\rangle_A |1\rangle_B + 2|2\rangle_A |2\rangle_B + 3|3\rangle_A |3\rangle_B$$

Type III_1

Is there an operational difference?

Types of infinite entanglement

Option A

$$\Psi_{AB} \propto |1\rangle_A|1\rangle_B + |2\rangle_A|2\rangle_B + |3\rangle_A|3\rangle_B$$

Type II_1

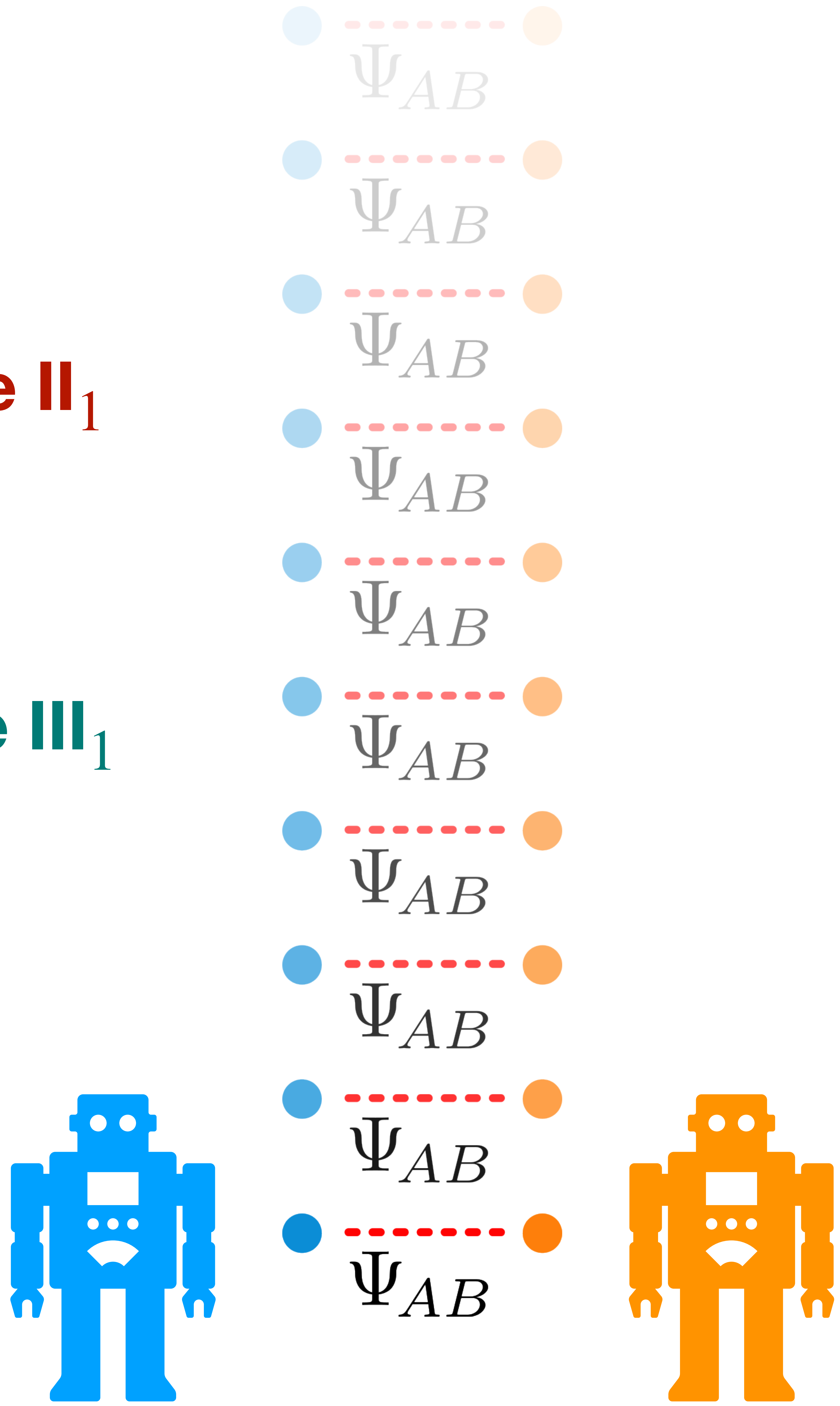
Option B

$$\Psi_{AB} \propto |1\rangle_A|1\rangle_B + 2|2\rangle_A|2\rangle_B + 3|3\rangle_A|3\rangle_B$$

Type III_1

Is there an operational difference?

Both have infinite 1-shot entanglement,
but only B can embezzle.



A General Theme

algebraic properties of $(\mathscr{M}_A, \mathscr{M}_B)$

?

properties in entanglement theory

Main Theorem: The type classification is in 1-to-1 correspondence with operational properties in entanglement theory.

L. van Luijk, A. S., R. F. Werner, H. Wilming, "Embezzlement of Entanglement, Quantum Fields, and the Classification of von Neumann Algebras," arXiv.2401.07299
L. van Luijk, A. S., R. F. Werner, H. Wilming, "Pure state entanglement and von Neumann algebras", C*ommun. Math. Phys.* 406, 296, 2025

Main Theorem: The type classification is in 1-to-1 correspondence with operational properties in entanglement theory.

	Defining property	Subtypes characterised by

L. van Luijk, A. S., R. F. Werner, H. Wilming, “Embezzlement of Entanglement, Quantum Fields, and the Classification of von Neumann Algebras,” arXiv.2401.07299
L. van Luijk, A. S., R. F. Werner, H. Wilming, “Pure state entanglement and von Neumann algebras”, *Commun. Math. Phys.* 406, 296, 2025

Main Theorem: The type classification is in 1-to-1 correspondence with operational properties in entanglement theory.

	Defining property	Subtypes characterised by
Type I	finite 1-shot entanglement	Schmidt rank

L. van Luijk, A. S., R. F. Werner, H. Wilming, “Embezzlement of Entanglement, Quantum Fields, and the Classification of von Neumann Algebras,” arXiv.2401.07299
L. van Luijk, A. S., R. F. Werner, H. Wilming, “Pure state entanglement and von Neumann algebras”, *Commun. Math. Phys.* 406, 296, 2025

Main Theorem: The type classification is in 1-to-1 correspondence with operational properties in entanglement theory.

	Defining property	Subtypes characterised by
Type I	finite 1-shot entanglement	Schmidt rank
Type II	LOCC not trivial	maximally entangled states

L. van Luijk, A. S., R. F. Werner, H. Wilming, “Embezzlement of Entanglement, Quantum Fields, and the Classification of von Neumann Algebras,” arXiv.2401.07299
L. van Luijk, A. S., R. F. Werner, H. Wilming, “Pure state entanglement and von Neumann algebras”, *Commun. Math. Phys.* 406, 296, 2025

Main Theorem: The type classification is in 1-to-1 correspondence with operational properties in entanglement theory.

	Defining property	Subtypes characterised by
Type I	finite 1-shot entanglement	Schmidt rank
Type II	LOCC not trivial	maximally entangled states
Type III	LOCC trivial	embezzlement of entanglement

L. van Luijk, A. S., R. F. Werner, H. Wilming, “Embezzlement of Entanglement, Quantum Fields, and the Classification of von Neumann Algebras,” arXiv.2401.07299
L. van Luijk, A. S., R. F. Werner, H. Wilming, “Pure state entanglement and von Neumann algebras”, *Commun. Math. Phys.* 406, 296, 2025

Applications to quantum many-body systems

Type III_1 factors and where to find them

Statistical Mechanics: critical fermions and infinite spin chains

Alice:
$$\mathscr{A}_L = \mathscr{A}_{(-\infty,0]}$$

$$\mathscr{A} = \bigotimes_{n\in\mathbb{Z}} M_2(\mathbb{C})$$

Bob:
$$\mathscr{A}_R = \mathscr{A}_{[1,\infty)}$$

Type III_1 factors and where to find them

Statistical Mechanics: critical fermions and infinite spin chains

Alice:
$$\mathscr{A}_L = \mathscr{A}_{(-\infty,0]}$$

$$\mathscr{A} = \bigotimes_{n\in\mathbb{Z}} M_2(\mathbb{C})$$

Bob:
$$\mathscr{A}_R = \mathscr{A}_{[1,\infty)}$$

transverse-field XY chain: $$H = -\frac{1}{2}\sum_n (X_n X_{n+1} + Y_n Y_{n+1} + 2h\, Z_n), \quad |h| < 1$$

Type III$_1$ factors and where to find them

Statistical Mechanics: critical fermions and infinite spin chains

Alice: $\mathscr{A}_L = \mathscr{A}_{(-\infty,0]}$

$$\mathscr{A} = \bigotimes_{n\in\mathbb{Z}} M_2(\mathbb{C})$$

Bob: $\mathscr{A}_R = \mathscr{A}_{[1,\infty)}$

transverse-field XY chain: $H = -\frac{1}{2}\sum_n (X_n X_{n+1} + Y_n Y_{n+1} + 2h\, Z_n), \quad |h| < 1$

If $(\mathscr{H}, \pi, |\Omega\rangle)$ = GNS rep. of the ground state $\omega : \mathscr{A} \to \mathbb{C}$, then

$$\left(\mathscr{H}, \mathscr{M}_A := \overline{\pi(\mathscr{A}_L)}^w, \mathscr{M}_B := \overline{\pi(\mathscr{A}_R)}^w\right)$$

is a standard bipartite system of **type III$_1$**. Hence, $|\Omega\rangle$ is embezzling.

T. Matsui, "The split property and symmetry breaking of the quantum spin chain", *Commun. Math. Phys.* 218, 393-416, 2001

Type III_1 factors and where to find them

Statistical Mechanics: critical fermions and infinite spin chains

Alice:
$$\mathscr{A}_L = \mathscr{A}_{(-\infty,0]}$$

$$\mathscr{A} = \mathrm{CAR}_e(\ell^2(\mathbb{Z}))$$

Bob:
$$\mathscr{A}_R = \mathscr{A}_{[1,\infty)}$$

Type III_1 factors and where to find them

Statistical Mechanics: critical fermions and infinite spin chains

Alice:
$\mathscr{A}_L = \mathscr{A}_{(-\infty,0]}$

$$\mathscr{A} = \mathrm{CAR}_e(\ell^2(\mathbb{Z}))$$

Bob:
$\mathscr{A}_R = \mathscr{A}_{[1,\infty)}$

quasi-free, translation-invariant, critical fermions: $H = \sum_{x,y} \sum_{l,m} h_{l,m}(x-y) a_l^\dagger(x) a_m(y)$

Type III$_1$ factors and where to find them

Statistical Mechanics: critical fermions and infinite spin chains

Alice:

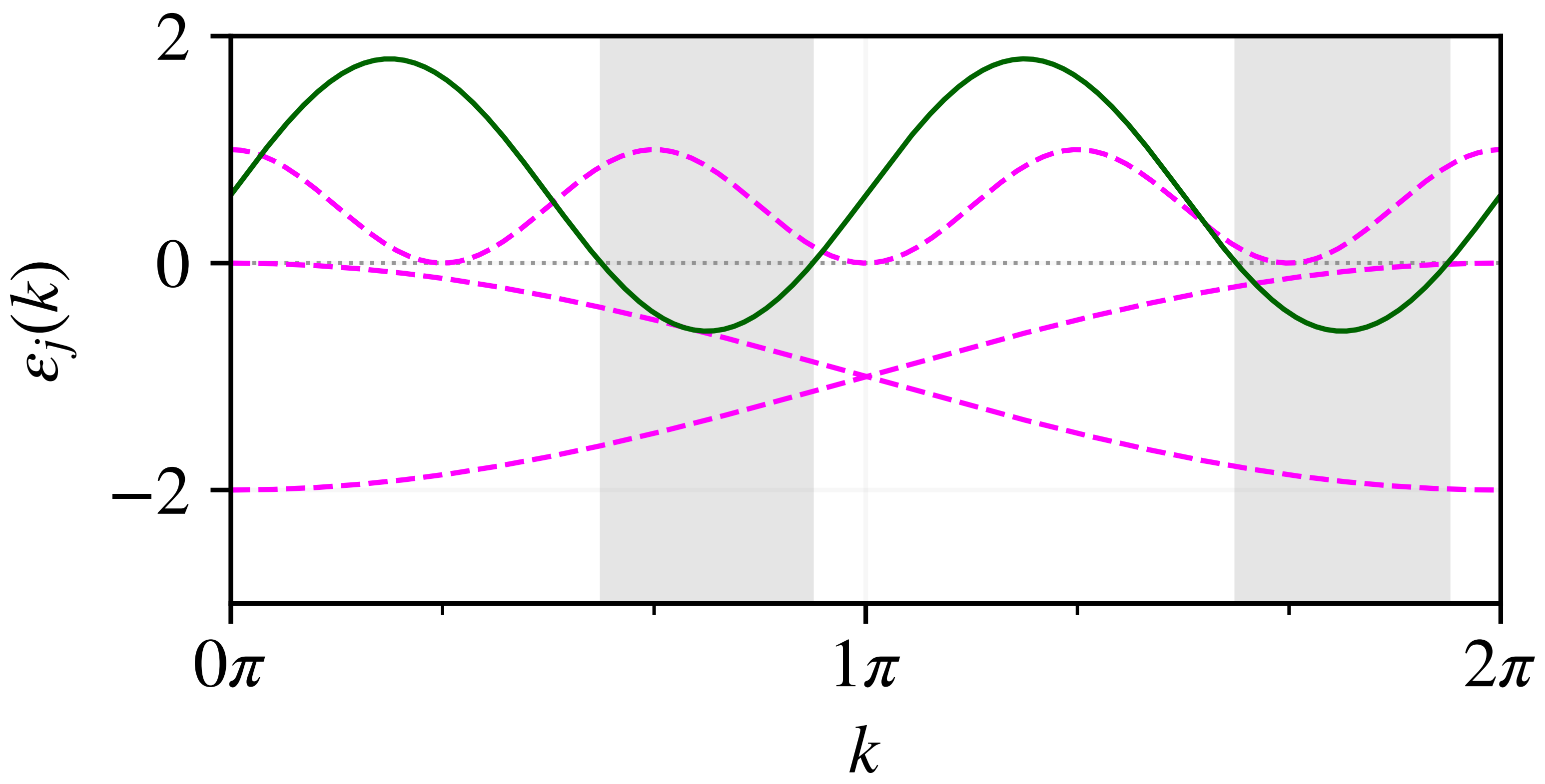
$$\mathscr{A}_L = \mathscr{A}_{(-\infty,0]}$$

$$\mathscr{A} = \mathrm{CAR}_e(\ell^2(\mathbb{Z}))$$

Bob:

$$\mathscr{A}_R = \mathscr{A}_{[1,\infty)}$$

quasi-free, translation-invariant, critical fermions: $H = \sum_{x,y} \sum_{l,m} h_{l,m}(x-y) a_l^\dagger(x) a_m(y)$



Type III$_1$ factors and where to find them

Statistical Mechanics: critical fermions and infinite spin chains

Alice:
$$\mathscr{A}_L = \mathscr{A}_{(-\infty,0]}$$

$$\mathscr{A} = \mathrm{CAR}_e(\ell^2(\mathbb{Z}))$$

Bob:
$$\mathscr{A}_R = \mathscr{A}_{[1,\infty)}$$

quasi-free, translation-invariant, critical fermions: $H = \sum_{x,y} \sum_{l,m} h_{l,m}(x-y) a_l^\dagger(x) a_m(y)$

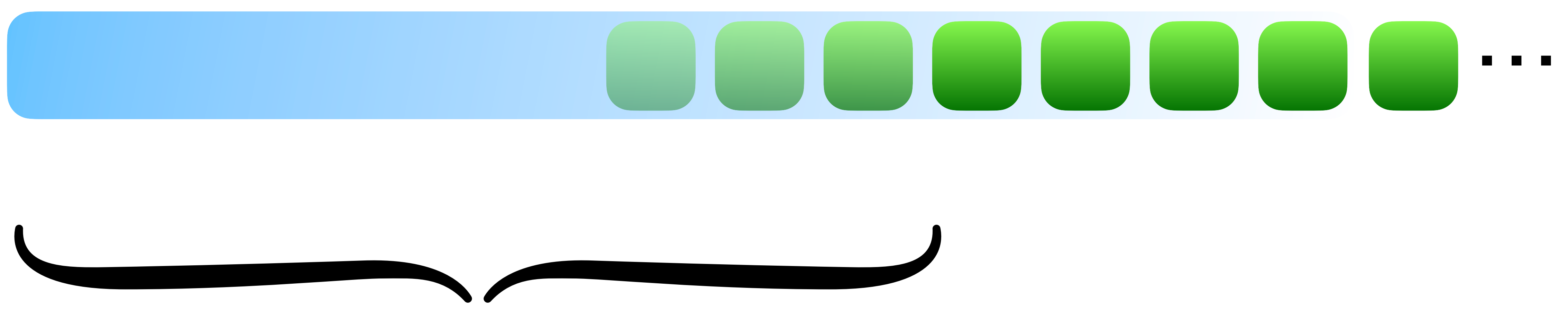
If $(\mathscr{H}, \pi, |\Omega\rangle)$ = GNS rep. of the ground state $\omega : \mathscr{A} \to \mathbb{C}$, then

$$\left(\mathscr{H}, \mathscr{M}_A := \overline{\pi(\mathscr{A}_L)}^{w}, \mathscr{M}_B := \overline{\pi(\mathscr{A}_R)}^{w}\right)$$

is a standard bipartite system of **type III$_1$**. Hence, $|\Omega\rangle$ is embezzling.

L. van Luijk, A. S., H. Wilming, "Critical fermions are universal embezzlers", *Nat. Phys. (2025)*

From embezzlers to embezzling families…



ω

$$\omega_{<n} \xrightarrow{\text{purification}} |\Omega_n\rangle$$

Theorem. Any hyperfinite embezzler induces an embezzling family.

Various many-body systems and quantum field theories are examples of hyperfinite von Neumann algebras.

L. van Luijk, A. S., H. Wilming, “Critical fermions are universal embezzlers”, *Nat. Phys. (2025)*

…and back again

Alice:

$$\mathscr{A}_L = \mathscr{A}_{(-\infty,0]}$$

Bob:

$$\mathscr{A}_R = \mathscr{A}_{[1,\infty)}$$

Theorem.
Finite-size ground states induce an embezzling family if a unique embezzling ground state in infinite volume exists.

Holds for very general quantum many-body systems

L. van Luijk, A. S., H. Wilming, “Critical fermions are universal embezzlers”, *Nat. Phys. (2025)*

Type III$_1$ factors and where to find them

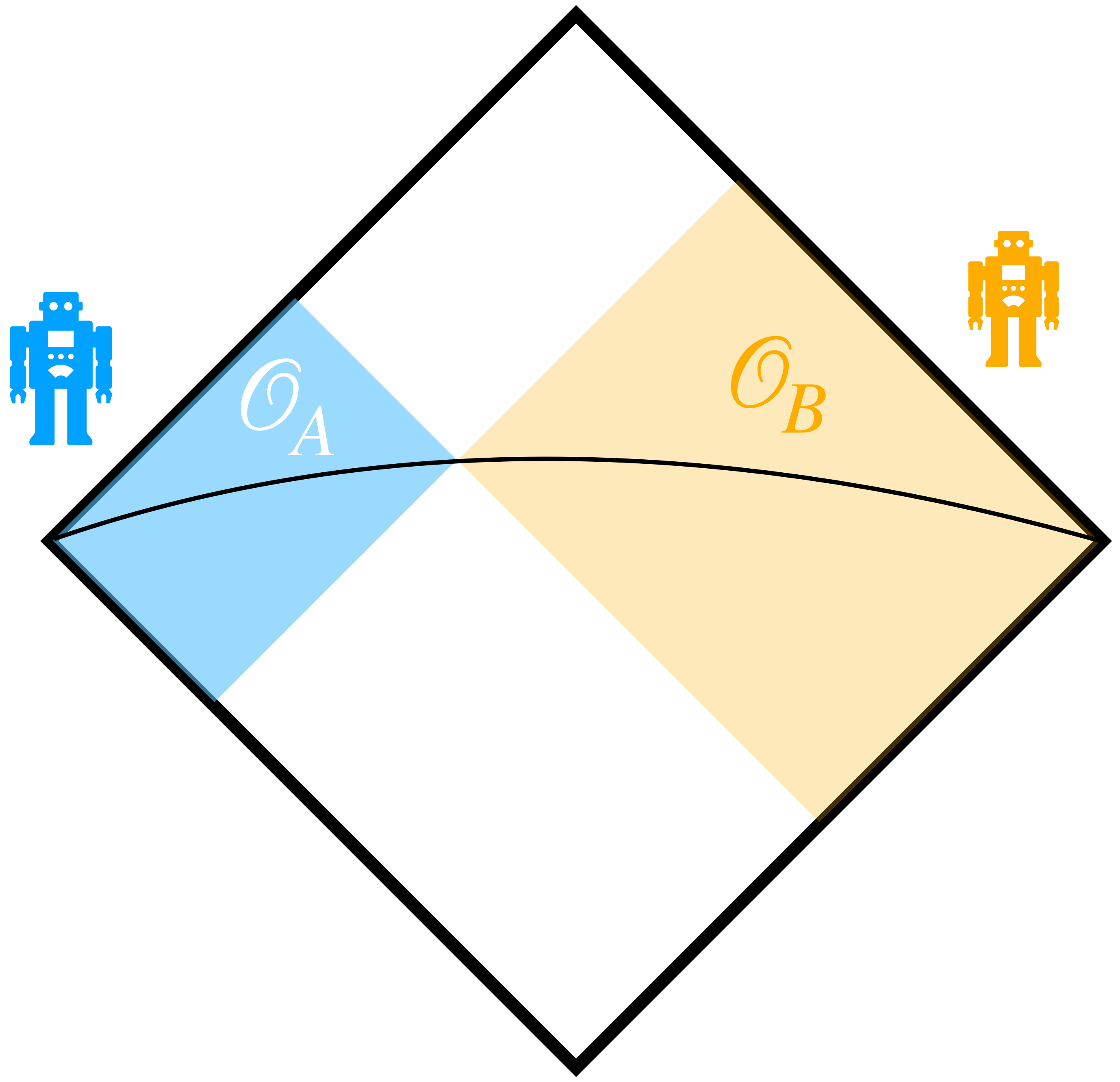
Relativistic Quantum Field Theory

$$\mathscr{O} \subseteq \mathrm{M} \quad \longmapsto \quad \mathscr{M}(\mathscr{O}) \subseteq \mathscr{B}(\mathscr{H})$$

open subset — von Neumann Algebra

M

Minkowski spacetime

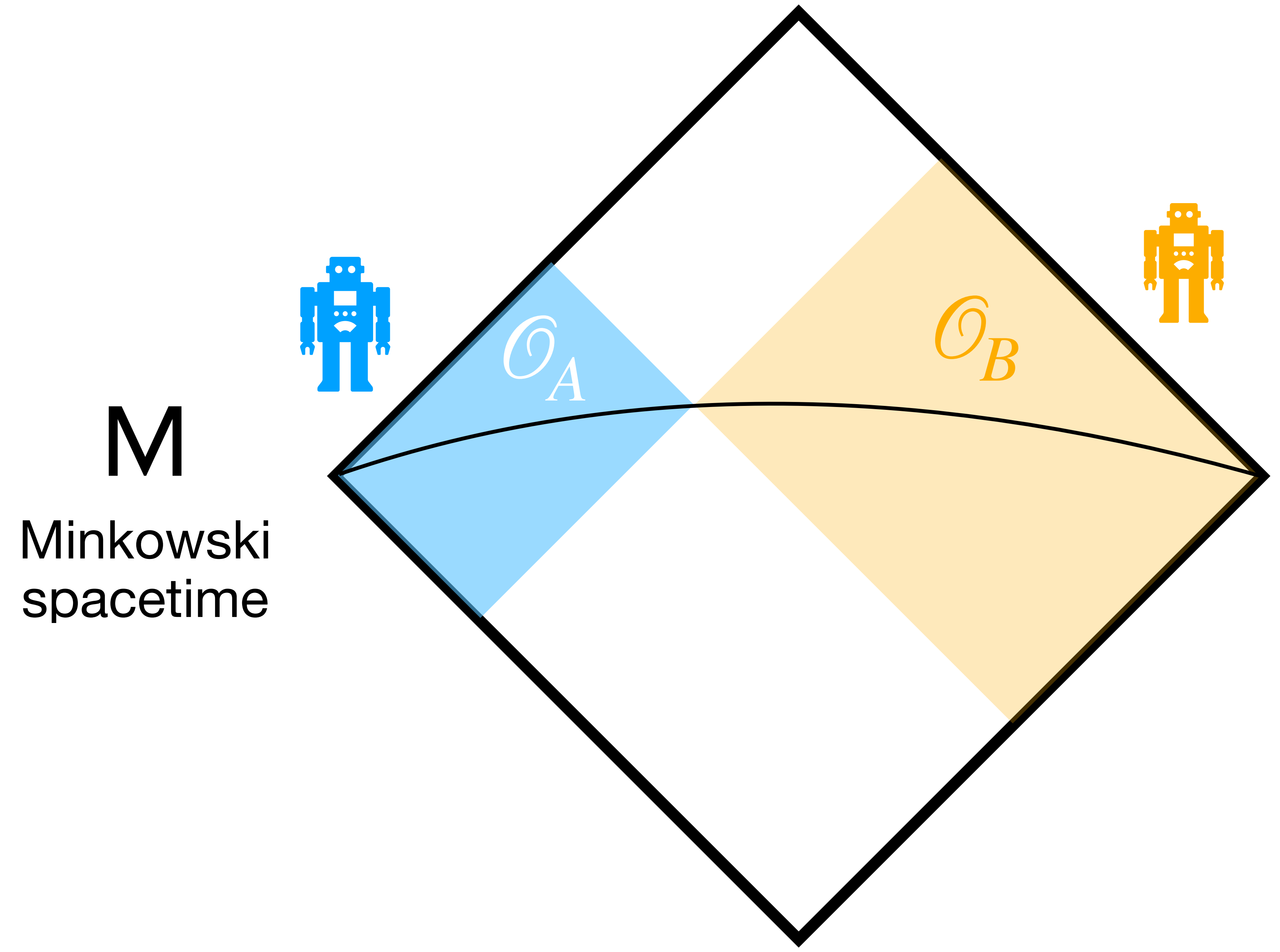


Type III$_1$ factors and where to find them

Relativistic Quantum Field Theory

$\mathscr{O} \subseteq \mathrm{M}$ $\longmapsto$ $\mathscr{M}(\mathscr{O}) \subseteq \mathscr{B}(\mathscr{H})$

open subset — von Neumann Algebra



General result in algebraic QFT:

$\mathscr{M}(\mathscr{O}_A) = \mathscr{M}(\mathscr{O}_B)'$ type III$_1$

$\downarrow$

Relativistic quantum fields are universal embezzlers.

W. Driessler, "Comments on Lightlike Translations and Applications in Relativistic Quantum Field Theory", *Communications in Mathematical Physics* 44, 133-41, 1975

K. Fredenhagen, "On the Modular Structure of Local Algebras of Observables", *Communications in Mathematical Physics* 97, 79-89, 1985

D. Buchholz, C. D'Antoni, and K. Fredenhagen, "The Universal Structure of Local Algebras", *Communications in Mathematical Physics* 111, 123-35, 1987

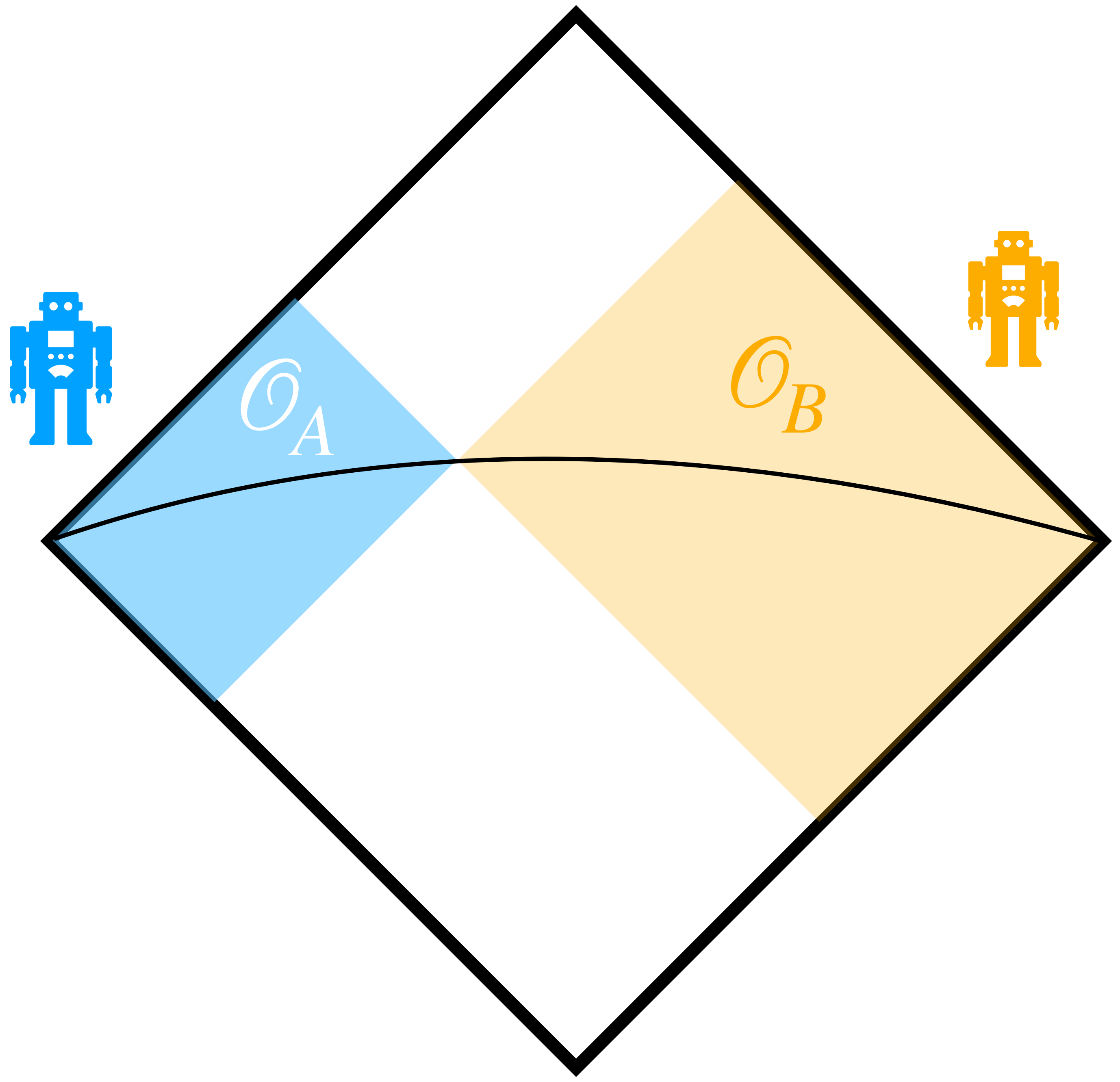
Type III$_1$ factors and where to find them

Relativistic Quantum Field Theory

$$\mathscr{O} \subseteq \mathrm{M} \longmapsto \mathscr{M}(\mathscr{O}) \subseteq \mathscr{B}(\mathscr{H})$$

open subset — von Neumann Algebra

M
Minkowski spacetime



General result in algebraic QFT:

$$\mathscr{M}(\mathscr{O}_A) = \mathscr{M}(\mathscr{O}_B)' \quad \text{type III}_1$$

$\Downarrow$

Relativistic quantum fields are universal embezzlers.

Bell inequalities (CHSH coefficient):

$$\beta(|\Omega\rangle, \mathscr{M}, \mathscr{M}') = \sup_{\substack{a_\pm \in \mathscr{M}, b_\pm \in \mathscr{M}' \\ -1 \leq a_\pm, b_\pm \leq 1}} \langle \Omega | (a_+ b_+ + a_+ b_- + a_- b_+ - a_- b_-) | \Omega \rangle$$

Theorem. Given a universal embezzler, every state fulfills

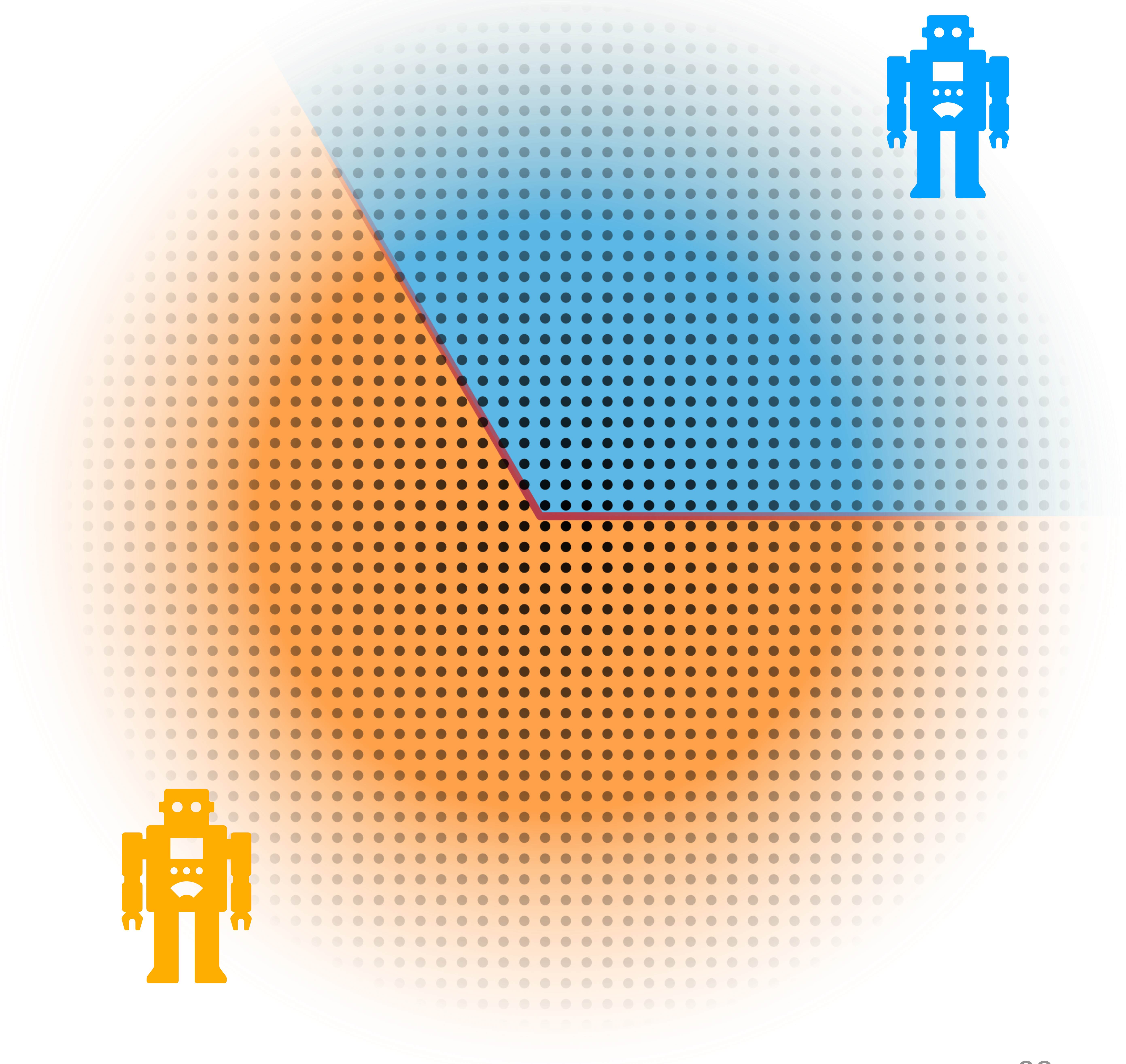
$$\beta\big(|\Omega\rangle, \mathscr{M}, \mathscr{M}'\big) = 2\sqrt{2}$$

W. Driessler, “Comments on Lightlike Translations and Applications in Relativistic Quantum Field Theory”, *Communications in Mathematical Physics* 44, 133-41, 1975

K. Fredenhagen, “On the Modular Structure of Local Algebras of Observables”, *Communications in Mathematical Physics* 97, 79-89, 1985

D. Buchholz, C. D’Antoni, and K. Fredenhagen, “The Universal Structure of Local Algebras”, *Communications in Mathematical Physics* 111, 123-35, 1987

Stephen J. Summers and Reinhard Werner, “The Vacuum Violates Bell’s Inequalities”, *Physics Letters A* 110, 257-59, 1985



Ground states of gapped phases of matter

There exist models that:

- are gapped,
- translation-invariant,
- in **trivial topological phase,**
- fulfill Haag duality ($\mathscr{M}_A = \mathscr{M}_B'$) ,

with $\mathscr{M}_A$ of type X for any

$$\mathrm{X} \in \{\mathrm{I}_\infty, \mathrm{II}_\infty, \mathrm{III}_\lambda\}, \quad 0 < \lambda \leq 1$$

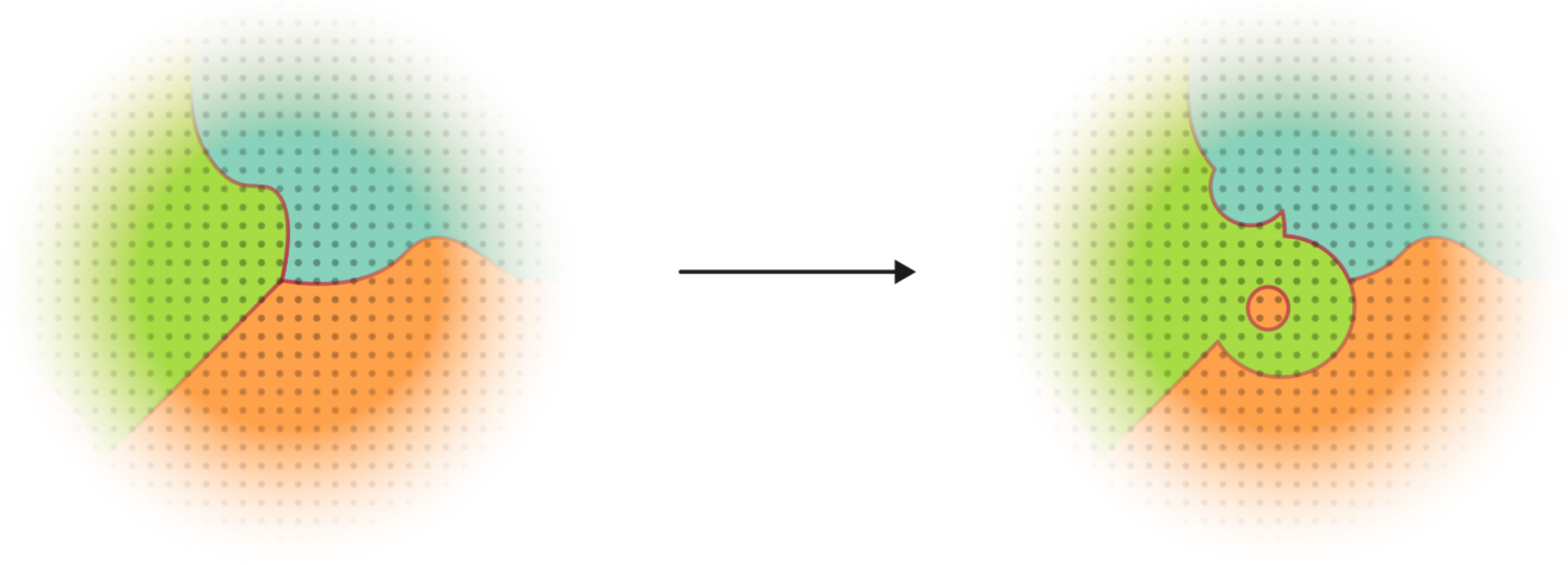
L. van Luijk, A. S., H. Wilming, "Large-scale structure of entanglement in quantum many-body systems", arXiv:2503.03833

Finite modifications of infinite subsystems

Preserved: Types of factors, Haag duality

L. van Luijk, A. S., H. Wilming, “Large-scale structure of entanglement in quantum many-body systems”, arXiv:2503.03833

Finite modifications of infinite subsystems

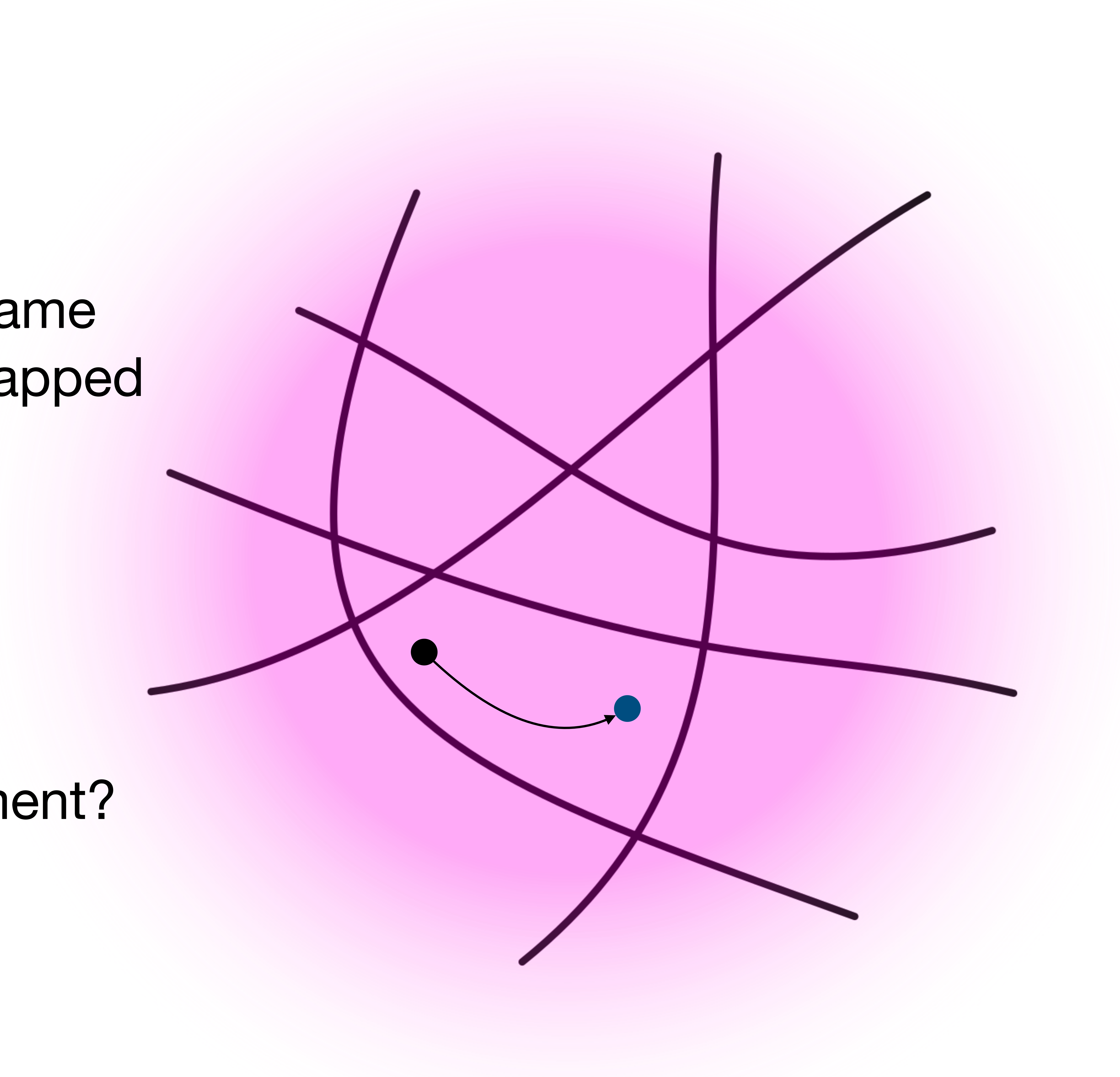


Preserved: $\underbrace{\text{Types of factors, Haag duality}}_{\textit{Large-scale structure of entanglement}}$

L. van Luijk, A. S., H. Wilming, "Large-scale structure of entanglement in quantum many-body systems", arXiv:2503.03833

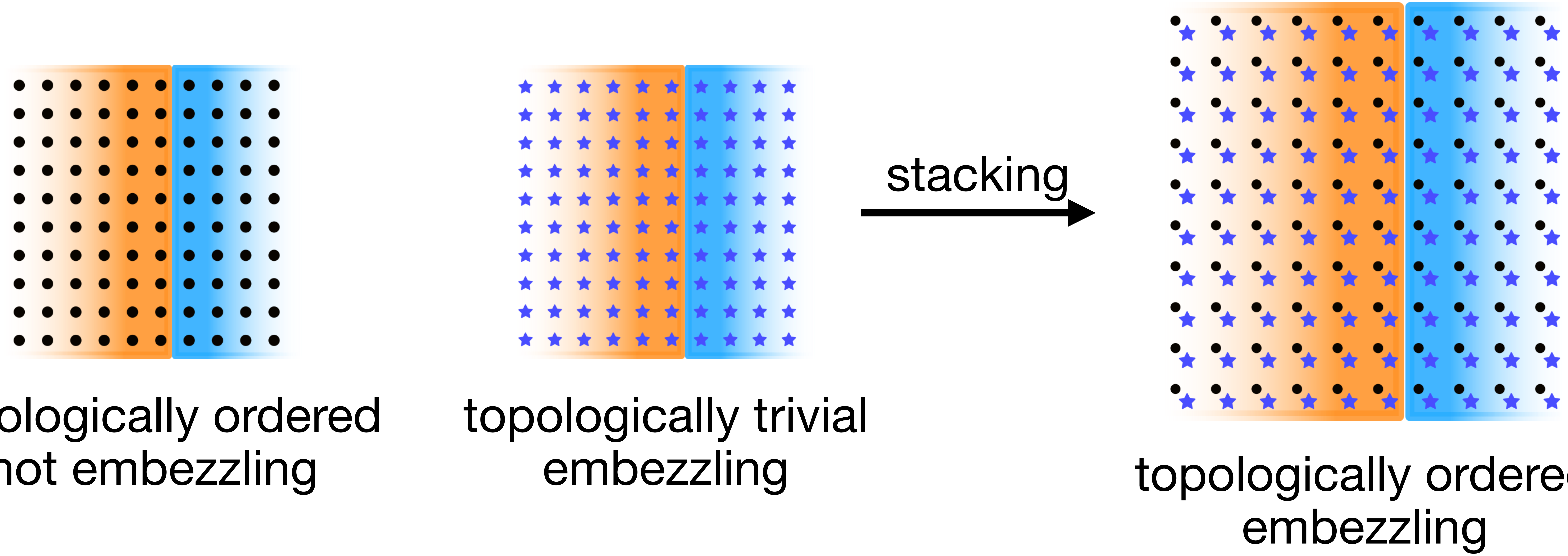
Gapped phases of matter

Two gapped ground states are in the same phase if they can be connected by a gapped path of local Hamiltonians.



Is there a relation between the phase and the bipartite large-scale entanglement?

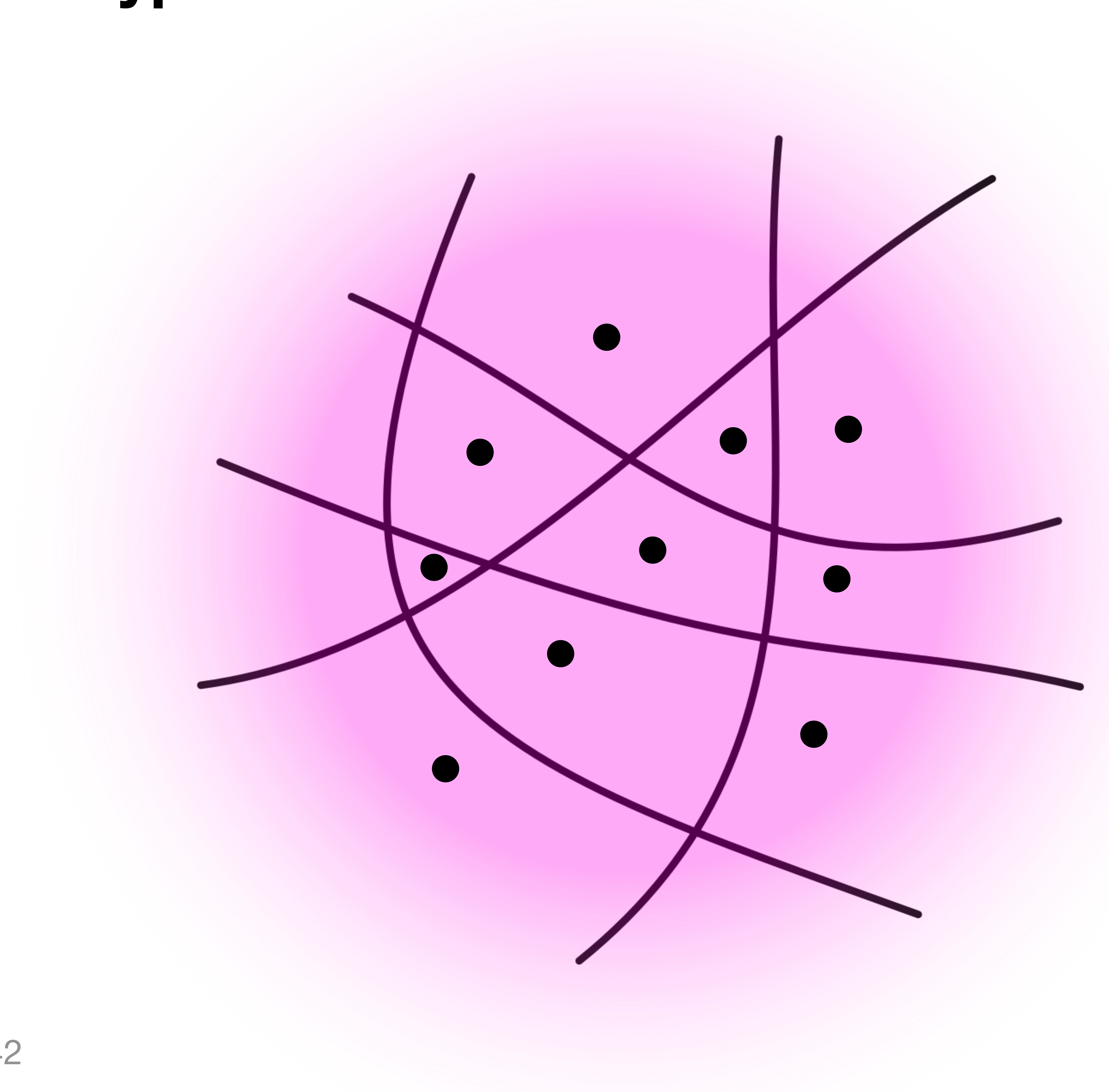
Universal embezzlers and gapped phases of matter



Corollary.

Every gapped phase of matter contains an embezzling ground state

Gapped phases of matter: minimal type

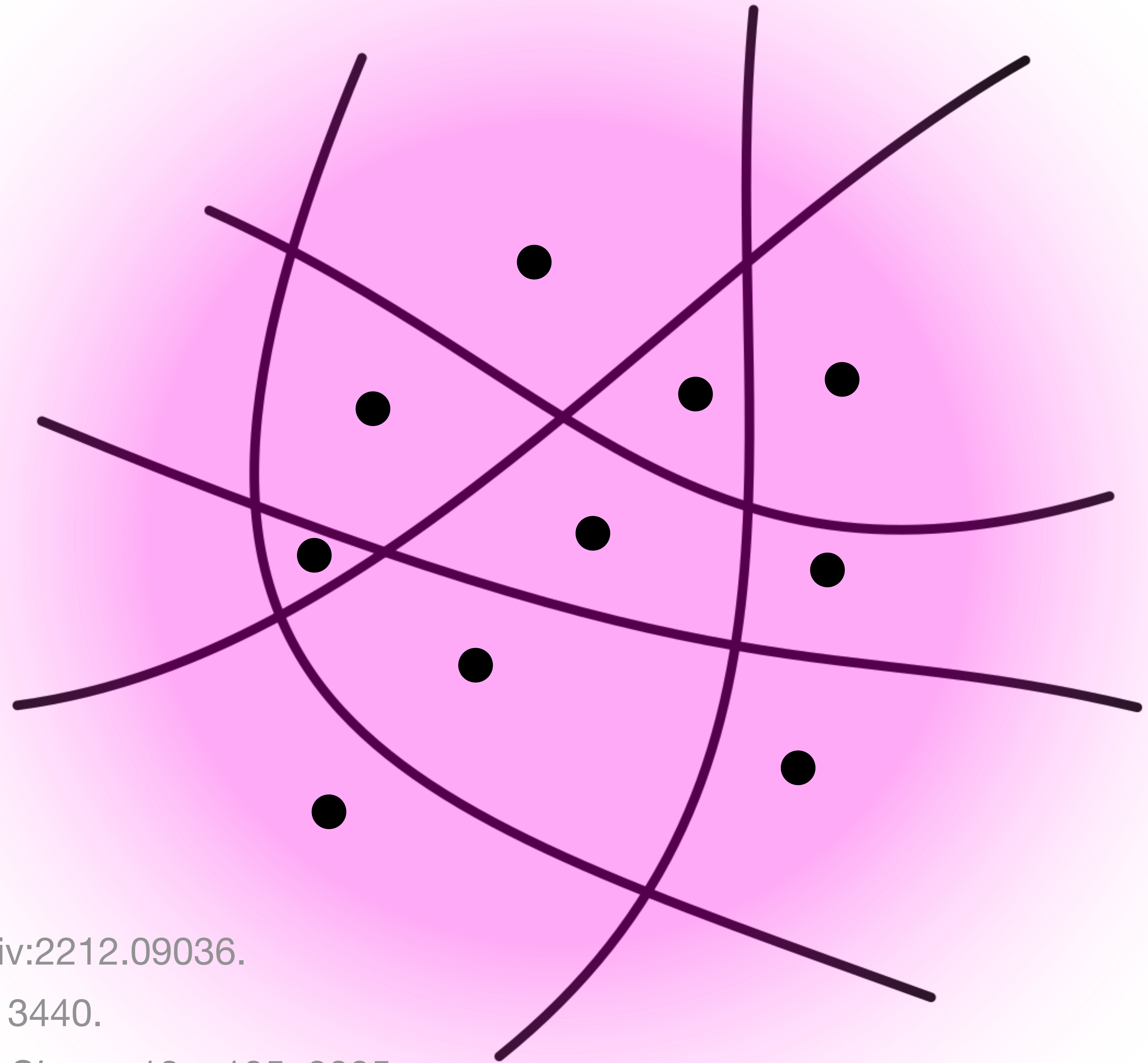


Gapped phases of matter: minimal type

In 2D:

Quantum doubles for finite group: Type II_∞

But: Levin-Wen models with same anyon content can yield type III



Y. Ogata, "Type of Local von Neumann Algebras in Abelian Quantum Double Model," arXiv:2212.09036.

M. Tomba et al., "Boundary Algebras of the Kitaev Quantum Double Model," arXiv:2309.13440.

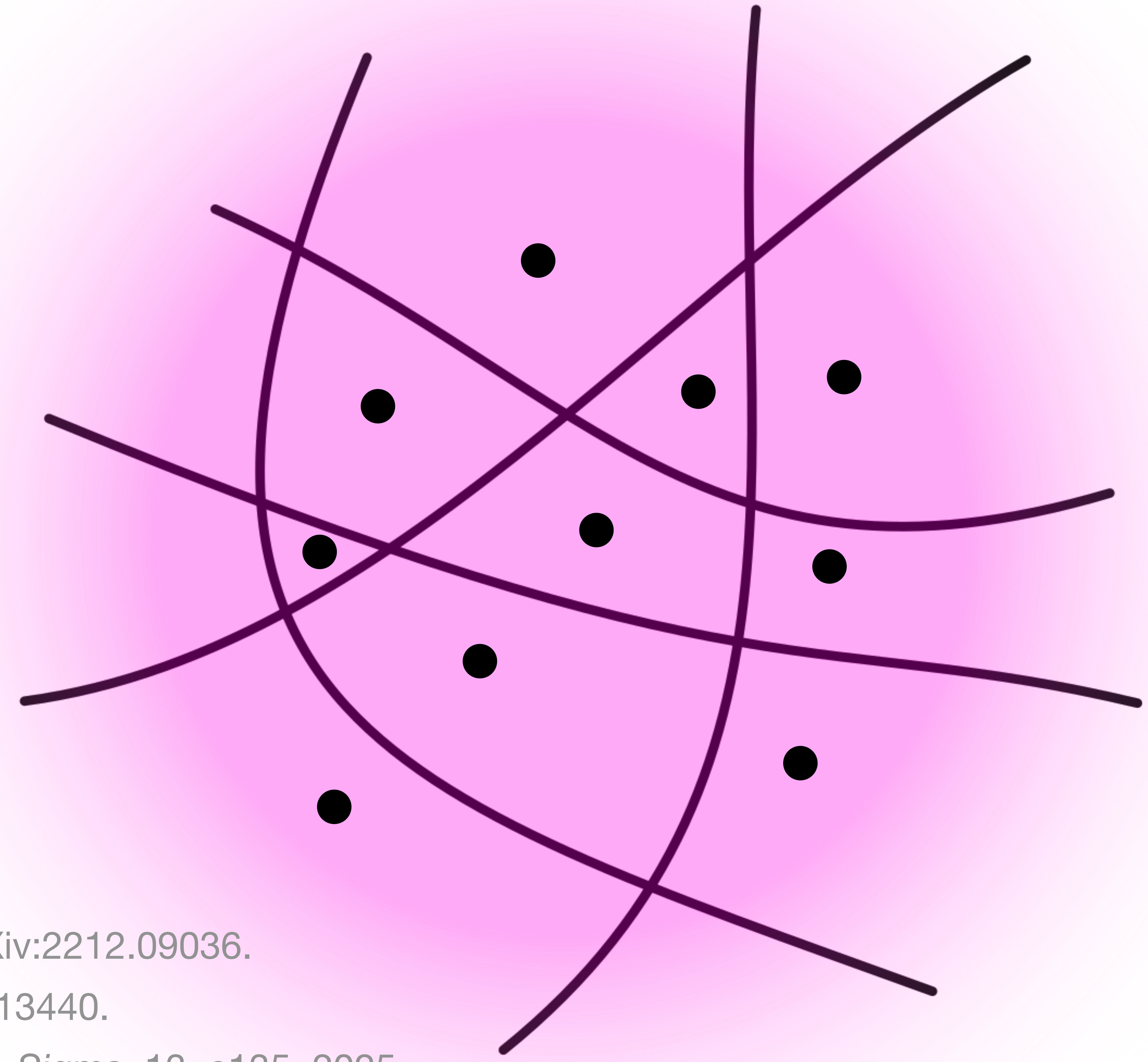
C. Jones et al., "Local topological order and boundary algebras", *Forum of Mathematics, Sigma*, 13, e135, 2025

Gapped phases of matter: minimal type

In 2D:

Quantum doubles for finite group: Type II_∞

But: Levin-Wen models with same anyon content can yield type III

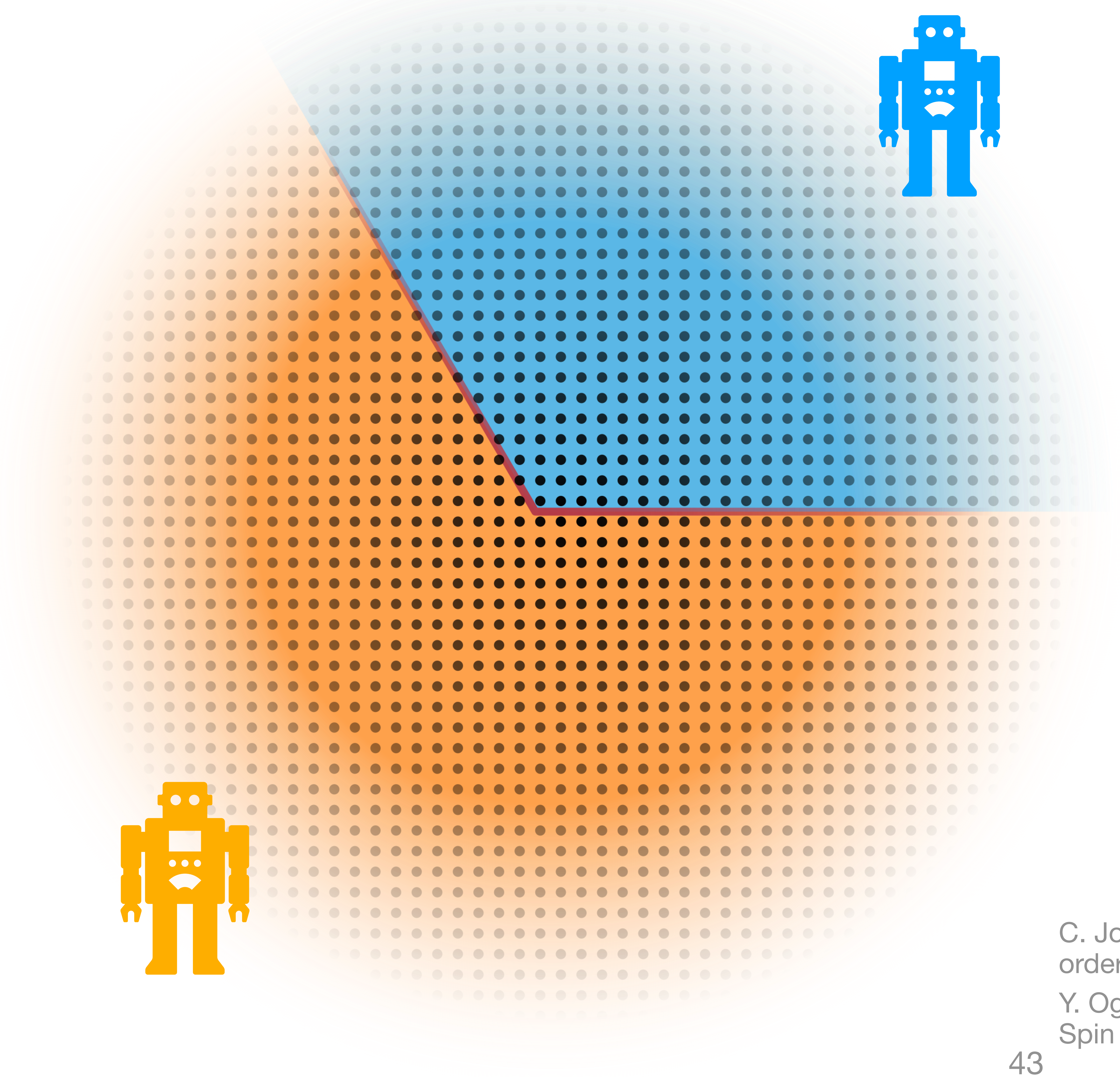


Open problem: Are there gapped phases in 2D that **require** type $III_{(1)}$.

Y. Ogata, “Type of Local von Neumann Algebras in Abelian Quantum Double Model,” arXiv:2212.09036.

M. Tomba et al., “Boundary Algebras of the Kitaev Quantum Double Model,” arXiv:2309.13440.

C. Jones et al., “Local topological order and boundary algebras”, *Forum of Mathematics, Sigma*, 13, e135, 2025



Ground states of gapped phases of matter

Levin-Wen models

Some quantum dim. $d_i > 1$: type III

subtype can be determined from fusion rules:

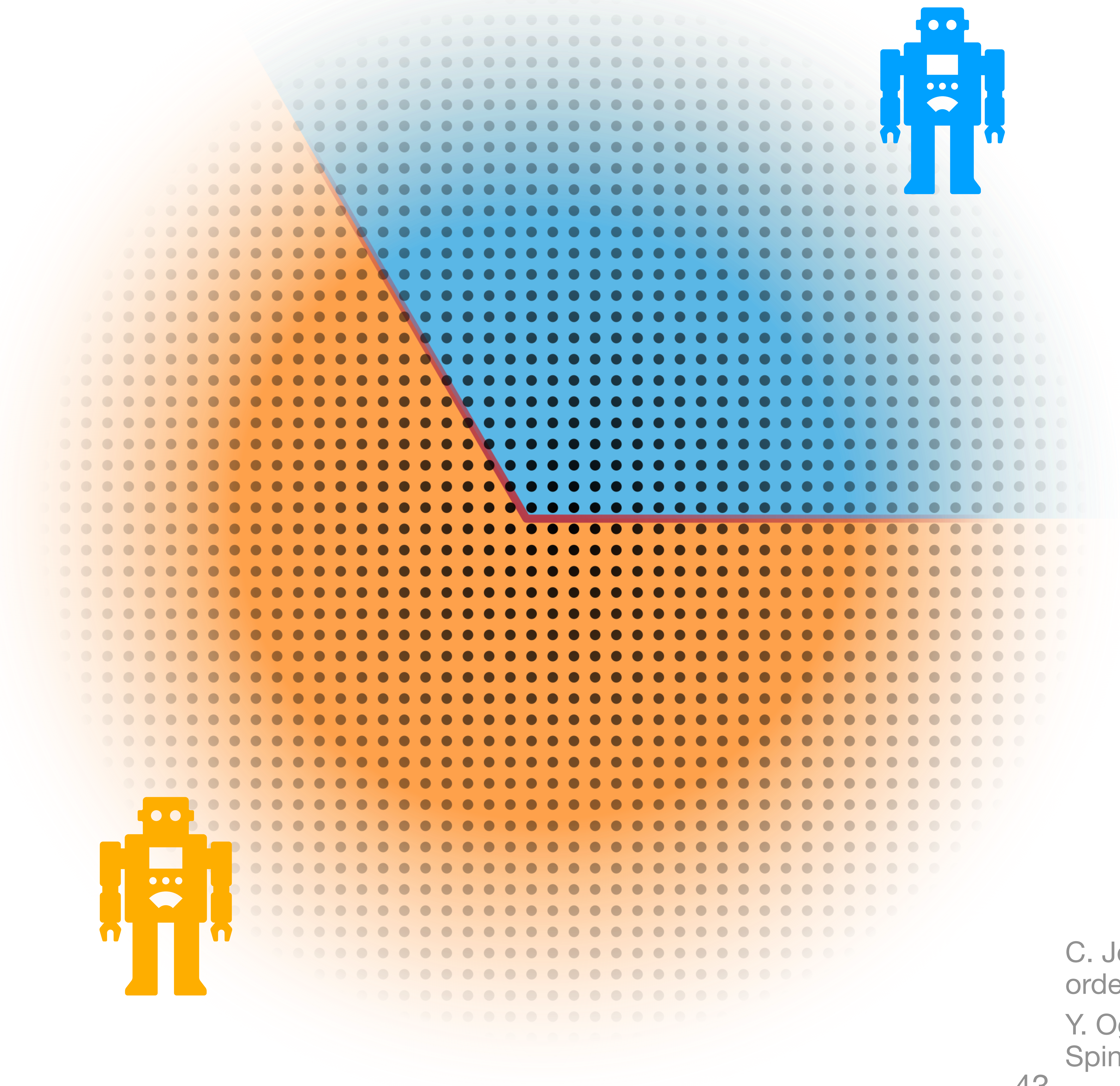
Ising fusion cat.: type $\text{III}_{1/2}$

Fibonacci fusion cat.: type $\text{III}_{1/\phi}$

suff. complex fusion rules: type III_1

C. Jones, P. Naaijkens, D. Penneys, D. Wallick, M. Izumi, “Local topological order and boundary algebras”, *Forum of Mathematics, Sigma*, 13, e135, 2025

Y. Ogata, D. Pérez-García, A. Ruiz-de-Alarcón, “Haag duality for 2D Quantum Spin Systems”, arXiv:2509.23734



Ground states of gapped phases of matter

Levin-Wen models

Some quantum dim. $d_i > 1$: type III

subtype can be determined from fusion rules:

Ising fusion cat.: type $\mathrm{III}_{1/2}$

Fibonacci fusion cat.: type $\mathrm{III}_{1/\phi}$

suff. complex fusion rules: type III_1

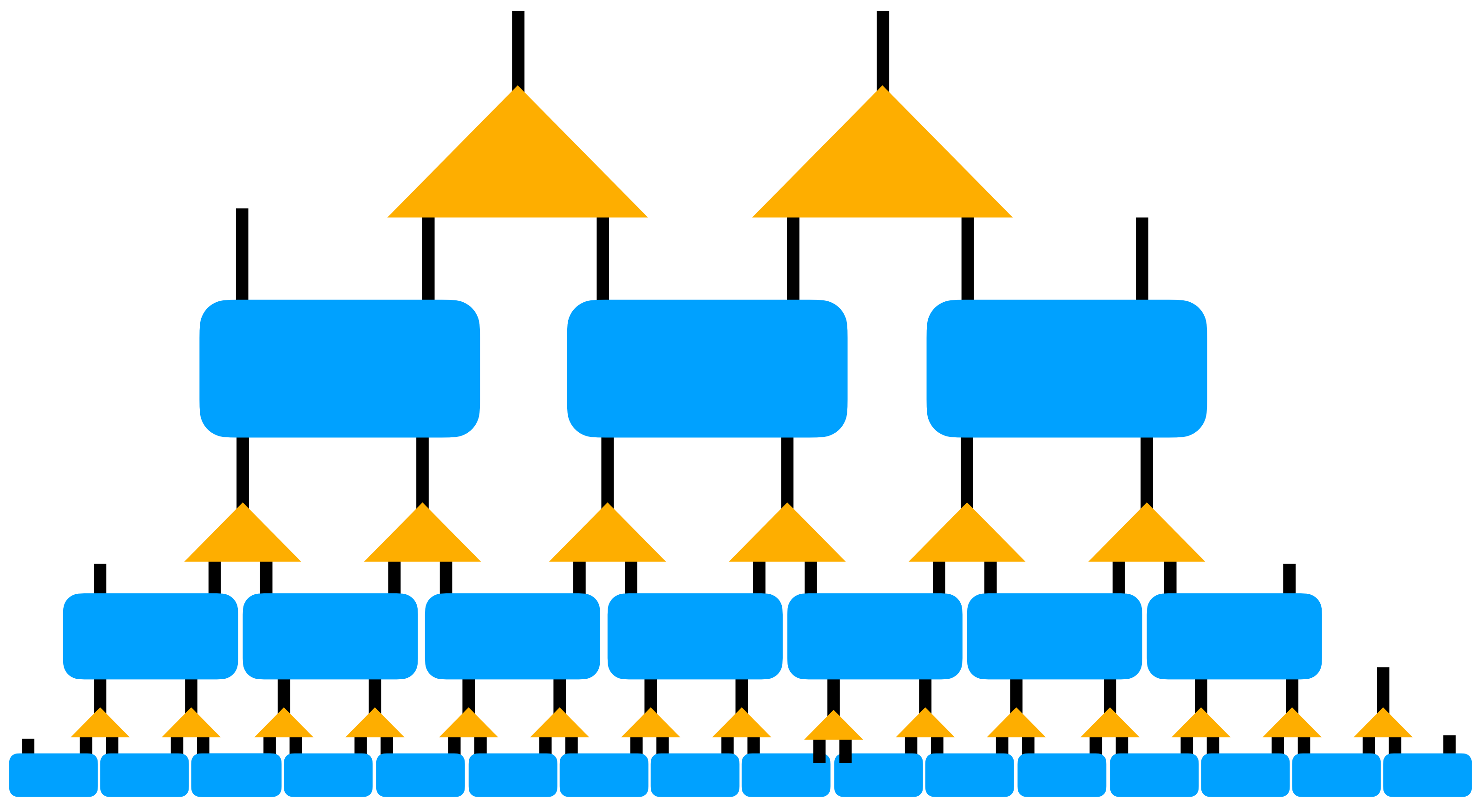
Haag duality holds by recent results

C. Jones, P. Naaijkens, D. Penneys, D. Wallick, M. Izumi, “Local topological order and boundary algebras”, *Forum of Mathematics, Sigma*, 13, e135, 2025

Y. Ogata, D. Pérez-García, A. Ruiz-de-Alarcón, “Haag duality for 2D Quantum Spin Systems”, arXiv:2509.23734

Outlook and other open problems

Outlook & open problems



1+1d CFTs yield universal embezzlers
(local algebras of type III_1)

Do all critical 1d many-body systems yield universal embezzlers?

Can we understand universal embezzlement explicitly in terms of MERA?

F. Gabbiani, J. Fröhlich, "Operator Algebras and Conformal Field Theory," *Commun. Math. Phys.* 155, 569-640, 1993

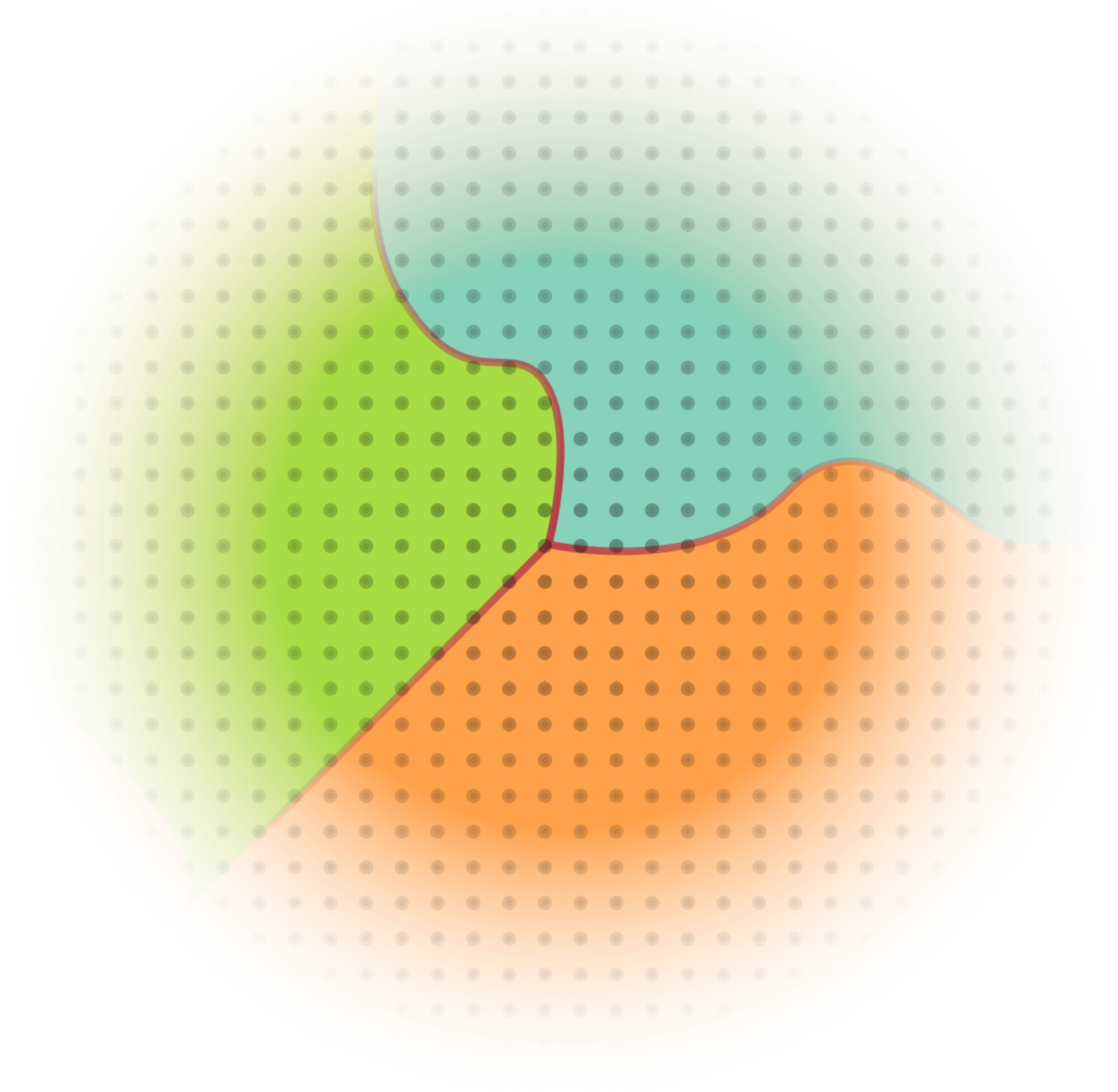
R. Brunetti, D. Guido, R. Longo, "Modular Structure and Duality in Conformal Quantum Field Theory", *Commun. Math. Phys.* 156, 201-219, 1993

G. Vidal, "Entanglement Renormalization," *Physical Review Letters* 99, 220405, 2007

Witteveen et al.,"Quantum Circuit Approximations and Entanglement Renormalization for the Dirac Field in 1+1 Dimensions", *Commun. Math. Phys.* 389, 75-120, 2022

T. J. Osborne, A. S., "Conformal field theory from lattice fermions", *Commun. Math. Phys.* 398, 219-289, 2023

Outlook & open problems



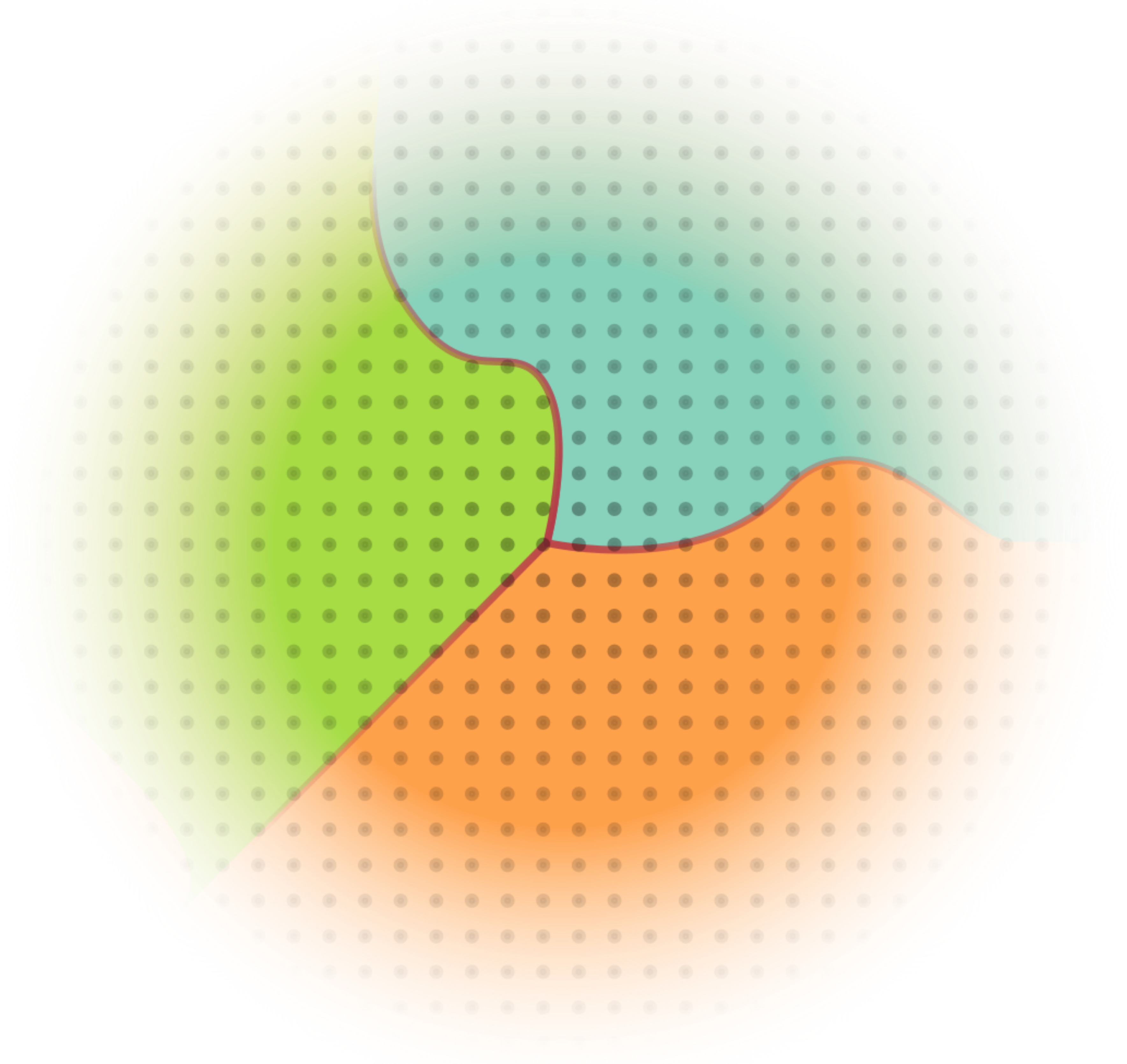
We have abstractly constructed **multipartite embezzling states**.

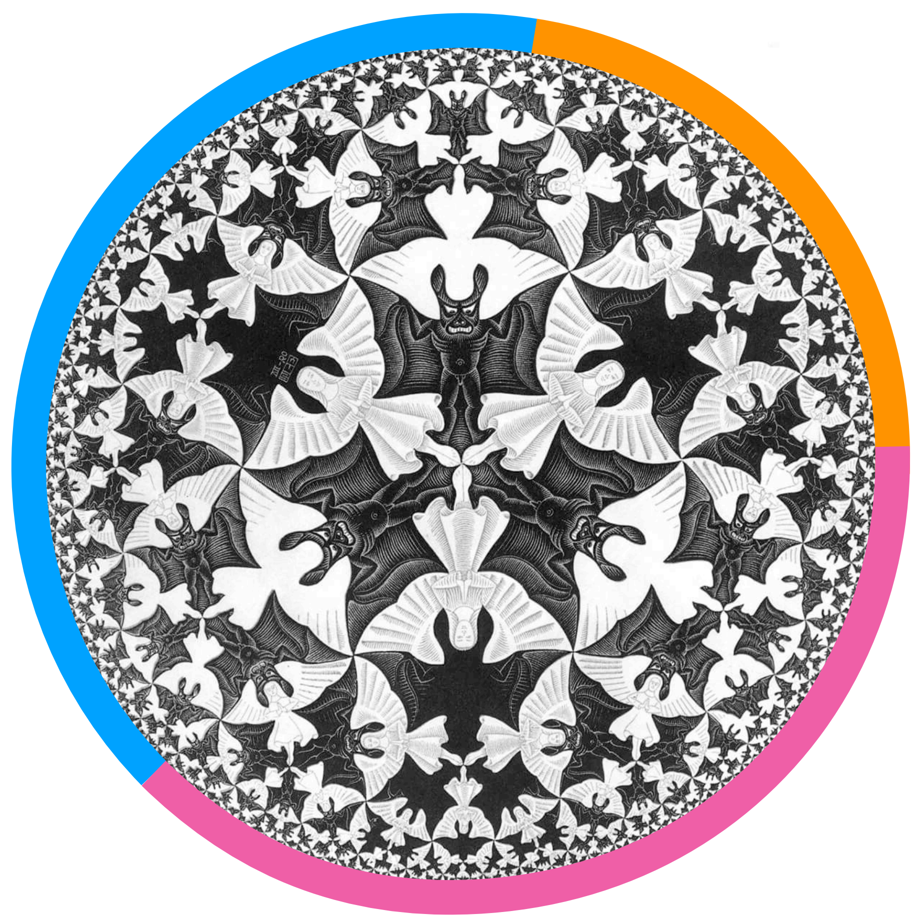
Approximation results by embezzling families transfer to the multipartite setting.

D. Leung, B. Toner, J. Watrous. “Coherent state exchange in multi-prover quantum interactive proof systems”. *Chicago Journal of Theoretical Computer Science* 19.1

L. van Luijk, A. S., H. Wilming, “Multipartite embezzlement of entanglement”, *Quantum* 9, 1818, 2025

Outlook & open problems





We have abstractly constructed **multipartite embezzling states.**

Approximation results by embezzling families transfer to the multipartite setting.

Can ground states of many-body systems or vacua of QFTs/CFTs be multipartite embezzling?

D. Leung, B. Toner, J. Watrous. "Coherent state exchange in multi-prover quantum interactive proof systems". *Chicago Journal of Theoretical Computer Science* 19.1

L. van Luijk, A. S., H. Wilming, "Multipartite embezzlement of entanglement", *Quantum* 9, 1818, 2025

Summary

1. Operational interpretation of Connes' classification of type III factors: Embezzlement

2. Universal embezzlers are ubiquitous in quantum many-body physics and quantum field theory

3. Embezzlement is a large-scale effect with detectable(?) effects in finite-size systems

Outlook and open problems

- What is the connection between embezzlement in CFTs and critical systems? MERA structure?
- Are embezzling unitaries localizable? What are local operations in the QFT?
- Explicitly determine the embezzling unitaries in QFT?
- What about the complexity of embezzlement?
- Can we drop or prove Haag duality ($\mathscr{M}_A \subsetneq \mathscr{M}'_B$)? The role of the index and conditional expectations.
- What is the effect of interactions and disorder?
- Strongly coupled quantum matter (e.g., Motzkin chain)?
- What about multi-partite embezzlement?

T. Schwartzman, "The complexity of entanglement embezzlement", *Phys. Rev. A* 112, 012415, 2025

Summary

1. Operational interpretation of Connes' classification of type III factors: Embezzlement

2. Universal embezzlers are ubiquitous in quantum many-body physics and quantum field theory

3. Embezzlement is a large-scale effect with detectable(?) effects in finite-size systems

Outlook and open problems

- What is the connection between embezzlement in CFTs and critical systems? MERA structure?
- Are embezzling unitaries localizable? What are local operations in the QFT?
- Explicitly determine the embezzling unitaries in QFT?
- What about the complexity of embezzlement?
- Can we drop or prove Haag duality ($\mathscr{M}_A \subsetneq \mathscr{M}_B'$)? The role of the index and conditional expectations.
- What is the effect of interactions and disorder?
- Strongly coupled quantum matter (e.g., Motzkin chain)?
- What about multi-partite embezzlement?

Thank you for your attention!

T. Schwartzman, "The complexity of entanglement embezzlement", *Phys. Rev. A* 112, 012415, 2025