When Born Probabilities Become Quantum Observables

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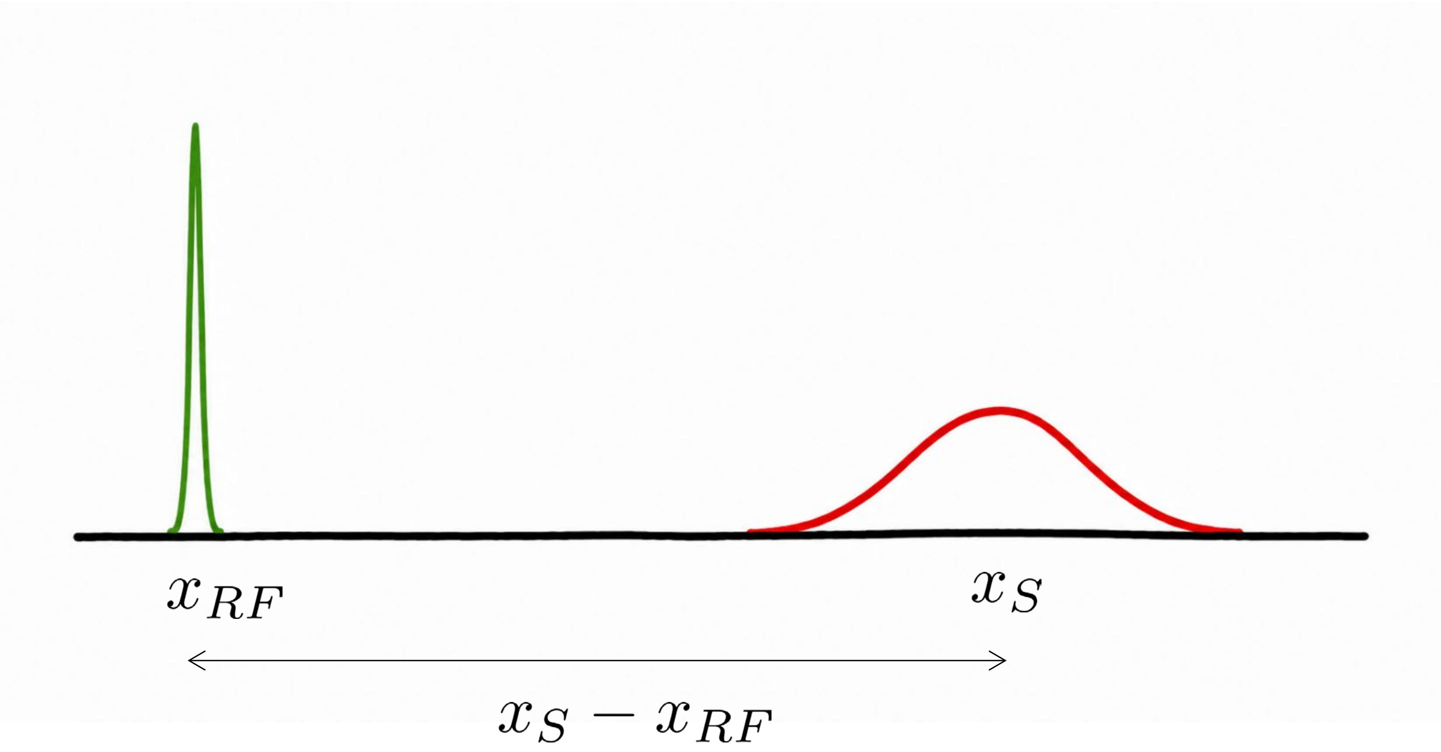
Outline

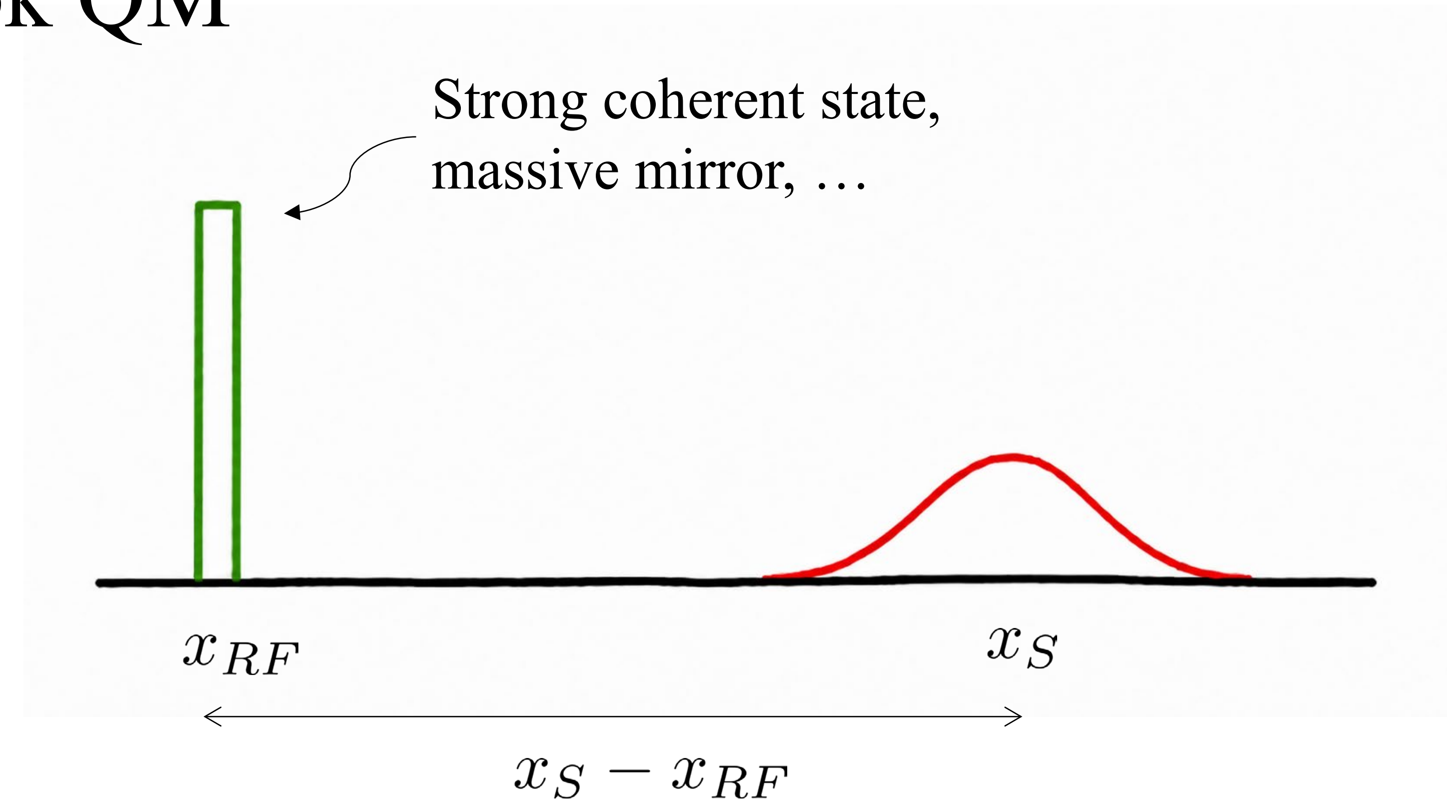
- **Textbook QM and ideal RFs**: individual outcomes of all measurements are not jointly definable, but probabilities are
- Relative-frequency operator
- **QM with non-ideal RFs**: relational relative frequencies are not jointly definable
- Heisenberg-type uncertainty relations
- Groups: SU(2), Heisenberg-Weyl, 1D Galilei
- Bell’s theorem for relational relative frequencies
- What is the notion of “quantum state” in quantum gravity?

Relational statements



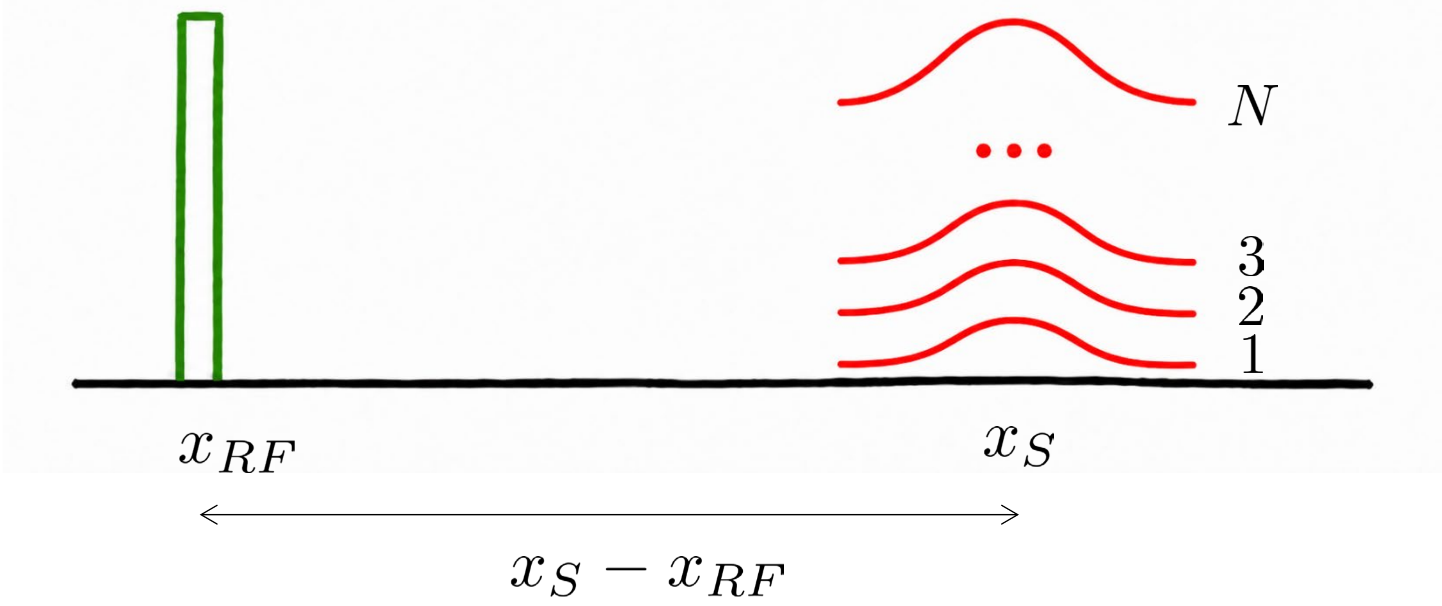
With no external RFs, all physical statements are ***relational***.

Textbook QM



Unbounded, massive, ideal RFs

Ideal RFs

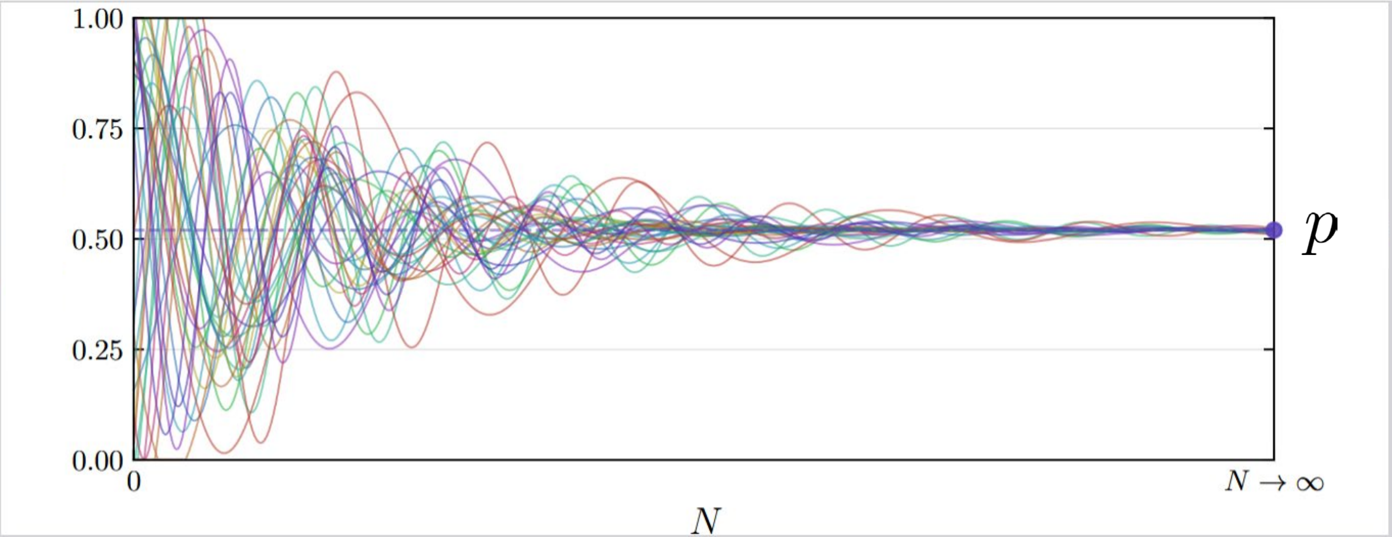


Ideal RFs remain unchanged throughout repeated runs of the experiment.

Experiments performed under “the same conditions”.

Probabilities

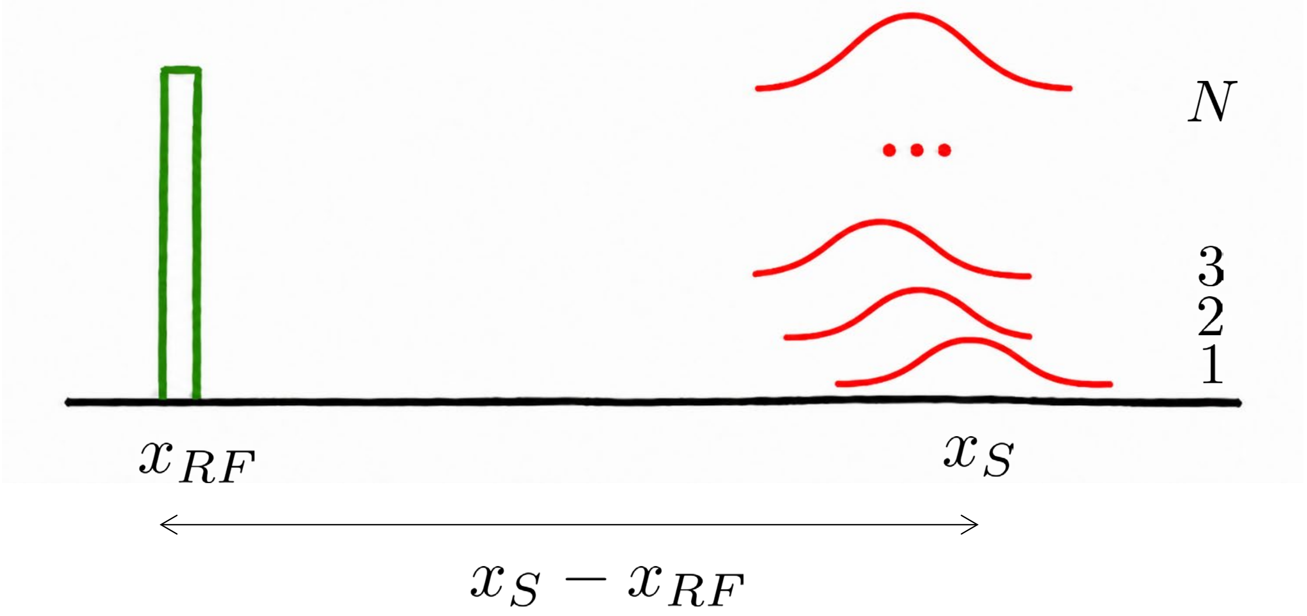
Relative frequencies in presence of ideal RFs



$$\lim_{n \to \infty} \frac{k}{N} \to p$$

Independent and identically distributed (i.i.d.) systems: $\hat{\sigma}^{\otimes N}$

Quantum de Finetti theorem



If the sequence is *exchangeable* and *extendable,* the state is a **classical mixture** of i.i.d. states:

$$\hat{\rho}^{(N)} = \int d\mu(\sigma)\, \hat{\sigma}^{\otimes N}$$

Quantum state

Unbounded, massive, ideal RFs

$$\hat{\rho} \equiv \begin{pmatrix} \mathrm{Tr}(\hat{\rho}\,\hat{P}_1) \\ \mathrm{Tr}(\hat{\rho}\,\hat{P}_2) \\ \vdots \\ \mathrm{Tr}(\hat{\rho}\,\hat{P}_i) \\ \vdots \end{pmatrix}$$

Probabilities for all conceivable measurements are **jointly definable**.
The quantum state = Schrödinger's "catalogue of expectations"

Relative frequency operator

i.i.d. state: the N-copy state:

$$|\psi_N\rangle = \underbrace{|\psi\rangle \otimes |\psi\rangle \otimes ... \otimes |\psi\rangle}_{N}$$

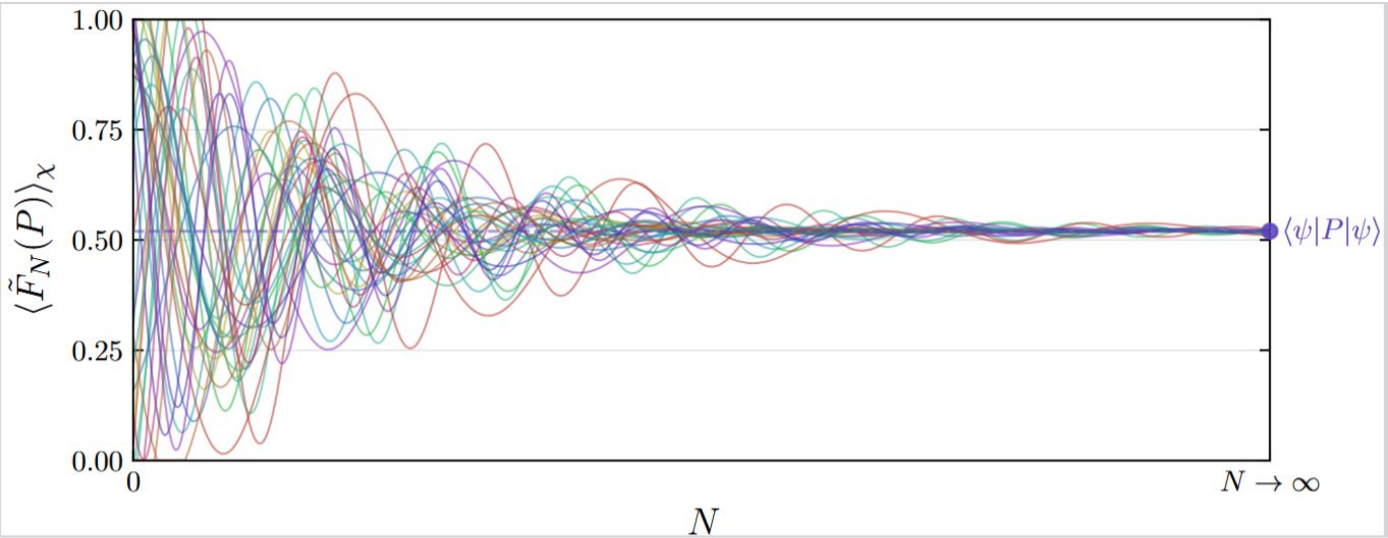
Relative frequency operator:

$$\begin{aligned} F_N(P) &:= \frac{1}{N}\sum_k P^k, \text{ with } P^k := I \otimes \cdots \otimes I \otimes \overset{k\text{th}}{P} \otimes I \otimes \cdots \otimes I. \\ &= \sum_{k=0}^{N} \frac{k}{N} Q_{k/N}, \text{ with } Q_{k/N} = \underbrace{P \otimes \cdots \otimes P}_{k} \otimes \underbrace{P^\perp \otimes \cdots \otimes P^\perp}_{N-k} + \text{permutations.} \end{aligned}$$

Finkelstein-Hartle-Herbut theorem

Theorem 1 (Finkelstein-Hartle-Herbut). Given an N-copy state $|\psi_N\rangle = |\psi\rangle^{\otimes N}$ and a projector P, the mean value of the frequency operator is $\langle\psi_N|F_N(P)|\psi_N\rangle = p$, where $p = \langle\psi|P|\psi\rangle$, and the variance of the frequency operator will vanish for large enough N:

$$\lim_{N\to\infty} \left\| \hat{F}^N(P)|\psi_N\rangle - p|\psi_N\rangle \right\| \to 0$$



Corollarlies

1. Relative frequencies are **jointly definable** for all measurements in the large ensemble limit:

$$\lim_{N\to\infty} \left\| [\hat{F}_N(\hat{P}_1), \hat{F}_N(\hat{P}_2)] |\psi_N\rangle \right\| = 0$$

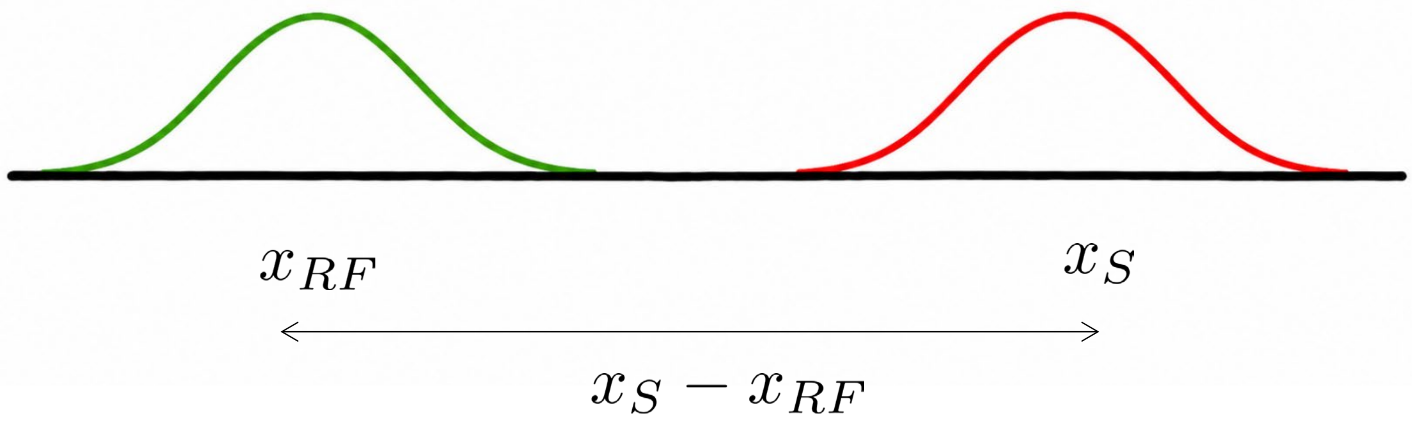
2. **Expectation value operators**

$$F_N(A) := \sum_i a_i F_N(P_i) = \frac{1}{N} \sum_k A^k$$

are also **jointly definable** for all measurements in the large ensemble limit:

$$\lim_{N\to\infty} \left\| [\hat{F}_N(\hat{A}_1), \hat{F}_N(\hat{A}_2)] |\psi_N\rangle \right\| = 0$$

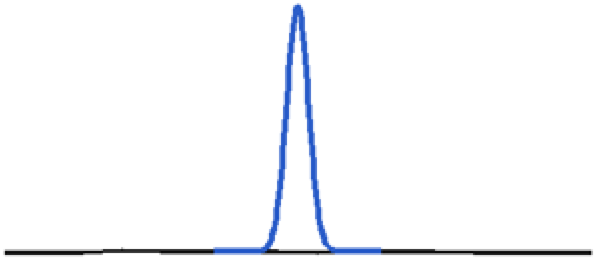
QM with non-ideal RFs



QRFs undergo “backreaction” over repeated runs of the experiment.

Fundamentally, experiments cannot be performed under “the same conditions.”

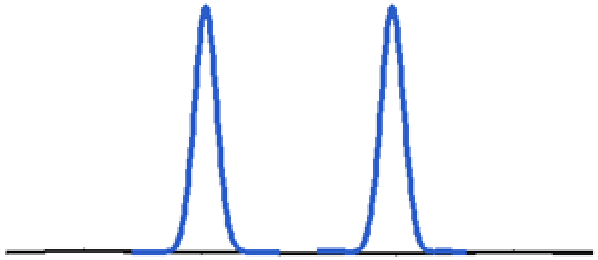
Ideal and non-ideal QRFs



$|g\rangle$

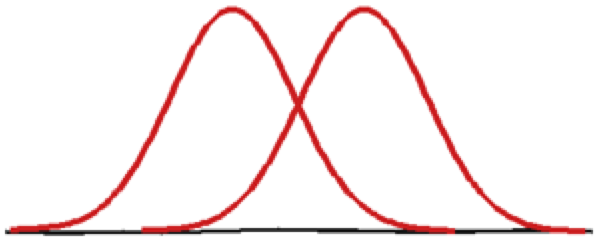
$\langle g|g'\rangle = \delta_{gg'}$

Classical-like



$|g\rangle + |g'\rangle$

Semi-classical



$\langle g|g'\rangle \neq \delta_{gg'}$

Quantum
(finite resources)

Relational relative frequency (RRF) operator

In absence of external ideal RFs for group G only "**relational observables**" can be measured. With covariant POVM element $E(g)$ of RF, the "**relational relative frequency operator**" is given by G-twirling of the frequency operator:

$$\hat{\tilde{\hat{F}}}_N(\hat{P}) = \int_G dg\, E(g)_R \otimes g[\hat{F}_N(\hat{P})]$$
$$= \frac{1}{N}\sum_k \int_G dg\, E(g)_R \otimes \left(\hat{U}(g)\hat{P}\hat{U}^\dagger(g)\right)^{(k)}$$

$$E(g) = \frac{1}{r}|g\rangle\langle g|, \text{ in general } \langle g|g'\rangle \neq \delta_{gg'}$$

Relational relative frequencies in non-ideal RFs

Theorem 2. Let $|\phi\rangle_A$ be the (pure) state assigned to Alice's quantum reference frame A, $|\psi_N\rangle = |\psi\rangle^{\otimes N} \in \mathcal{H}_a^{\otimes N}$ be an N-copy state of the system a, P_1 and P_2 be projectors on $\mathcal{H}_a$ and $E(g)$ the covariant POVM element on $\mathcal{H}_A$ associated to the group element g. Then, the relational relative frequency operators $\widetilde{F}_N(P_1)$ and $\widetilde{F}_N(P_2)$ acting on $|\chi\rangle = |\phi\rangle \otimes |\psi_N\rangle$ satisfy

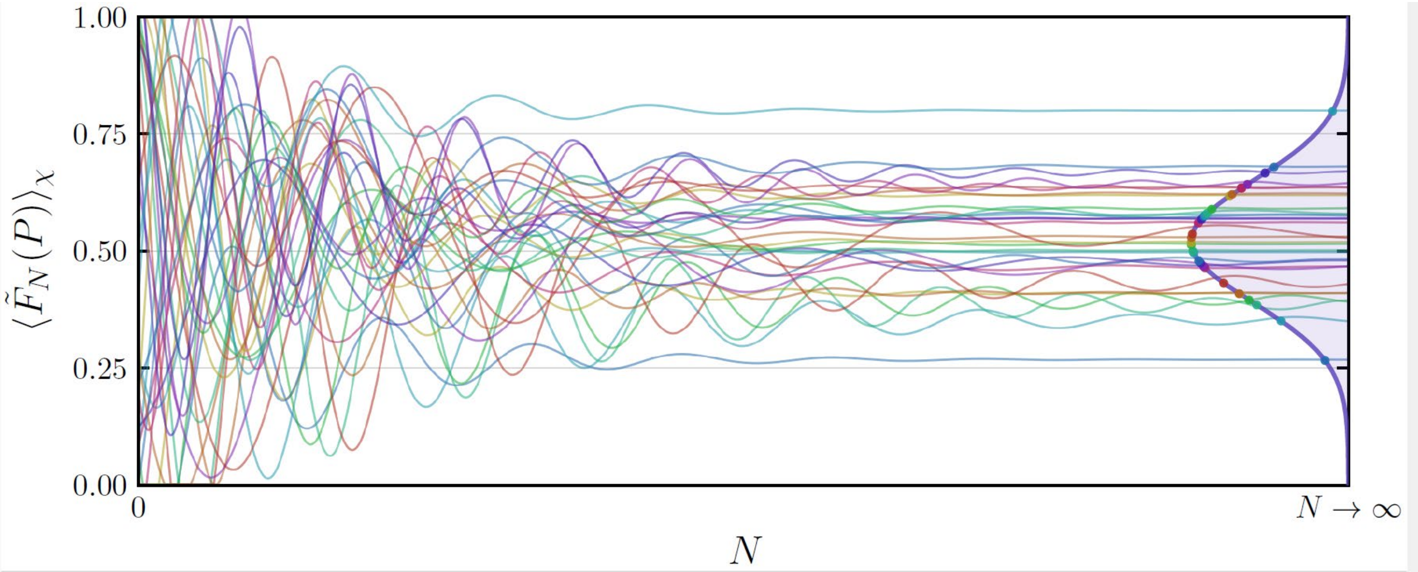
$$\lim_{N\to\infty} \left\| \left[\widetilde{F}_N(P_1), \widetilde{F}_N(P_2) \right] |\chi\rangle \right\| = \| [M_{P_1}, M_{P_2}] \, |\phi\rangle \| \neq 0$$

with

$$M_{P_1} := \int_G dg \, \langle\psi| U(g) P_1 U^\dagger(g) |\psi\rangle E(g), \qquad M_{P_2} := \int_G dg \, \langle\psi| U(g) P_2 U^\dagger(g) |\psi\rangle E(g).$$

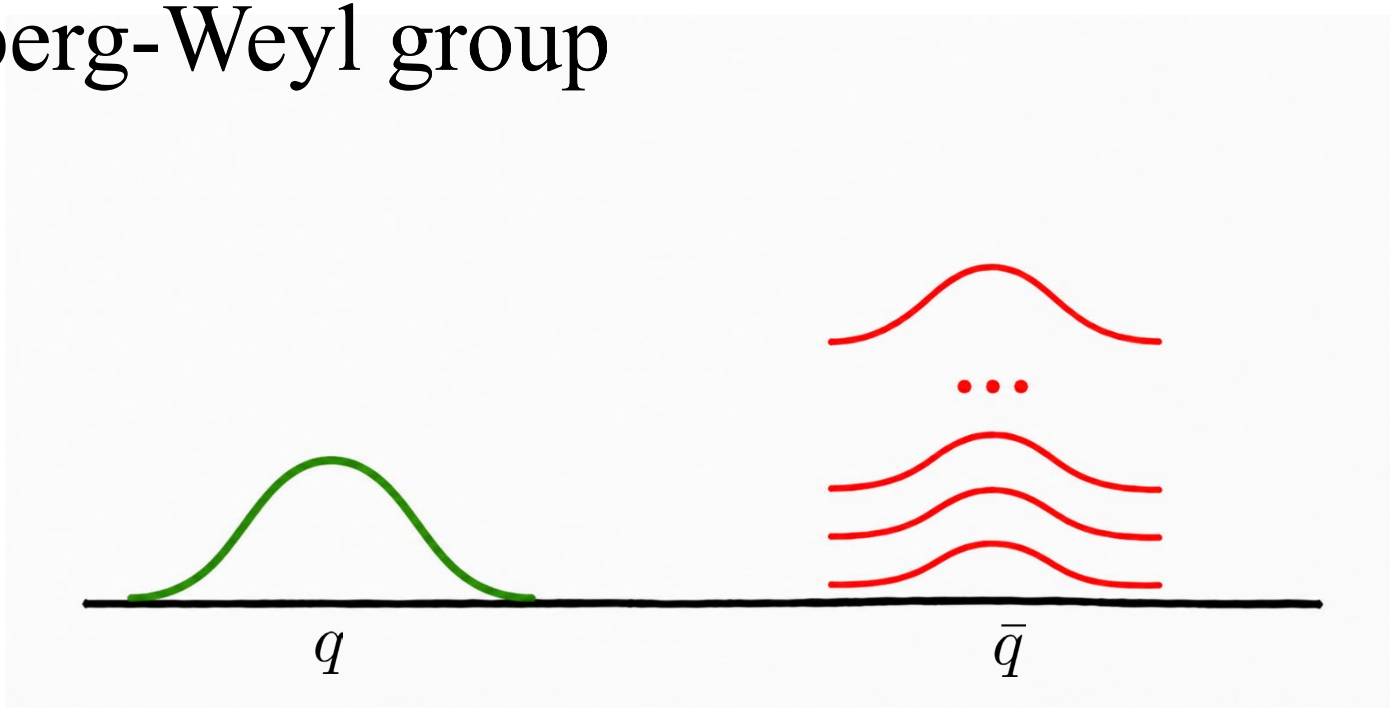
They do commute if the RFs are ideal, or there are *N* independent RFs for *N* systems.

The limit of many repetitions



Even in the limit of infinitely many repetitions, the distribution of the relative frequency does not become sharp; a residual **nonzero standard deviation** remains.

Heisenberg-Weyl group



$$\widetilde{F}_N(X_\vartheta) = \mathbb{I}_A \otimes F_N(X_\vartheta) - X_\vartheta \otimes \mathbb{I}_{a^N}$$

$$\widetilde{F}_N(X_\vartheta)\left(|q\rangle_\vartheta \otimes |\vec{q}\rangle_\vartheta\right) = (\bar{q} - q)|q\rangle_\vartheta \otimes |\vec{q}\rangle_\vartheta \qquad \bar{q} := \frac{1}{N}\sum_{k=1}^{N} q_k$$

First take the mean value over the N systems, and then measure its relative distance from the RF.

Heisenberg uncertainty relations

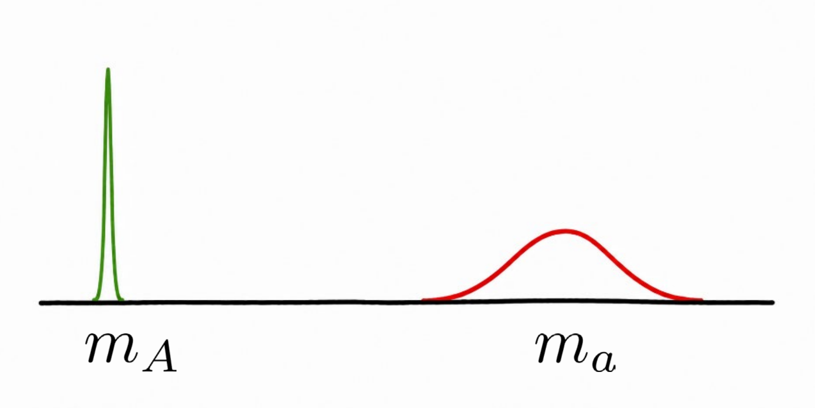
$$\Delta^2_\chi\left(\widetilde{F}_N(A_1)\right)\Delta^2_\chi\left(\widetilde{F}_N(A_2)\right) \geq \frac{1}{4}\left|\left\langle\left[\widetilde{F}_N(A_1),\widetilde{F}_N(A_2)\right]\right\rangle_\chi\right|^2$$

$\Downarrow$

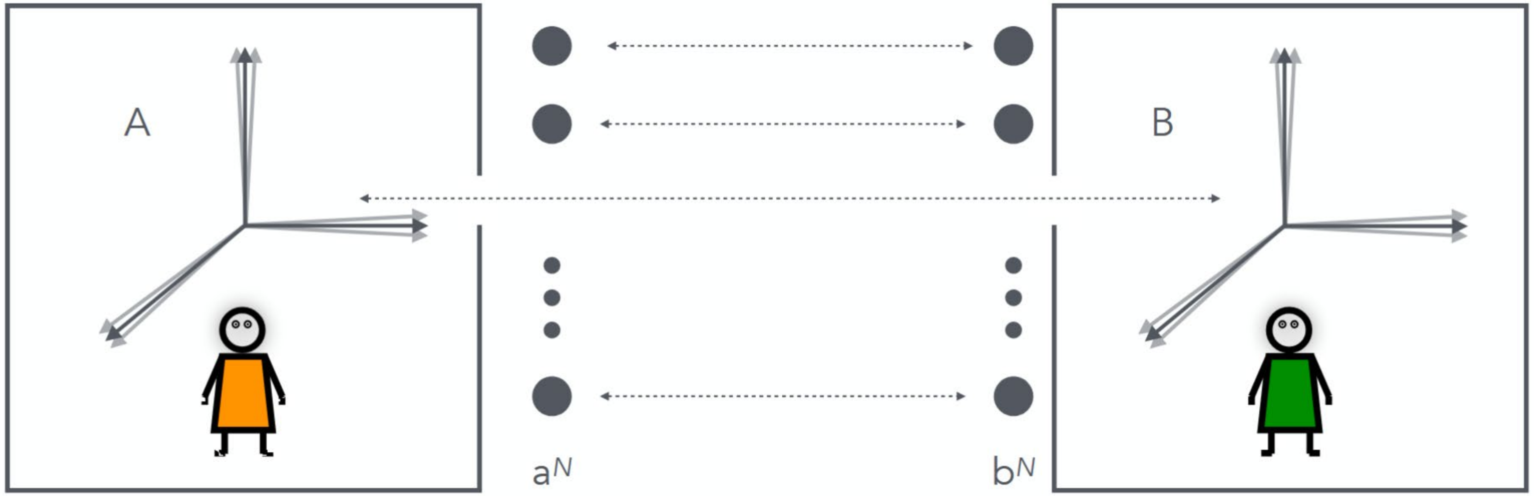
- SU(2): $\lim_{N\to\infty}\Delta^2_\chi\widetilde{F}_N(P_v)\Delta^2_\chi\widetilde{F}_N(P_w) \geq \frac{1}{324} \approx 0.003, \quad = 0$
  $j = 1/2, \quad j \to \infty$

- Heisenberg-Weyl: $\Delta^2_\Psi(\widetilde{F}_N(X))\Delta^2_\Psi(\widetilde{F}_N(P)) \geq \frac{\hbar^2}{4}$

- 1D Galilei: $\Delta^2_\chi(\widetilde{F}_N(P))\Delta^2_\chi(\widetilde{F}_N(K)) \geq \frac{\hbar^2}{4}\frac{m_a^4}{m_A^2}$



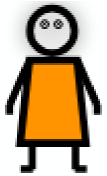
Bell‘s theorem



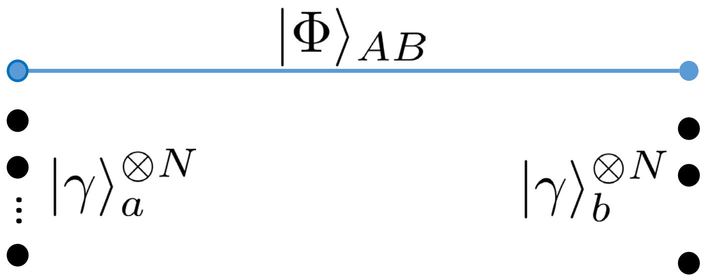
$$p_{QM}(x, y|a, b) \neq \int d\lambda \rho(\lambda) p(x|a, \lambda) p(y|b, \lambda)$$

QM is not “locally causal”. There exits **no joint probability** for the outcomes of all conceivable measurements.

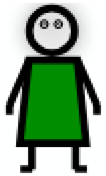
Bell‘s theorem



$\mathrm{sgn}\left(\widetilde{F}_N(A) - \mu\right)$



$$|\Psi\rangle = |\Phi\rangle_{AB} \otimes |\gamma,\gamma\rangle_{ab}^{\otimes N}$$

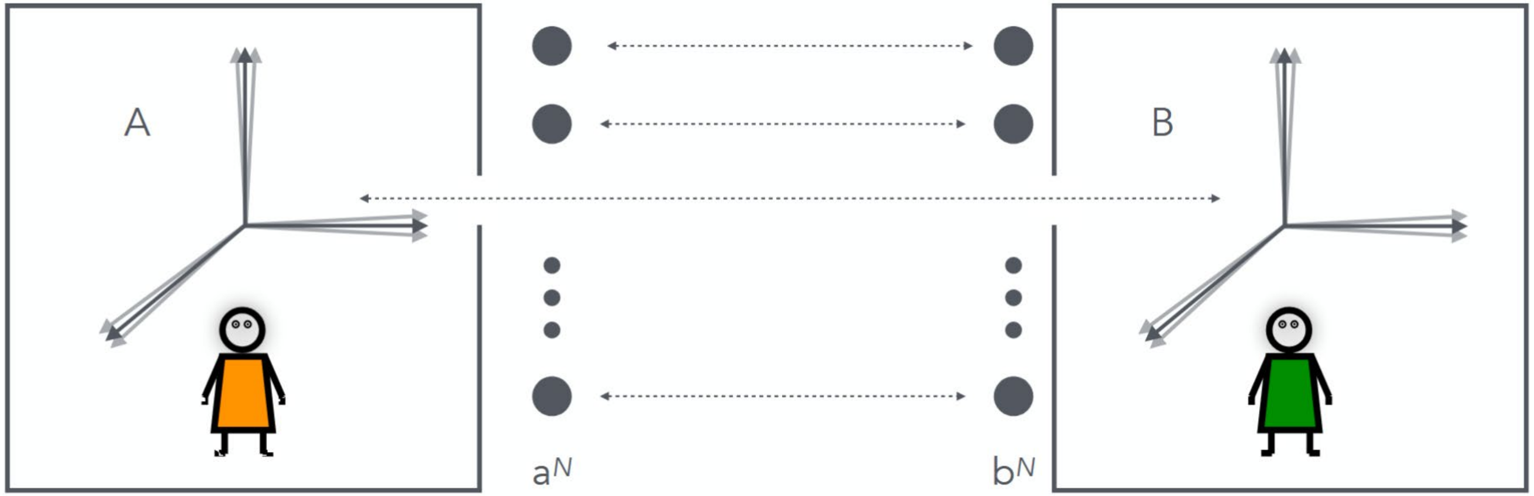


$\mathrm{sgn}\left(\widetilde{F}_N(B) - \mu\right)$

$$C = \textstyle\sum_{a,b=0}^{1} (-1)^{ab} E(\vartheta_a, \varphi_b) \leq 2 \text{ LHV}$$

- Heisenberg-Weyl: $|\Phi\rangle_{AB} \propto \sum_{k=0}^{\infty} (k+1) r^k \, |k\rangle_A \otimes |k\rangle_B,\quad r \approx 0.572 \quad \langle\Psi|C|\Psi\rangle \approx 2.048$
- SU(2): $|\Phi^+\rangle = \frac{1}{\sqrt{2}} (|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B) \quad \langle\Psi|C|\Psi\rangle = 2\sqrt{2}$

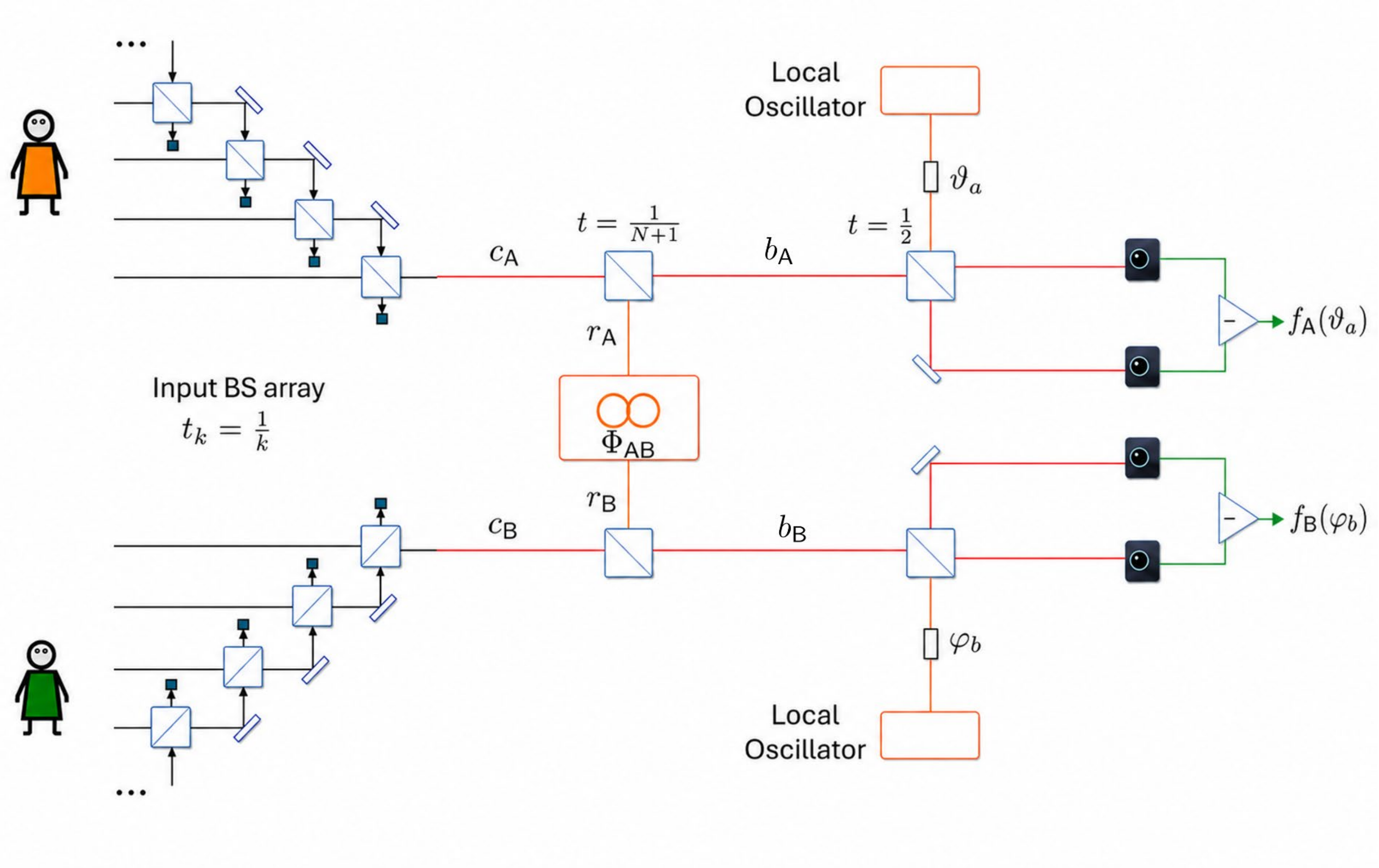
Bell‘s theorem



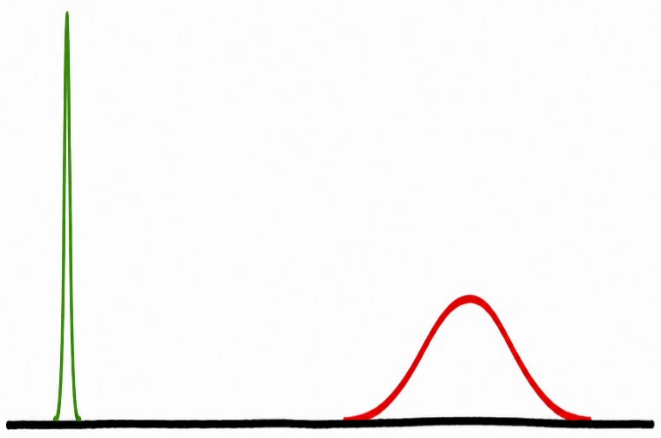
$$p(f_A, f_B | x, y) \neq \int d\lambda \rho(\lambda) p(f_A | x, \lambda) p(f_B | y, \lambda)$$

For non-ideal QRFs, the probabilities are **fundamentally indefinite.** Thus, these "*probabilities of probabilities*" are not mere ignorance about underlying "true probabilities".

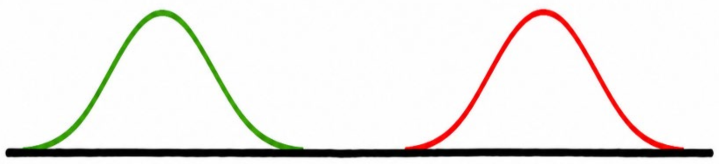
Optical experimental implementation



Tension QT - Gravity



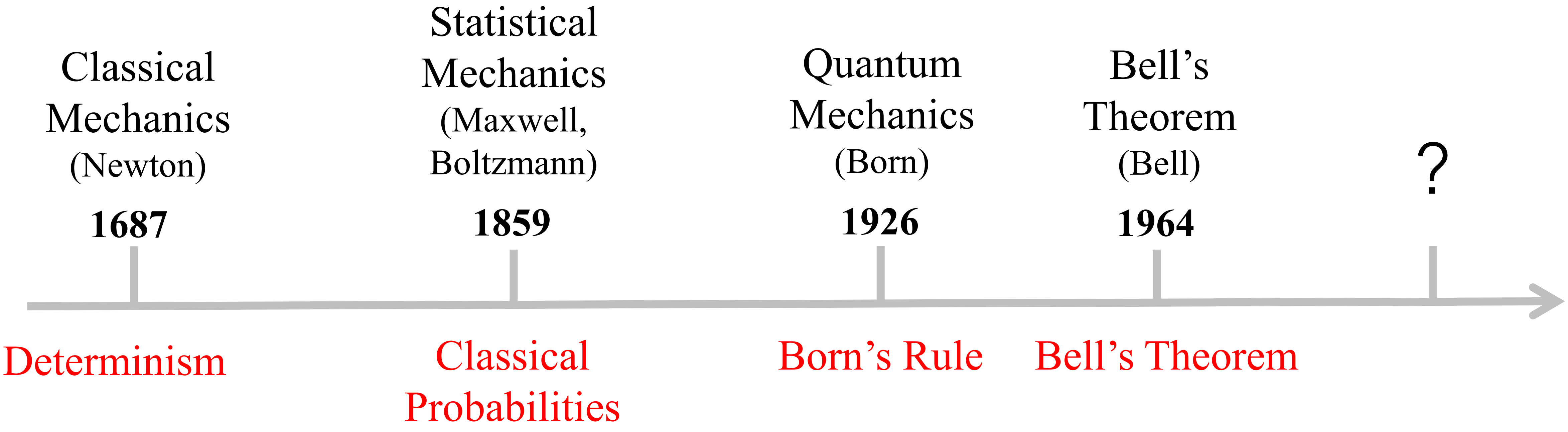
Massive, unbounded RF to ensure repeatability of the experiment and the stability of the probabilities.



Light or ideally no-mass RF to ensure no gravitational backreaction (BH-formation)

$$\dim \mathcal{H}_{\max} = \exp\left(\frac{A}{4\, l_P^2}\right)$$

The history of (un)predictability in physics



Conclusions

- With non-ideal reference frames, relational relative frequencies remain unsharp even for infinitely many runs.
- They obey Heisenberg-type uncertainty relations.
- Their values are Bell-indefinite, not merely unknown.
- Such a regime is plausibly generic in quantum gravity.
- Revision of the notion of "state" in presence of finite resources or quantum gravity.

Thank you!