Classical Overlap and Area-Type Scaling from Symplectic Reduction

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(with Kristina Giesel & Oliver Friedrich)

(arxiv.org/abs/2602.09100)

FAU^2 Workshop: Quantum Gravity Meets Quantum Information

July 9, 2026

The QI question

→ If we are given N qubits, do we know they live in 2^N - dimensional Hilbert space?

- More precisely,
  - a pair of anti-commuting Hermitian unitary operators X and Z defines a qubit.
  - If $\|[X_i, Z_j]\| = 0$, $\forall\, i \neq j \in \{1, 2, \cdots, N\}$ then we know we are in 2^N dimensional Hilbert space.

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- In reality, we hardly have perfect qubits + we have limited computational resources and finite experimental precision, so the actual Hilbert space dimension of such qubits can be difficult to verify.

- From the theory building POV - you might be tempted to dismiss this as a nuisance.

  “ *Physical qubits are very close to exact ones, so I can just write them down in exact tensor factors and I expect the deviations from the exact results would be small.* ”

  - this line-of-thought is fine when *small* changes actually lead to *small* effects.

Overlapping Qubits

- HOWEVER, if you allow the Pauli algebra to overlap, i.e., $\|[P_i, P_j]\| < \epsilon$ then with n exact qubits, then it is possible to spoof $N = \mathcal{O}(\exp(\epsilon^2 n))$ qubits!

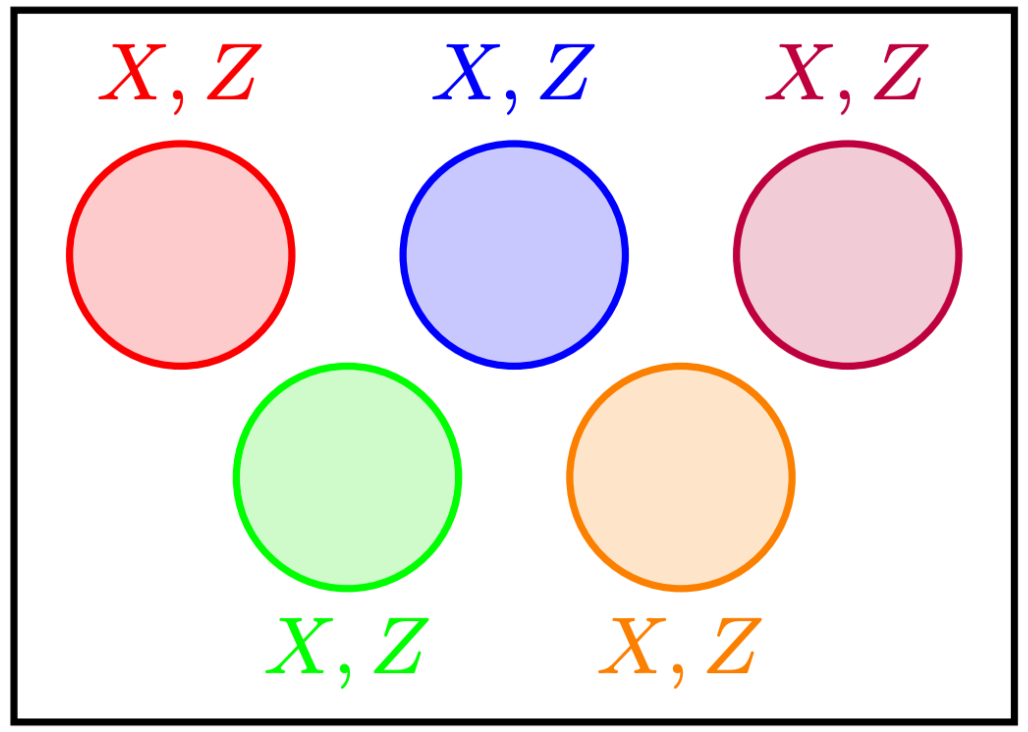
($e^{n\epsilon^2/4}$ many unit vectors can be chosen in $\mathbb{R}^{2n}$ so that any pair $|u\rangle, |v\rangle$, $|\langle u | v\rangle| \leq \epsilon$.) (Johnson-Lindenstrauß lemma)

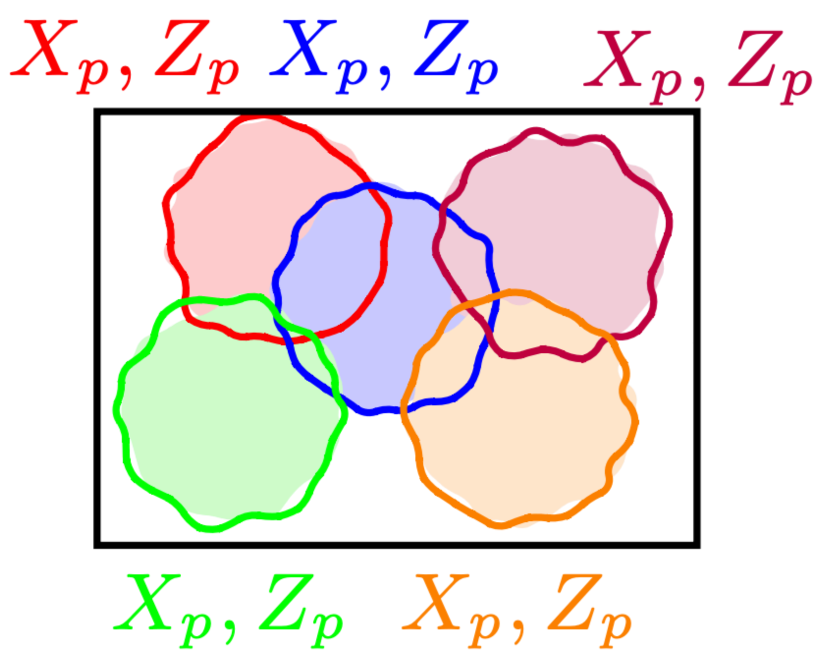
Construction (Chao, Reichardt, Sutherland, Vidick 2017):

$$\{(O_1, O_1^\dagger), \cdots, (O_n, O_n^\dagger)\} \longrightarrow \widetilde{O}_k := \sum_{i=1}^{n} \mathbf{v}_{ki} O_i, \quad k \in [N] \longrightarrow \|\{\widetilde{O}_k, \widetilde{O}_l^\dagger\}\| = \begin{cases} 1, & k = l \\ \mathbf{v}_k^\top \cdot \mathbf{v}_l \leq \epsilon, & k \neq l \end{cases}$$

(Fermion algebra)

Draw N random unit vectors, $\mathbf{v}_k \in \mathbb{R}^n$ and define





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  (Fermion algebra) — Draw N random unit vectors, $\mathbf{v}_k \in \mathbb{R}^n$ and define

- Verification protocol $\sim$ poly(N); but with N being exponentially larger than n, it's operationally expensive.

- Furthermore, gravity itself bounds the entropy/information one can have within a region.

We can be fooled into thinking that the Hilbert space is larger than it actually is!

Reminds us of a problem in QG

- Volume vs area scaling

  Suppose we naively put a QFT inside a region. Then the Hilbert space seems enormous:

$$\mathscr{H}_{\mathrm{QFT}} = \bigotimes_{x \in \mathrm{lattice}} \mathscr{H}_x \longrightarrow \text{ gives volume-scaling entropy, but BH entropy} \propto \text{Area.}$$

  Proposal by (Cao, Chemissany, Jahn, Zimborás 2024): could think of EFT dofs as overlapping qubits spoofing a system with a larger apparent Hilbert space.

nature communications

Article

https://doi.org/10.1038/s41467-024-55463-9

Overlapping qubits from non-isometric maps and de Sitter tensor networks

Received: 11 January 2024

Accepted: 13 December 2024

ChunJun Cao[1,2,3], Wissam Chemissany[4], Alexander Jahn[2,5] & Zoltán Zimborás[6,7,8]

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OPEN ACCESS
IOP Publishing
Classical and Quantum Gravity
Class. Quantum Grav. **41** (2024) 195003 (68pp)
https://doi.org/10.1088/1361-6382/ad6e4d

Holographic phenomenology via overlapping degrees of freedom

Oliver Friedrich[1,2,*], ChunJun Cao[3,4,5], Sean M Carroll[6,7], Gong Cheng[5,8] and Ashmeet Singh[9]

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$$\hat{\psi}(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3 E_{\mathbf{p}}} \left\{ \hat{c}_{\mathbf{p}}(t) u(\mathbf{p})\ e^{i\mathbf{p}\mathbf{x}} + \text{anti-particle part...} \right\}$$

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$$\hat{\psi}(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3 E_{\mathbf{p}}} \left\{ \tilde{C}_{\mathbf{p}}(t) u(\mathbf{p})\, e^{i\mathbf{p}\mathbf{x}} + \text{anti-particle part...} \right\}$$

$$\|\{\tilde{C}_{\mathbf{p}}, \tilde{C}^\dagger_{\mathbf{q}}\}\| = v^\top_{\mathbf{p}} \cdot v_{\mathbf{q}} \leq \epsilon$$

But, $\{\tilde{C}_{\mathbf{p}}, \tilde{C}^\dagger_{\mathbf{p}}\} = 1$

$$\begin{pmatrix} \tilde{C}_1 \\ \vdots \\ \tilde{C}_N \\ \tilde{C}^\dagger_1 \\ \vdots \\ \tilde{C}^\dagger_N \end{pmatrix} = \begin{pmatrix} \\ \mathbf{V} \\ \text{matrix of random vectors} \\ \\ \end{pmatrix} \begin{pmatrix} \hat{c}_1 \\ \vdots \\ \hat{c}_n \\ \hat{c}^\dagger_1 \\ \vdots \\ \hat{c}^\dagger_n \end{pmatrix}$$

$N \propto$ Volume

$n \propto$ Area

(assumed)

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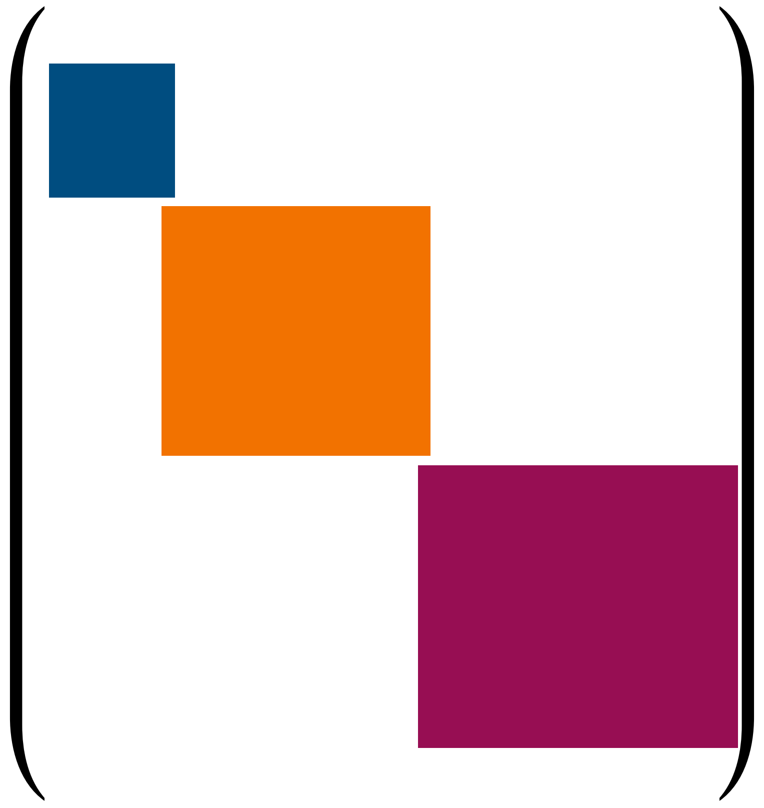
$$\| \{ \tilde{C}_{\mathbf{p}}, \tilde{C}^\dagger_{\mathbf{q}} \} \| = v_{\mathbf{p}}^\top \cdot v_{\mathbf{q}} \leq \epsilon(|\mathbf{p}|)\, \delta_{|\mathbf{p}|, |\mathbf{q}|}$$

$$\begin{pmatrix} \tilde{C}_1 \\ \vdots \\ \tilde{C}_N \\ \tilde{C}_1^\dagger \\ \vdots \\ \tilde{C}_N^\dagger \end{pmatrix} =$$



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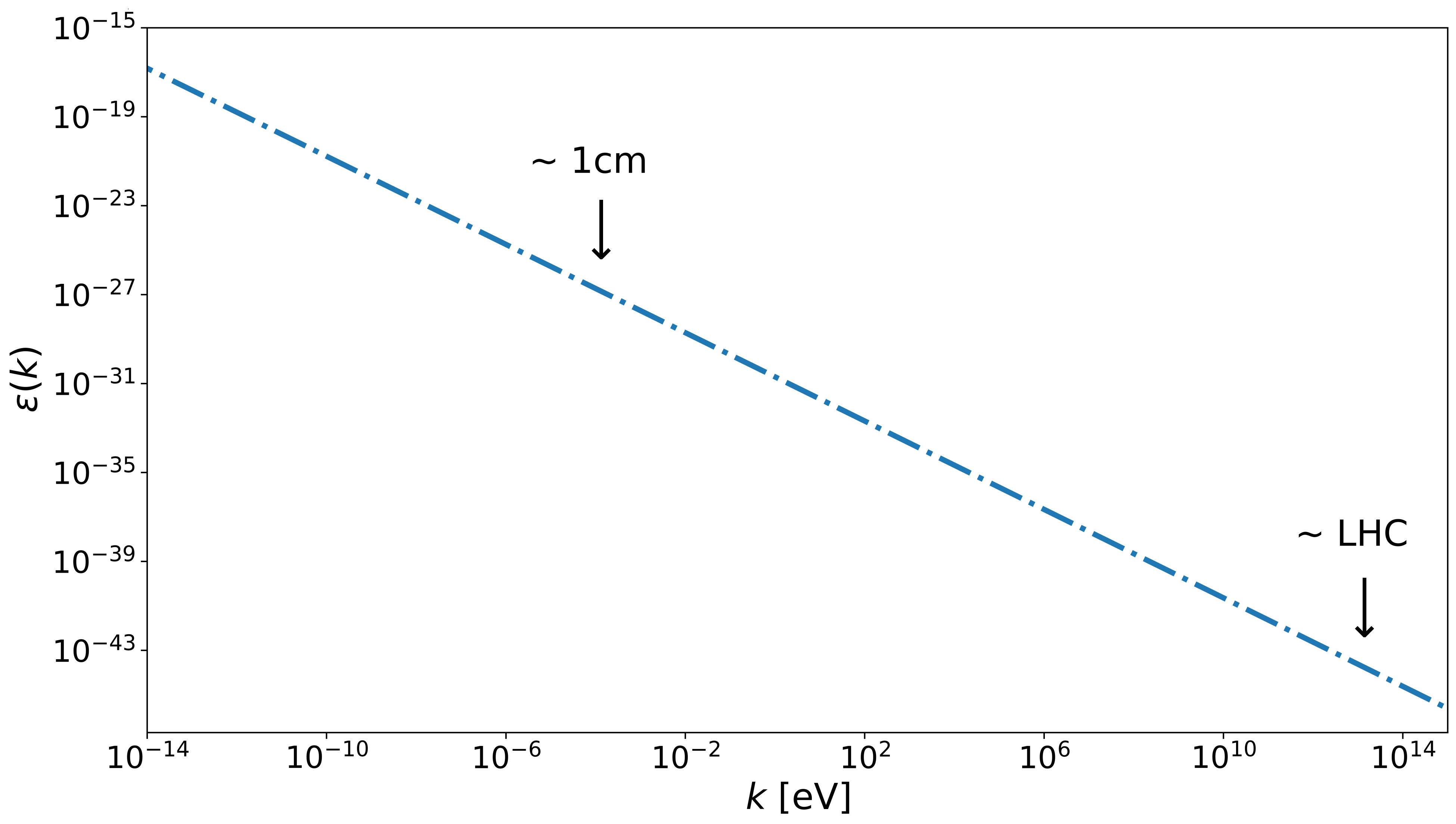

$$\|\{\tilde{C}_{\mathbf{p}}, \tilde{C}^{\dagger}_{\mathbf{q}}\}\| = v_{\mathbf{p}}^{\top} \cdot v_{\mathbf{q}} \le \epsilon(|\mathbf{p}|)\, \delta_{|\mathbf{p}|,|\mathbf{q}|}$$

Bridge to present work

- At least at the kinematics level, the construction seems to be doing pretty well.

- Beyond kinematics $\rightarrow$ dynamics? (Friedrich, Cao, Carroll, Cheng, Singh 2024) did primitive analysis and even made connections with observations.

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- Beyond kinematics → dynamics? (Friedrich, Cao, Carroll, Cheng, Singh 2024) did primitive analysis and even made connections with observations.

This work:

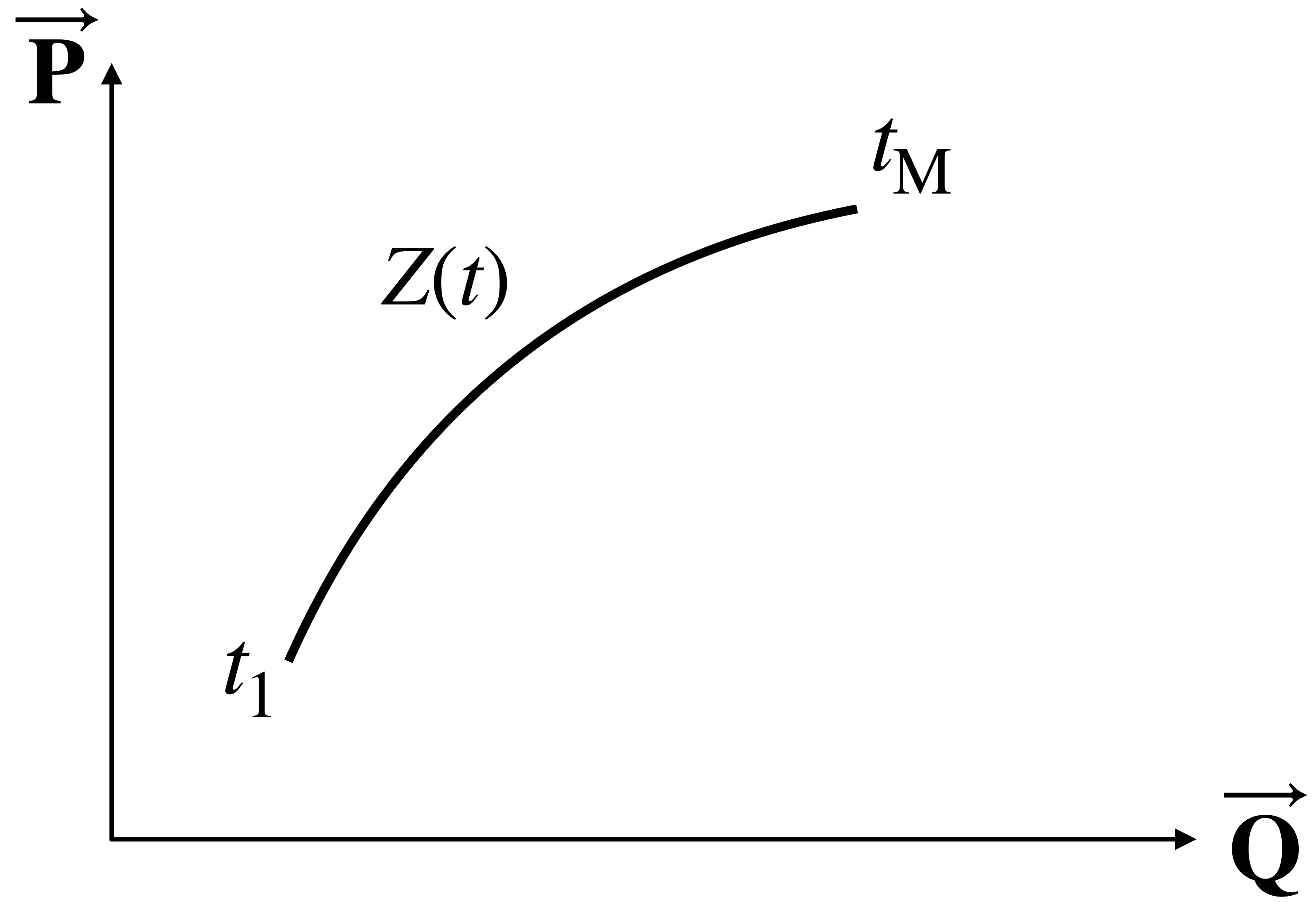
- We asked how $\mathbf{V}$ should be structured so that dynamics sector could also be captured?

- To start simple we consider a scalar field → classical scalar field.

- Is there a classical analog for these overlaps?

Classical picture

Consider a Hamiltonian, $H(\overrightarrow{\mathbf{Q}}, \overrightarrow{\mathbf{P}})$ on a $2N$ dimensional phase space.



$$Z(t) = \begin{pmatrix} Q_1(t) \\ \vdots \\ Q_N(t) \\ P_1(t) \\ \vdots \\ P_N(t) \end{pmatrix}$$

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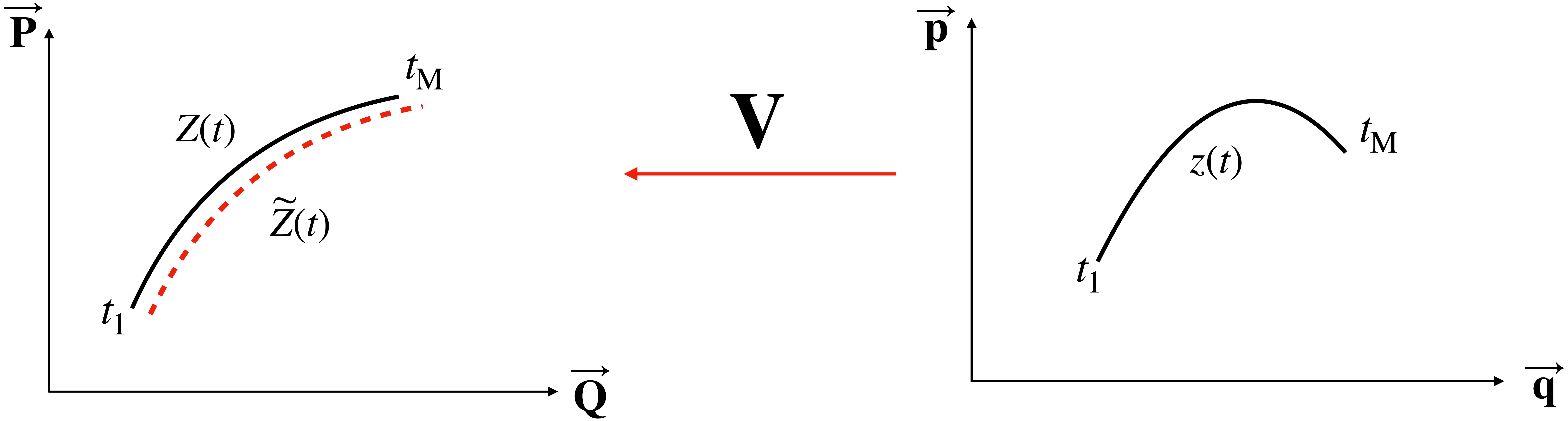


$$Z(t) = \begin{pmatrix} Q_1(t) \\ \vdots \\ Q_N(t) \\ P_1(t) \\ \vdots \\ P_N(t) \end{pmatrix} \approx \underbrace{\begin{pmatrix} \widetilde{Q}_1(t) \\ \vdots \\ \widetilde{Q}_N(t) \\ \widetilde{P}_1(t) \\ \vdots \\ \widetilde{P}_N(t) \end{pmatrix}}_{\widetilde{Z}(t)} = \begin{pmatrix} \\ \mathbf{V}_{2N\times 2n} \\ \\ \end{pmatrix} \begin{pmatrix} q_1(t) \\ \vdots \\ q_n(t) \\ p_1(t) \\ \vdots \\ p_n(t) \end{pmatrix}$$

$$n \;<\; N$$

not fixed yet; would like to determine it such that the reconstruction is as perfect as it can be.

Loss function

We can turn this into an optimization problem.

Construct the snapshot matrix

$$\mathbb{X}_s = \begin{pmatrix} \vec{\mathbf{Q}}(t_1) & \cdots & \vec{\mathbf{Q}}(t_M) \\ \vec{\mathbf{P}}(t_1) & \cdots & \vec{\mathbf{P}}(t_M) \end{pmatrix}_{2N \times M} \qquad t_1 \ll t_M$$

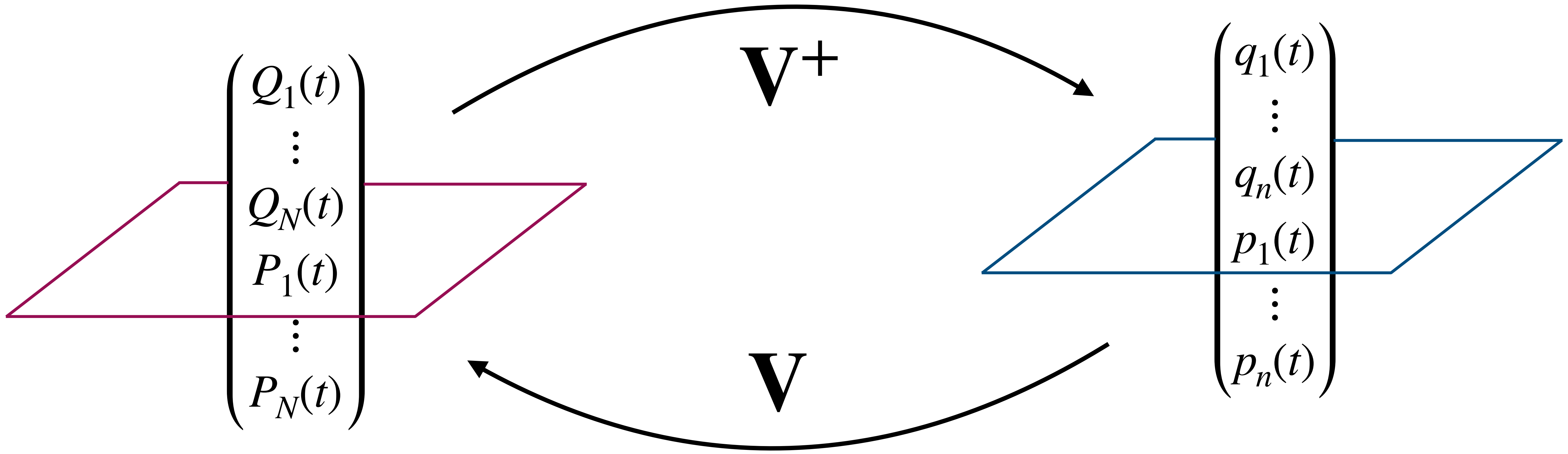
If it's possible to reconstruct the columns of this matrix from $n\,(< N)$ many dofs, then

Minimize

For a given $n < N$,

$$\|\mathbb{X}_s - \mathbf{V}\mathbf{V}^{+}\mathbb{X}_s\|_F$$

over matrices in $\mathbb{R}^{2N \times 2n}$



1. Compress → Retrieve ≈ Minimal loss, i.e., $\mathbf{V}\mathbf{V}^{+} \approx \mathbb{I}_{2N}$, where $\mathbf{V}^{+}\mathbf{V} = \mathbb{I}_{2n}$.

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2. Columns are obtained by solving Hamiltonian EOMs

$$\dot{Z}(t) = \mathbb{J}_{2N} \nabla H(\vec{\mathbf{Q}}, \vec{\mathbf{P}}) \quad \longrightarrow \quad \dot{z}(t) = \mathbb{J}_{2n} \nabla \widetilde{H}(\vec{\mathbf{q}}, \vec{\mathbf{p}}), \qquad \mathbb{J}_{2k} := \begin{pmatrix} 0 & \mathbb{I}_k \\ -\mathbb{I}_k & 0 \end{pmatrix}$$

$\implies$ $\mathbf{V} \in \mathbb{R}^{2N\times 2n}$ must preserve this symplectic structure

$$\mathbf{V}^\top \mathbb{J}_{2N} \mathbf{V} = \mathbb{J}_{2n} \text{ and take } \mathbf{V}^+ := \mathbb{J}_{2n}^\top \mathbf{V}^\top \mathbb{J}_{2N}$$

$\implies$ The lower dim. Hamiltonian, $\widetilde{H}(z) = H(Z \to \mathbf{V}z)$.

Minimize

For a given $n < N$,

$$\|\mathbb{X}_s - \mathbf{V}\mathbf{V}^+\mathbb{X}_s\|_F$$

over matrices in $\mathbb{R}^{2N\times 2n}$

$\downarrow$

$$\|\mathbb{X}_s - \mathbf{V}\mathbf{V}^+\mathbb{X}_s\|_F$$

over symplectic matrices in $\mathbb{R}^{2N\times 2n}$

Symplectic Model Order reduction (SMOR)

(Peng, Mohseni 2014, Hesthaven, Pagliantini, Ripamonti 2021)

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3. Specific to Scalar field theory, we impose orthogonality $\mathbf{V}^\top \mathbf{V} = \mathbb{I}_{2n} \implies \mathbf{V}^+ = \mathbf{V}^\top$.

→ $\widetilde{H}(z) = H(\mathbf{V}z)$ becomes independent of $\mathbf{V}$.

That is, the lower dim. model becomes independent of the reconstruction map, only the reconstruction of states depends on $\mathbf{V}$.

Minimize

For a given $n < N$,

$$\|\mathbb{X}_s - \mathbf{V}\mathbf{V}^+\mathbb{X}_s\|_F$$

over matrices in $\mathbb{R}^{2N\times 2n}$

↓

$$\|\mathbb{X}_s - \mathbf{V}\mathbf{V}^+\mathbb{X}_s\|_F$$

over symplectic matrices in $\mathbb{R}^{2N\times 2n}$

↓

$$\boxed{\|\mathbb{X}_s - \mathbf{V}\mathbf{V}^+\mathbb{X}_s\|_F}$$

over ortho-symplectic matrices in $\mathbb{R}^{2N\times 2n}$

Optimal embedding

The minimizer of the loss function can be analytically obtained (Buchfink, Bhatt, Haasdonk 2019).

Do the SVD of the complex snapshot matrix

$$\mathbb{X}_s = \begin{pmatrix} \boxed{\vec{\mathbf{Q}}(t_1) \quad \cdots \quad \vec{\mathbf{Q}}(t_M)} \\ \boxed{\vec{\mathbf{P}}(t_1) \quad \cdots \quad \vec{\mathbf{P}}(t_M)} \end{pmatrix}_{2N\times M} \longrightarrow \mathbb{X}_c = \begin{pmatrix} \mathbf{Q}(t_1) & \cdots & \mathbf{Q}(t_M) \\ \vdots & \ddots & \vdots \\ \mathbf{Q}(t_1) & \cdots & \mathbf{Q}(t_M) \end{pmatrix}_{N\times M} + i \begin{pmatrix} \mathbf{P}(t_1) & \cdots & \mathbf{P}(t_M) \\ \vdots & \ddots & \vdots \\ \mathbf{P}(t_1) & \cdots & \mathbf{P}(t_M) \end{pmatrix}_{N\times M} = \mathbb{U}\Sigma\mathbb{W}^*$$

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For a given $n < N$, take the first n left singular vectors $\mathbb{U}_n$ and let $\mathbf{\Phi} = \mathrm{Re}(\mathbb{U}_\mathrm{n})$, $\mathbf{\Psi} = \mathrm{Im}(\mathbb{U}_\mathrm{n})$, then

$$\mathbf{V}_\mathrm{opt} = \begin{pmatrix} \mathbf{\Phi} & -\mathbf{\Psi} \\ \mathbf{\Psi} & \mathbf{\Phi} \end{pmatrix}_{2N\times 2n} \xrightarrow{\text{with reconstruction error}} \|\mathbb{X}_s - \mathbf{V}_\mathrm{opt}\mathbf{V}^+_\mathrm{opt}\mathbb{X}_s\|_F = \sqrt{\sum_{j=n+1}^{\mathrm{rank}(\mathbb{X}_\mathrm{c})} \sigma_j^2}$$

$\Longrightarrow$ If we set $n = \mathrm{rank}(\mathbb{X}_\mathrm{c})$ then the reconstruction is exact ($\forall\, t$ if the Hamiltonian is quadratic).

*It'd be best if $\mathrm{rank}(\mathbb{X}_\mathrm{c}) \ll \mathrm{N}$, but generally $\mathrm{rank}(\mathbb{X}_\mathrm{c}) \sim \mathrm{N}$.

Free classical scalar field theory

Consider a scalar field in a box.

$$L = \int_{\mathscr{B}} d^3x \, \frac{1}{2}\left[(\partial_t \phi)^2 - (\nabla \phi)^2 - m_0^2 \phi^2\right] \longrightarrow H = \frac{1}{2} \sum_{|\mathbf{k}| \leq \Lambda_{\mathrm{UV}}} \left(P_{\mathbf{k}}^2 + \Omega_{\mathbf{k}}^2 Q_{\mathbf{k}}^2\right) \qquad \Omega_{\mathbf{k}} = \sqrt{|\,\mathbf{k}\,|^2 + m_0^2}$$

Box: $[-L_{\mathrm{IR}}/2,\, L_{\mathrm{IR}}/2]^3$

$$N(L_{\mathrm{IR}}, \Lambda_{\mathrm{UV}}) \approx \frac{(L_{\mathrm{IR}} \Lambda_{\mathrm{UV}})^3}{6\pi^2}$$

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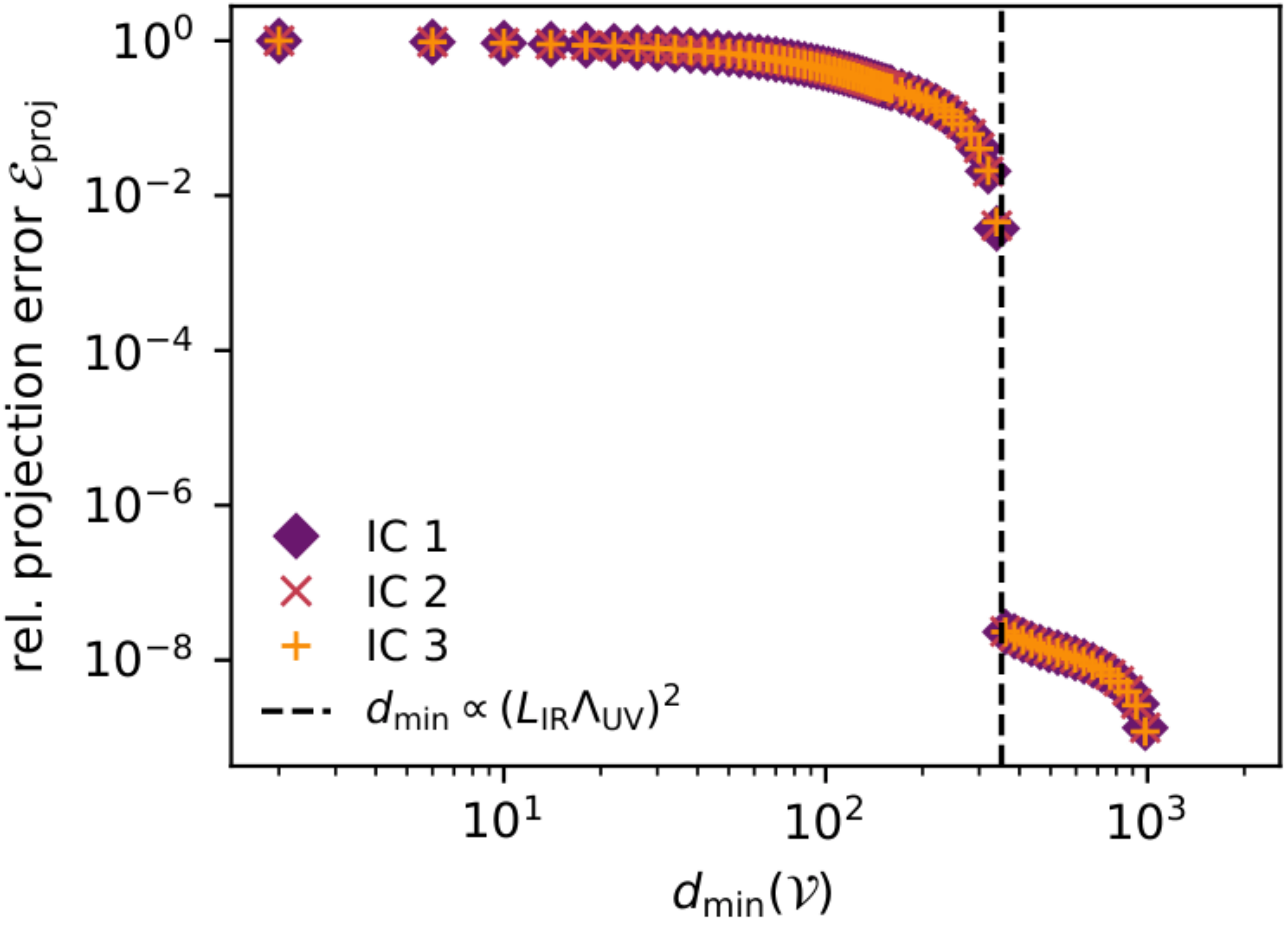
This is a simple theory in terms of dynamics. **Solutions:** $Q_{\mathbf{k}}(t) = A_{\mathbf{k}} \cos(\Omega_{\mathbf{k}}\, t) + B_{\mathbf{k}} \sin(\Omega_{\mathbf{k}}\, t)$

$$P_{\mathbf{k}}(t) = C_{\mathbf{k}} \cos(\Omega_{\mathbf{k}}\, t) + D_{\mathbf{k}} \sin(\Omega_{\mathbf{k}}\, t)$$

Construct $\mathbb{X}_c = \begin{pmatrix} \mathbf{Q}(t_1) & \cdots & \mathbf{Q}(t_M) \\ \vdots & \ddots & \vdots \\ \mathbf{Q}(t_1) & \cdots & \mathbf{Q}(t_M) \end{pmatrix}_{N \times M} + i \begin{pmatrix} \mathbf{P}(t_1) & \cdots & \mathbf{P}(t_M) \\ \vdots & \ddots & \vdots \\ \mathbf{P}(t_1) & \cdots & \mathbf{P}(t_M) \end{pmatrix}_{N \times M}$ and numerically compute the error $\dfrac{\|\mathbb{X}_s - \mathbf{V}\mathbf{V}^+ \mathbb{X}_s\|_F}{\|\mathbb{X}_s\|_F}$.

Free classical scalar field theory



On y-axis: $\dfrac{\|\mathbb{X}_s - \mathbf{V}\mathbf{V}^{+}\mathbb{X}_s\|_F}{\|\mathbb{X}_s\|_F}$, for projection onto lower dimensional symplectic subspace.

The cliff scales with the **area** of the box.
No matter what the initial state.

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Consider a scalar field in a box.

$$L = \int_{\mathscr{B}} d^3x \, \frac{1}{2}\left[(\partial_t \phi)^2 - (\nabla \phi)^2 - m_0^2 \phi^2\right] \longrightarrow H = \frac{1}{2} \sum_{|\mathbf{k}| \le \Lambda_{\mathrm{UV}}} \left(P_{\mathbf{k}}^2 + \Omega_{\mathbf{k}}^2 Q_{\mathbf{k}}^2\right) \qquad \Omega_{\mathbf{k}} = \sqrt{|\mathbf{k}|^2 + m_0^2}$$

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$\Longrightarrow$ Columns of $\mathbb{X}_c = \begin{pmatrix} \mathbf{Q}(t_1) & \cdots & \mathbf{Q}(t_M) \\ \vdots & \ddots & \vdots \\ \mathbf{Q}(t_1) & \cdots & \mathbf{Q}(t_M) \end{pmatrix}_{N\times M} + i \begin{pmatrix} \mathbf{P}(t_1) & \cdots & \mathbf{P}(t_M) \\ \vdots & \ddots & \vdots \\ \mathbf{P}(t_1) & \cdots & \mathbf{P}(t_M) \end{pmatrix}_{N\times M}$ are spanned by sine and cosine functions.

$\Longrightarrow$ Rank($\mathbb{X}_c$) $\le 2n_\Omega$, i.e., 2 x (# of unique frequencies realized on the lattice).

Counting the # of unique frequencies

Consider a scalar field in a box.

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$$N(L_{\mathrm{IR}}, \Lambda_{\mathrm{UV}}) \approx \frac{(L_{\mathrm{IR}} \Lambda_{\mathrm{UV}})^3}{6\pi^2}$$

For the flat box case:

$$n_{\Omega} = \#\text{ of unique } \Omega_{\mathbf{k}} = \sqrt{|\mathbf{k}|^2 + m_0^2} \longrightarrow \#\text{ of unique solutions of } k_1^2 + k_2^2 + k_3^2 \leq \Lambda_{\mathrm{UV}}^2 \text{ , where } k_i = \frac{2\pi n_i}{L_{\mathrm{IR}}} \, .$$

Using Legendre's three square problem one can obtain:

$$\#\text{ of dofs required} = \mathrm{Rank}(\mathbb{X}_c) = 2 n_{\Omega} \approx \mathscr{O}(1) \frac{|\partial \mathscr{B}|}{l_{\mathrm{UV}}^2} + \mathscr{O}(\log(L_{\mathrm{IR}} \Lambda_{\mathrm{UV}})) \, .$$

Counting the # of unique frequencies

For maximally symmetric spacetimes with spherical boundary:

$$\gamma_{ij}dx^i\, dx^j = \; d\rho^2 + S_K^2(\rho)\big(\, d\theta^2 + \sin^2\theta \;\, d\varphi^2\big) \qquad S_K(\rho) = \begin{cases} R_c \sin(\rho/R_c), & K > 0, \\ \rho, & K = 0, \\ R_c \sinh(\rho/R_c), & K < 0. \end{cases}$$

$$L = \frac{1}{2}\int_{\mathscr{B}} d^3x\, \sqrt{\gamma}\left[(\partial_t\phi)^2 - \gamma^{ij}\partial_i\phi\partial_j\phi - m_0^2\phi^2\right] \longrightarrow H = \frac{1}{2}\sum_{|k_{nl}|\leq\Lambda_{\mathrm{UV}}}\left(P_{nlm}^2 + \Omega_{nl}^2 Q_{nlm}^2\right)$$

$n_\Omega =$ # of unique $\Omega_{nlm} = \sqrt{k_{nl}^2 + m_0^2} \longrightarrow$ # of unique eigenvalues of $-\Delta_\gamma \leq \Lambda_{\mathrm{UV}}$ $\quad (-\Delta_\gamma)f = -\dfrac{1}{\sqrt{\gamma}}\partial_i(\sqrt{\gamma}\gamma^{ij}\partial_j f)$

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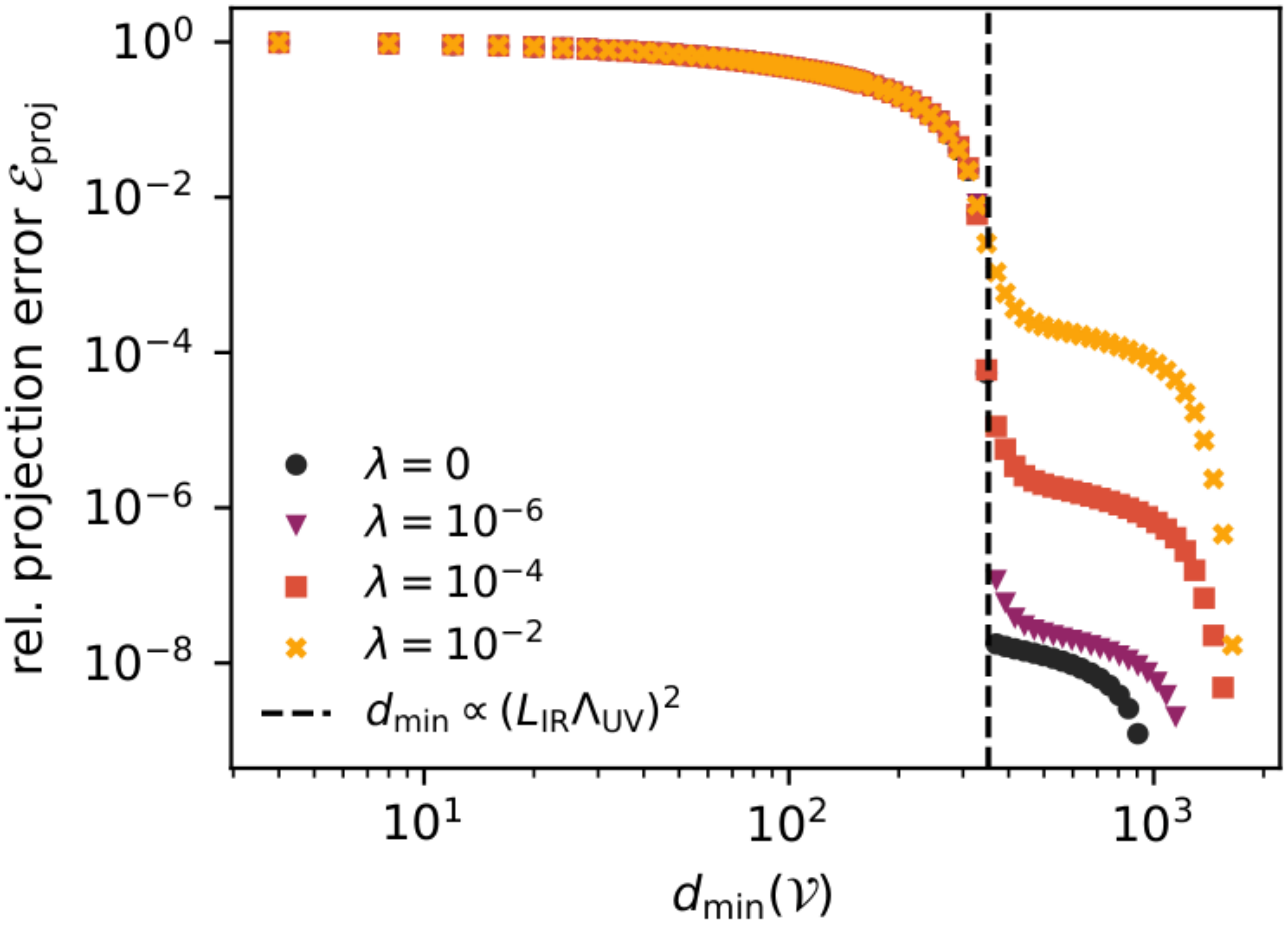
Using WKB approximation methods one can obtain:

$$\#\text{ of dofs required} = \mathrm{Rank}(\mathbb{X}_c) = 2n_\Omega \approx \mathcal{O}(1)\frac{\rho_{\rm IR}^2}{l_{\rm UV}^2} \qquad\Bigg|\qquad \mathcal{O}(1)\frac{4\pi R_c^2}{l_{\rm UV}^2}\sinh^2\!\left(\frac{\rho_{\rm IR}}{2R_c}\right) \qquad\Bigg|\qquad \mathcal{O}(1)\frac{4\pi R_c^2}{l_{\rm UV}^2}\sin^2\!\left(\frac{\rho_{\rm IR}}{2R_c}\right)$$

Flat $(K=0)$ Open $(K<0)$ Closed $(K>0)$

Interacting case



Projection error with $\lambda\phi^4$ interactions.

The cliff remains anchored to the free field case. The initial state was chosen such that the interactions don't become dominant over the evolution duration.

$$\varepsilon(t) := \frac{\lambda \int_{\mathscr{B}} \phi^4 \, d^3x}{\sum_\alpha \left(|P_\alpha(t)|^2 + \Omega_\alpha^2 |Q_\alpha(t)|^2 \right)}$$

Evolution duration: $\max\limits_{0 \leq t \leq T_{\rm obs}} \varepsilon(t) \ll 1.$

Classical overlaps

Dynamics also fixes the elements of the embedding map (depends on the initial state) and naturally gives a shell overlap structure.

$$\begin{pmatrix} Q_1(t) \\ \vdots \\ Q_N(t) \\ P_1(t) \\ \vdots \\ P_N(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Phi}_{N\times 2n_\Omega} & -\boldsymbol{\Psi}_{N\times 2n_\Omega} \\ \boldsymbol{\Psi}_{N\times 2n_\Omega} & \boldsymbol{\Phi}_{N\times 2n_\Omega} \end{pmatrix} \cdot \begin{pmatrix} q_1(t) \\ \vdots \\ q_{2n_\Omega}(t) \\ p_1(t) \\ \vdots \\ p_{2n_\Omega}(t) \end{pmatrix} \longrightarrow \begin{pmatrix} A_1(t) \\ \vdots \\ A_N(t) \\ A_1^\dagger(t) \\ \vdots \\ A_N^\dagger(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Phi} + i\boldsymbol{\Psi} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Phi} - i\boldsymbol{\Psi} \end{pmatrix} \cdot \begin{pmatrix} a_1(t) \\ \vdots \\ a_{2n_\Omega}(t) \\ a_1^\dagger(t) \\ \vdots \\ a_{2n_\Omega}^\dagger(t) \end{pmatrix}$$

$$\longrightarrow \{A_{\mathbf{k}}, A^\dagger_{\mathbf{k}'}\} = \boldsymbol{\Upsilon}\boldsymbol{\Upsilon}^\dagger_{\mathbf{k}\mathbf{k}'}\delta_{|\mathbf{k}|,|\mathbf{k}'|}\,, \qquad \{A_{\mathbf{k}}, A_{\mathbf{k}'}\} = \{A^\dagger_{\mathbf{k}}, A^\dagger_{\mathbf{k}'}\} = 0 \quad \text{where, } \Upsilon = \boldsymbol{\Phi} + i\boldsymbol{\Psi}$$

Summary

- Broadly we are asking a modest question:
  How accurately can the std. QFT (esp. for the relevant purposes) be already spoofed with less dofs than we conventionally take?

- Asked a resource-accuracy question for Hamiltonian field theory:
  how many canonical degrees of freedom are needed to reproduce the field evolution?

- Used **SMOR** so that the reduction is not just data fitting:
  the low dim. model remains an autonomous Hamiltonian system.

- For a free scalar field, the answer is controlled by the **spectrum of the Hamiltonian.**

- In flat space this gives an **area-type threshold**; curved maximally symmetric backgrounds give systematic deviations.

- Reconstructing the full field variables from the low dim. phase space induces **classical overlap** among apparent modes.

Outlook

- Clarify state dependence: different trajectories may require different reconstruction maps. Can we do better?
- Develop the quantum version: quantize the reduced symplectic sector and study the induced operator overlap.
- Compare SMOR with Krylov-type complexity as two ways of extracting dynamically relevant subspaces.

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Thank you!