What triangulations at criticality in TGFT?

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FAU^2 workshop on quantum gravity across scales:
from physics at the Planck scale to effective theories

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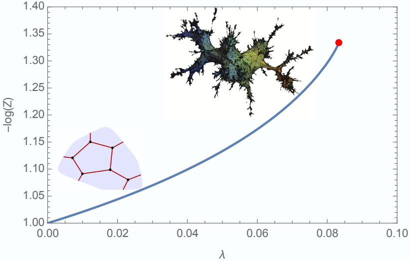
Tensorial Group Field Theory (TGFT)

TGFT: Lattice Quantum Gravity generated by auxiliary fields Φ:

$$Z^{\mathrm{QG}} = \int \mathcal{D}g\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_{\mathrm{GR}}[g]} \xrightarrow[\mathcal{M}\to\mathcal{C}]{\text{discretize}} \sum_{\mathcal{C}} Z^{\mathrm{QG}}_{\mathcal{C}} \xrightarrow[|\mathcal{C}|\to\infty]{\text{continuum limit}} \quad ??$$

$$\updownarrow$$

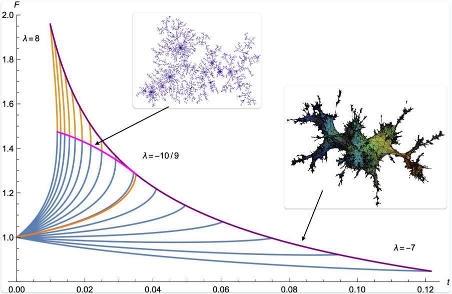
$$\int \mathcal{D}\Phi\, \mathrm{e}^{-S_{\mathrm{TGFT}}[\Phi]} = \sum_{\Gamma} \lambda^{n_\Gamma} A_\Gamma \xrightarrow[\lambda\to\lambda^\star]{\text{criticality}} \text{continuum GR}$$



- generalization of matrix-model description of $2D$ Liouville QG to $D > 2$
- $\Phi(g_i)$ excites geometric d.o.f. g_c on $\Gamma \cong \mathcal{C}$
- $\to$ ensembles of lattices $\mathcal{C}$ with geometry g_c
- various lattice formulations $Z^{\mathrm{QG}}_{\mathcal{C}}$ possible: *spin-foams, Regge, dynamical triangulations,...*

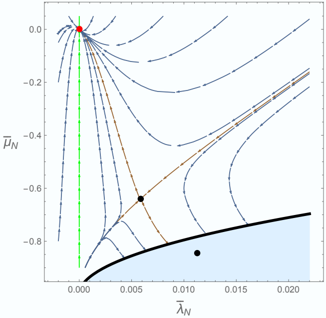
More couplings - which continuum limit/critical point to choose??

Matrix/Tensor models



[Lionni/JT '1707]

Tensorial group field theory

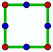


[Ben Geloun, Martini, Oriti '1601]

Field theory: Non-perturbative, critical regime via RG flow for given initial condition $\Gamma_{k=\Lambda}$

TGFT phase space is very large...

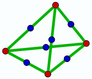
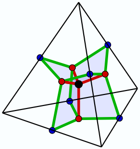
Matrix diagrams Γ are gluings of polygons $\rightarrow$ discrete surfaces

$$\mathrm{Tr}(M^4) = M_{ij}\delta_{jk}M_{kl}\delta_{lm}M_{mn}\delta_{no}M_{op}\delta_{pi} \cong$$ 

Diagramms Γ of rank-D tensors $\Phi_{a_1 \ldots a_D}$ are gluings of D-dim. polyhedra

$$\mathrm{Tr}_{\gamma}(\Phi) = \Phi_{ijk}\Phi_{lmn}\Phi_{opq}\Phi_{rst}\delta_{il}\delta_{jo}\delta_{kr}\delta_{mp}\delta_{ns}\delta_{qt} \cong$$

(subscript γ drawn as: )

 $\cong$ 

Tensor "traces" Tr_γ are bijective to graphs γ

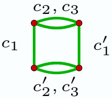
- Phase space spanned by couplings λ_γ for any graphs $\gamma =$ , , , , ...
- D-valent graphs for single $\Phi(g_1, ..., g_D)$ (but more can be generated [Oriti, Ryan, JT '1409])
- compatibility with full kinematic Hilbert space of LQG [Kaminski, Kisielowski, Lewandowski '0909]

- Phase-space point $\vec{\lambda} = (\lambda_\gamma)$ describes theory/measure $\mathrm{e}^{-\sum_\gamma \lambda_\gamma \mathrm{Tr}_\gamma(\Phi)}$
- favours/disfavours triangulations with some building blocks γ over others

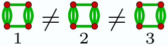
What ensembles of triangulations? $=$ What theories at critical points?

- there are stable regimes (under the RG flow) of specific interaction types
- large-N techniques for tensorial $(\mathrm{U}(N)/\mathrm{O}(N)$ invariant) interactions

$$\mathrm{Tr}_{\square}(T) = \Phi_{c_1 c_2 c_3} \Phi_{c_1' c_2 c_3} \Phi_{c_1 c_2' c_3'} \Phi_{c_1' c_2' c_3'} =$$



- even more phase space directions: "coloured" graphs



Today: Focus on the RG dynamics of colours

first results with L. Juliano in a simple stable regime ("cyclic melonic")
[Phys. Lett. B 860 (2025) 139218]

Critical points via functional Renormalization Group (fRG)

Theory at scale k given by generating function with k-dep. IR-regulator $\mathcal{R}_k$

$$e^{W_k[J]} = \int D\Phi \, e^{-S[\Phi]-(\Phi,\mathcal{R}_k\Phi)+(J,\Phi)}$$

Scale-dependent effective action via Legendre transform w.r.t. $\varphi = \frac{\delta W_k[J]}{\delta J}$

$$\Gamma_k[\varphi] \equiv \sup_J\{(J,\varphi) - W_k[J]\} - (\varphi, \mathcal{R}_k\varphi)$$

RG flow determined by functional equation [Wetterich'93, Morris'94]

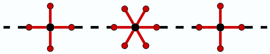
$$\boxed{k\partial_k\Gamma_k[\varphi] = \frac{1}{2}\mathrm{Tr}\frac{k\partial_k\mathcal{R}_k}{\Gamma_k^{(2)}[\varphi]+\mathcal{R}_k}}$$

Interpolates between microsc. theory $k = \Lambda^{\mathrm{UV}}$ and quantum effective action $\Gamma = \lim_{k\to 0}\Gamma_k$

Some useful generality: PF expansion yields Bell polynomials

"PF expansion": split Gaussian part $\mathcal{K}$ in Γ_k such that with $P^{-1} = \mathcal{K} + \mathcal{R}_k$

$$\frac{1}{\Gamma_k^{(2)}[\varphi] + \mathcal{R}_k} = \frac{1}{P^{-1} + F[\varphi]} = \sum_{l \geq 0} (-PF[\varphi])^l P = \ldots +$$



$+ \ldots$

- multinomial expansion for commutative series $F[\varphi]$

$$F[\varphi]^l = (a_1\varphi^2 + a_2\varphi^4 + \ldots)^l = \sum_{s_1+s_2+\ldots=l} \binom{l}{s_1, s_2, \ldots} a_1^{s_1} a_2^{s_2} \cdots \varphi^{2\sum_{j\geq 1} j s_j}$$

- $\rightarrow$ at order φ^{2n}, sum over partitions $\sigma \vdash n$ of length $|\sigma| = l$

$$[\varphi^{2n}] \frac{1}{P^{-1} + F[\varphi]} = P \sum_{l \geq 1} (-P)^l \sum_{\substack{\sigma \vdash n \\ |\sigma| = l}} \binom{l}{s_1, s_2, \ldots} a_1^{s_1} a_2^{s_2} \cdots$$

- ordinary Bell polynomials (up to factor $l!/n!$)

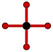
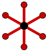
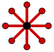
$$\widehat{B}_{n,l}(a_1, a_2, \ldots, a_{n-l+1}) = \sum_{\substack{\sigma \vdash n \\ |\sigma| = l}} \binom{n}{s_1, s_2, \ldots, s_{n-l+1}} \prod_{j=1}^{n-l+1} (a_j)^{s_j}$$

Flow equations

General PF expansion for series $F[\varphi] = \sum_{j\geq 1} a_j \varphi^{2j}$:

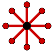
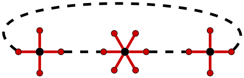
$$k\partial_k \Gamma_k[\varphi] = \frac{1}{2}\mathrm{Tr}\left[k\partial_k \mathcal{R}_k \sum_{n\geq 1} \frac{\varphi^{2n}}{n!} P \sum_{l\geq 1} l!(-P)^l \widehat{B}_{n,l}\left(a_1, a_2, ..., a_{n-l+1}\right)\right]$$

Example: scalar field theory

$$\Gamma_k[\varphi] = \int \varphi\mathcal{K}\varphi + \sum_{n\geq 2} \frac{\lambda_{2n}}{2n}\varphi^{2n} = \int \varphi\mathcal{K}\varphi + \frac{\lambda_4}{4}$$  $$+ \frac{\lambda_6}{6}$$  $$+ \frac{\lambda_8}{8}$$  $$+ ...$$

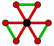
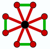
Then $a_j = (2j+1)\lambda_{2j+2}$ (since $\Gamma_k^{(2)} \equiv \mathcal{K} + F[\varphi] = \mathcal{K} + \sum_{n\geq 2}(2n-1)\lambda_{2n}\varphi^{2n-2}$):

$$\boxed{k\partial_k \lambda_{2n} = P \sum_{l\geq 1} l!(-P)^l \widehat{B}_{n,l}(3\lambda_4, 5\lambda_6, ...) \equiv \beta_{2n}(3\lambda_4, 5\lambda_6, ...)}$$

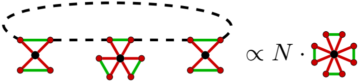
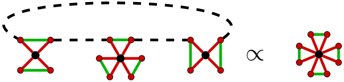
$k\partial_k$  $= ... +$  $+ ...$

Vector theory

Vector-valued scalar field φ_a, $a = 1, ..., N$:

$$\Gamma_k[\varphi] = \int \varphi_a \mathcal{K} \varphi_a + \sum_{n\geq 2} \frac{\lambda_{2n}}{2n} (\varphi_a \varphi_a)^n = \int \varphi_a \mathcal{K} \varphi_a + \frac{\lambda_4}{4}$$  $$+ \frac{\lambda_6}{6}$$  $$+ \frac{\lambda_8}{8}$$  $$+ \ldots$$

$\Gamma_k^{(2)}$ has two parts: $F_{ab}(\varphi) = \sum_{j\geq 1} \lambda_{2j+2} \left(2j \varphi_a \varphi_b + \varphi^2 \delta_{ab}\right) (\varphi^2)^{j-1}$

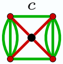
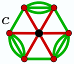
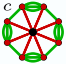
- $\mathrm{Tr}(\delta^l) = N$ for one term: $\mathrm{Tr}[(\delta_{ab}\varphi^2)^3 \varphi^2] \cong$ 
- all others are N^0: $\mathrm{Tr}[(\delta_{ab}\varphi^2)^2 \varphi^2 (\varphi_b \varphi_a)] \cong$ 

$$k\partial_k \lambda_{2n} = \beta_{2n}(3\lambda_4, 5\lambda_6, \ldots) + (N-1)\beta_{2n}(\lambda_4, \lambda_6, \ldots)$$

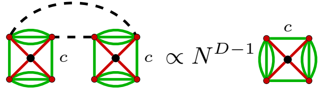
Universality classes for various N from balance of factors $2j+1$ vs. $N-1$

"Tensor theory ≈ (Vector theory)^D "

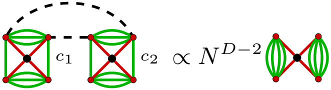
Tensor field $\varphi_{a_1 \dots a_D}$: "cyclic melons" dominate at large N

$$\Gamma_k[\varphi] = \int \varphi \mathcal{K} \varphi + \sum_{c=1}^{r} \Bigg(\frac{\lambda_4^c}{4}$$  $$+ \frac{\lambda_6^c}{6}$$  $$+ \frac{\lambda_8^c}{8}$$  $$+ \dots \Bigg)$$

- LO N^{D-1} only if all λ_{2j}^c of same colour c:



- different $c_1 \neq c_2$ generate other stuff, but NLO:



→ At large N, same combinatorics as large-N vector theory for each colour c:

$$k\partial_k \lambda_{2n}^{c} \underset{N\to\infty}{=} \beta_{2n}(\lambda_4^{c}, \lambda_6^{c}, \dots)$$

Anomalous dimension in TGFT

Propagating tensor d.o.f. $a_c = j_c/V$ (yields tensor scale $N_k = V \cdot k$):

Field renormalization Z_c for each colour *necessary*: $\mathcal{K} = \sum_{c=1}^{r} Z_{c,k}(j_c/V)^2 + \mu_k$

$$\to \text{(optimized) regulator: } \mathcal{R}_k(\boldsymbol{j}) = \left(\bar{Z}_k k^2 - k^2 \sum_{c=1}^{r} Z_{c,k}(j_c/N_k)^2\right) \theta\left(N_k^2 - \sum_{c=1}^{D} j_c^2\right)$$

Each colour c contributes a part $\eta_c = -\frac{k\partial_k Z_c}{\bar{Z}}$ to the anomalous dimension

$$\bar{\eta} = -\frac{k\partial_k \bar{Z}}{\bar{Z}} = \frac{1}{D}\sum_{c=1}^{r} \eta_c \,, \qquad \bar{Z} = \frac{1}{D}\sum_{c=1}^{D} Z_c$$

because

$$k\partial_k \mathcal{R}_k(\boldsymbol{j}) = 2\bar{Z}k^2 \left(1 - \frac{\bar{\eta}}{2} + \sum_c \frac{\eta_c}{2} \frac{j_c^2}{N_k^2}\right) \theta\left(N_k^2 - \sum_{c=1}^{D} j_c^2\right)$$

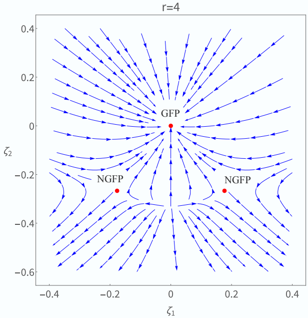
Coupling of fRG equations at quadratic order

Flow equations at φ^2 couple the colour sectors: (with $F(\eta_c) = 1 - \frac{1}{2}\frac{\bar\eta+\eta_c}{d_D+2}$)

$$k\partial_k\tilde\mu_k - (\bar\eta - 2)\tilde\mu_k = \sum_{c=1}^{D} F(\eta_c)\beta_2(\tilde\mu_k, \tilde\lambda^c_{2i})$$

$$k\partial_k\tilde\lambda^c_{2n} + d_D\tilde\lambda^c_{2n} - n(d_D + \bar\eta - 2)\tilde\lambda^c_{2n} = F(\eta_c)\beta_{2n}(\tilde\mu_k, \tilde\lambda^c_{2i})$$

- **isotropic** case $\tilde\lambda^c_{2n} = \tilde\lambda_{2n}$ agrees with previous results, same isotropic fixed points
- **"s-regimes"**: only $1 \le s \le D$ couplings $\tilde\lambda^c_{2n} \neq 0$
- fixed points with multiplicity $\binom{D}{s}$



Isotropic flow equation in an s-regime: (with $F_s(\bar\eta) = 1 - \frac{\frac{D}{s}+1}{d_D+2}\,\frac{\bar\eta}{2}$)

$$k\partial_k\tilde\mu_k - (\bar\eta - 2)\,\tilde\mu_k = s\,F_s(\bar\eta)\,\beta_2(\tilde\mu_k, \tilde\lambda_{2i})$$

$$k\partial_k\tilde\lambda_{2n} + d_D\tilde\lambda_{2n} - n\,(d_D + \bar\eta - 2)\,\tilde\lambda_{2n} = F_s(\bar\eta)\,\beta_{2n}(\tilde\mu_k, \tilde\lambda_{2i})$$

- dimension $d_D = d_G(D-1)$ and scaling not changed, but factors D/s
- critical exponents θ: Gaussian directions θ^{G} and "angular/radial" $\theta^{\mathrm{A}}, \theta^{\mathrm{R}}$
- $\rightarrow$ only one Wilson-Fisher type fixed point for $s = D$ (single relevant direction)
- $s < D$ fixed points: candidates for asymptotic safety (finite # rel. directions)

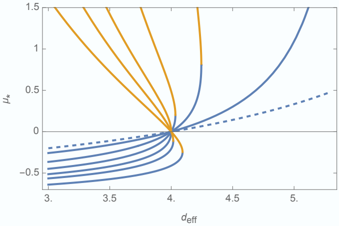
$d_D = 3$ with $\bar\eta = 0$ at order φ^{24}:

s	$\tilde\mu_k^*$	$10\tilde\lambda_{2\cdot2}^*$	$10^2\tilde\lambda_{2\cdot3}^*$	θ_1	θ_2^{G}	θ_2^{A}	θ_2^{R}	θ_3^{G}	θ_3^{A}	θ_3^{R}
1	-0.38835	2.9058	16.738	1.0000	1	/	-1.0000	0	/	-3.0000
2	-0.55944	1.0858	3.2449	0.64994	1	-0.38149	-1.5265	0	-2.6136	-3.5635
3	-0.65573	0.51811	0.94544	0.51716	1	-0.38149	-1.7654	0	-2.6136	-3.9317
4	-0.71749	0.28633	0.35186	0.45217	/	-0.38149	-1.8761	/	-2.6136	-4.1011

Fixed points in critical dimension

Anomalous dimension yields new fixed points:

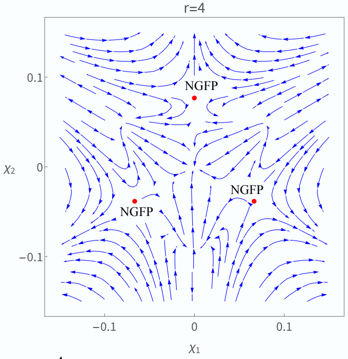
- several branches connected to $\tilde{\mu}_k = 0$ at $d_D = 4$
- "Wilson-Fisher" fixed point also for $d_D \geq 4$



$d_D = 4$ with flowing $\bar{\eta}$ at order φ^{20}:

s	$\tilde{\mu}_k^*$	$\bar{\eta}^*$	$10\,\tilde{\lambda}^*_{2\cdot 2}$	$10^2\,\tilde{\lambda}^*_{2\cdot 3}$	θ_1	θ_2^{G}	θ_2^{A}	θ_2^{R}	θ_3^{G}	θ_3^{A}	θ_3^{R}
1	-0.38160	-0.13109	2.9186	16.530	0.78152	0.26217	/	-2.4369	-1.6067	/	-3.4757
1	-0.47879	-0.23236	2.6014	15.580	0.79536	0.46472	/	-2.4250	-1.3029	/	-3.0706
1	-0.55347	-0.39334	2.2071	13.091	1.0362	0.78669	/	-1.9567	-0.81997	/	-9.3463
2	-0.28277	-0.11869	1.5374	2.2712	1.0565	0.23739	-0.74750	-3.3581	-1.6439	-1.2248	-3.5003
2	-0.62457	-0.45014	1.0357	3.2407	-0.32774	0.90029	-2.2771	-5.4425	-0.64957	-3.8345	-7.3678
3	-0.72775	-0.69033	0.45113	0.88234	1.4936	0.54809	1.3807	-1.6337	-1.2387	0.071000	-4.9059
3	-0.87521	-1.0897	0.12352	0.19475	2.0970	2.1794	1.5334	-1.9772	1.2691	0.35881	-14.163
3	-0.96742	-1.5470	0.0099619	0.0057458	-2.3200	3.0941	-0.52954	-37.022	2.6411	-6.0245	-106.83
4	-0.95561	-1.3584	0.013206	0.0071868	-0.61347	2.7168	-2.4397	-15.942	2.0752	-22.734	-49.439
5	-0.94714	-1.4194	0.014634	0.0071649	0.54992	/	-0.56823	-8.1386	/	-12.171	-28.966

- any kind of anisotropic fixed points possible
- at φ^4 truncation only (s_1, s_2) type fixed points (s_1 couplings $\tilde\lambda^c_{2n} = \tilde\lambda^1_{2n}$, s_2 couplings $\tilde\lambda^c_{2n} = \tilde\lambda^2_{2n}$)
- at higher orders, any partition types $(s_1, s_2, ...)$
- but calculation challenging, convergence?



some (s_1, s_2) type fixed points at φ^4:

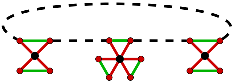
(s_1, s_2)	$\tilde\mu^*_k$	η^{1*}	η^{2*}	$\tilde\lambda^{1*}_{2\cdot 2}$	$\tilde\lambda^{2*}_{2\cdot 2}$	θ_1	θ_2^{G}	θ_2^{A}	θ_2^{R}
(4, 1)	-0.87423	5.3773	7.5628	0.33816	0.063274	87.055	/	14.251	-6.6718
(3, 2)	-0.88956	2.4321	9.5360	-0.019852	0.030340	64.833	/	10.547	-4.8224
(3, 2)	-0.87656	4.4527	8.2973	-0.19367	0.049911	82.660	/	13.519	-6.3059
(2, 2)	-0.86633	2.6650	10.396	-0.036422	0.041362	53.711	-10.449	10.449	-4.6924
(3, 1)	-0.85024	4.6088	9.7456	-0.43859	0.060429	69.156	-13.538	13.538	-6.2342
(2, 1)	-0.82707	3.1416	11.844	-0.095352	0.063919	44.279	-10.732	10.732	-4.7174

Conclusions

- TGFT provides a completion of regularized QG, continuum limit at criticality
- Criticality/Continuum geometry is approached dynamically via RG flow
- Already the simplest (melonic) regime provides a plethora of fixed points

Outlook

- Big open question: characterize continuum geometry at fixed points!
- Intermediate challenge: step by step mapping of the huge phase space
- current puzzle: role of cyclic interactions - an artifact of fRG?



Thanks for your attention!