Dynamics of semiclassical states in the single-vertex model of quantum-reduced loop gravity

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Plan of the talk

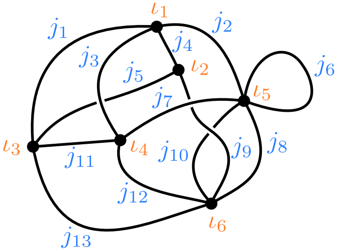
1. Quick review of quantum-reduced loop gravity
2. A Hamiltonian operator for quantum-reduced loop gravity
3. Effective dynamics of a homogeneous and isotropic universe
4. Quantum dynamics of semiclassical states

A brief reminder of loop quantum gravity

An orthonormal basis on the kinematical Hilbert space:

$$\Psi_\Gamma(h_{e_1},\ldots,h_{e_N}) = \prod_{e\in\Gamma} \mathcal{D}^{(j_e)}_{m_e n_e}(h_e)$$

$$\mathcal{D}^{(j)}_{mn}(h) := \sqrt{2j+1}D^{(j)}_{mn}(h)$$



Gauss constraint $\rightarrow SU(2)$ intertwiners at nodes

Elementary operators: Holonomy and flux

$$\widehat{D^{(s)}_{m'n'}(h_e)} D^{(j)}_{mn}(h_e) = \sum_{km''n''} C^{(j\ s\ k)}_{mm'm''} C^{(j\ s\ k)}_{nn'n''} D^{(k)}_{m''n''}(h_e)$$

$$\hat{J}^{(v,e)}_i D^{(j)}(h_e) = \begin{cases} iD^{(j)}(h_e)\tau^{(j)}_i & (e \text{ begins at } v) \\ -i\tau^{(j)}_i D^{(j)}(h_e) & (e \text{ ends at } v) \end{cases} \qquad \tau_i = -\frac{i}{2}\sigma_i$$

Geometric operators: Area, volume, ...

Quantum-reduced loop gravity

The Hilbert space of quantum-reduced loop gravity is spanned by the "reduced spin network states" [Alesci, Cianfrani 2013]

$$\prod_{e\in\Gamma_\square} \mathcal{D}^{(j_e)}_{j_e j_e}(h_e)_{i_e} \qquad \text{where} \quad \mathcal{D}^{(j)}_{mn}(h)_{i_e} := {}_{i_e}\langle jm|\mathcal{D}^{(j)}(h)|jn\rangle_{i_e}$$

They are characterized by the following conditions:

- The state is defined on a cubical graph $\Gamma_\square$
- The magnetic numbers take the maximal (or minimal) value with respect to the basis
$$\hat{J}^2|jm\rangle_i = j(j+1)|jm\rangle_i \qquad \hat{J}_i|jm\rangle_i = m|jm\rangle_i$$
where $i_e = x$, y or z is chosen according to the direction of the edge e
- $j_e \gg 1$ for every edge e

Quantum-reduced loop gravity = "LQG in diagonal gauge" ($E^a_i = 0$ for $a \neq i$)

$$M = \int d^3x \, \frac{\sum_{a\neq i}(E^a_i)^2}{\sqrt{q}} \qquad \hat{M}|\Psi_j\rangle = \mathcal{O}\left(\frac{1}{\sqrt{j}}\right)$$

Operators on the reduced Hilbert space

For a large class of LQG operators, the action of the operator on a reduced spin network state $|\Psi_0\rangle$ has the structure [I.M. 2020]

$$\hat{\mathcal{O}}|\Psi_0\rangle \;=\; f(j)|\Psi\rangle + g(j)|\Phi\rangle$$

where $|\Psi\rangle \in \mathcal{H}_{\text{reduced}}$, and for large j,

$$f(j) \gg g(j)$$

This suggests that operators of quantum-reduced loop gravity can be obtained from operators of full loop quantum gravity by dropping the small "offending" terms:

$${}^{R}\hat{\mathcal{O}}|\Psi_0\rangle := f(j)|\Psi\rangle$$

The reduced operator ${}^{R}\hat{\mathcal{O}}$ is:

- A well-defined operator on the reduced Hilbert space
- A good approximation of the action of the full operator $\hat{\mathcal{O}}$ on the state $|\Psi_0\rangle$:

$$\frac{\big|\big|\hat{\mathcal{O}}|\Psi_0\rangle - {}^{R}\hat{\mathcal{O}}|\Psi_0\rangle\big|\big|}{\big|\big|\hat{\mathcal{O}}|\Psi_0\rangle\big|\big|} \ll 1$$

- Typically very simple in comparison with the corresponding full operator

The kinematical structure of the quantum-reduced model

Loop quantum gravity

$$\left(\prod_{v\in\Gamma}\iota_v\right)\cdot\left(\prod_{e\in\Gamma}\mathcal{D}^{(j_e)}(h_e)\right)$$

– States defined on arbitrary graphs

– $SU(2)$ intertwiners at nodes

$$\widehat{D^{(s)}_{m'n'}(h_e)}D^{(j)}_{mn}(h_e)$$

$$=\sum_{km''n''}C^{(j\ s\ k)}_{mm'm''}C^{(j\ s\ k)}_{nn'n''}D^{(k)}_{m''n''}(h_e)$$

– SU(2) multiplication law

$$\hat{J}^{(v,e)}_i D^{(j)}(h_e)=\begin{cases} iD^{(j)}(h_e)\tau^{(j)}_i \\ -i\tau^{(j)}_i D^{(j)}(h_e)\end{cases}$$

Quantum-reduced loop gravity

$$\prod_{e\in\Gamma_\square}\mathcal{D}^{(j_e)}_{j_ej_e}(h_e)_{i_e}$$

– States defined on cubical graphs

– No (non-trivial) intertwiner structure

$$\widehat{{}^R D^{(s)}_{mn}(h_e)_{i_e}}\mathcal{D}^{(j)}_{jj}(h_e)_{i_e}$$

$$=\delta_{mn}\mathcal{D}^{(j+m)}_{j+m\ j+m}(h_e)_{i_e}$$

– U(1)-like multiplication law

$${}^R\hat{J}^{(v,e)}_i\mathcal{D}^{(j)}_{jj}(h_e)_{i_e}=\pm\delta_{i,i_e}j\mathcal{D}^{(j)}_{jj}(h_e)_{i_e}$$

Dynamics: Hamiltonian constraint

In the canonical formulation of LQG, the dynamics is governed by the Hamiltonian constraint operator. As the classical starting point, we take

$$C(N) = \frac{1}{\beta^2}\int d^3x\, N\biggl(\frac{\epsilon^{ij}{}_k E^a_i E^b_j F^k_{ab}}{\sqrt{|\det E|}} + (1+\beta^2)\sqrt{|\det E|}\,{}^{(3)}R\biggr)$$

The operator obtained upon quantization can be interpreted in two different ways:

– As a Hamiltonian constraint in the vacuum theory, where it determines the space of physical states through the condition

$$\hat{C}(N)|\Psi\rangle = 0$$

– As a physical Hamiltonian in a model of gravity coupled to a suitable reference matter field (irrotational dust) [Brown, Kuchař 1994; Giesel, Thiemann 2012; Husain, Pawłowski 2013]:

$$i\frac{d}{dT}|\Psi(T)\rangle = \hat{H}_{\rm phys}|\Psi(T)\rangle$$

where

$$\hat{H}_{\rm phys} = \hat{C}(1)$$

Hamiltonian: Euclidean part

To obtain a Hamiltonian operator for the quantum-reduced model, we start with an operator defined on a cubical graph in the framework of full LQG.

$$C_E(N) = \int d^3x\, N \frac{\epsilon^{ij}{}_k E^a_i E^b_j F^k_{ab}}{\sqrt{|\det E|}}$$

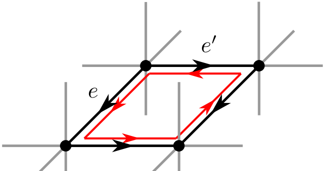
We take the Euclidean part of the Hamiltonian to be represented by the operator
[Alesci, Assanioussi, Lewandowski, I.M. 2015; Yang, Ma 2015]

$$\hat{C}_E(N)|\Psi_\Gamma\rangle = \sum_{v\in\Gamma} N(v)\hat{C}^v_E|\Psi_\Gamma\rangle$$

$$\hat{C}^v_E = \sum_{e \nparallel e' \text{ at } v} \epsilon^{ij}{}_k \operatorname{Tr}\big(\tau^{(s)}_k \widehat{D^{(s)}(h_{\alpha_{ee'}})}\big)\hat{J}^{(v,e)}_i \hat{J}^{(v,e')}_j \widehat{\mathcal{V}^{-1}_v}$$

$$\widehat{\mathcal{V}^{-1}_v} = \lim_{\epsilon\to 0} \frac{\hat{V}_v}{\hat{V}^2_v + \epsilon^2}$$

In the context of quantum-reduced loop gravity, we use a graph-preserving regularization for the loop $\alpha_{ee'}$. In the present work we also fix $s = 1/2$.



Hamiltonian: Lorentzian part

$$C_L(N) = \int d^3x\, N \sqrt{|\det E|}\, {}^{(3)}R$$

The integrated scalar curvature can be quantized as a well-defined operator on the Hilbert space of a fixed cubical graph in LQG [Lewandowski, I.M. 2022].

The Ricci scalar is first expressed as

$${}^{(3)}R = {}^{(3)}R\big(E^a_i, \mathcal{D}_a E^b_i, \mathcal{D}_a \mathcal{D}_b E^c_i\big)$$

where $\mathcal{D}_a E^b_i = \partial_a E^b_i + \epsilon_{ij}{}^k A^j_a E^b_k$. The covariant derivatives can be regularized on a cubical graph using parallel transported (gauge covariant) flux variables

$$\widetilde{E}(S, x_0) = \int_S d^2\sigma\, n_a(\sigma) h_{x(\sigma)\to x_0} E^a_i\big(x(\sigma)\big)\tau^i h^{-1}_{x(\sigma)\to x_0}$$

For example

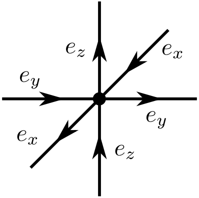
$$\Delta_a E^b(v) := \frac{\widetilde{E}\big(S^b(v^+_a), v\big) - \widetilde{E}\big(S^b(v^-_a), v\big)}{2\epsilon} = \epsilon^2 \mathcal{D}_a E^b_i(v)\tau^i + \mathcal{O}(\epsilon^3)$$

This results in an operator of the form

$$\hat{C}_L(N)\Big|_{\mathcal{H}_{\Gamma_\square}} = \sum_{v\in\Gamma_\square} N(v)\hat{\mathcal{R}}_v\Big(\hat{E}^a(v), \widehat{\Delta_a E^b}(v), \widehat{\Delta_{ab} E^c}(v)\Big)$$

The single-vertex model

Although the action of the Hamiltonian is explicitly computable on the entire reduced Hilbert space, we will now consider a very simple model, which is obtained by choosing a graph containing just a single six-valent node.



We assume that the spatial manifold has the topology of a three-torus (or has periodic boundary conditions) so the graph is formed by three closed mutually orthogonal edges.

The state space of the model is spanned by the orthonormal basis states

$$|j_x j_y j_z\rangle = \mathcal{D}^{(j_x)}_{j_x j_x}(h_{e_x})_x \mathcal{D}^{(j_y)}_{j_y j_y}(h_{e_y})_y \mathcal{D}^{(j_z)}_{j_z j_z}(h_{e_z})_z$$

Hamiltonian for the single-vertex model

To obtain the Hamiltonian constraint for the single-vertex model, we compute

$$\hat{C}(N)|j_x j_y j_z\rangle = {}^R\hat{C}(N)|j_x j_y j_z\rangle + \text{lower order}$$

The result is

$$\begin{aligned}{}^R\hat{C}_E(N)|j_x j_y j_z\rangle = -\frac{1}{\beta^2}N(v)\Bigg[&\sqrt{\frac{j_x j_y}{j_z}}\hat{s}^{(1)}(e_x)\hat{s}^{(1)}(e_y)\\&+\sqrt{\frac{j_x j_z}{j_y}}\hat{s}^{(1)}(e_x)\hat{s}^{(1)}(e_z)+\sqrt{\frac{j_y j_z}{j_x}}\hat{s}^{(1)}(e_y)\hat{s}^{(1)}(e_z)\Bigg]|j_x j_y j_z\rangle\end{aligned}$$

$$\begin{aligned}{}^R\hat{C}_L(N)|j_x j_y j_z\rangle = -16\frac{1+\beta^2}{\beta^2}N(v)\Bigg[&\frac{j_x^{3/2}}{\sqrt{j_y j_z}}\big[\hat{s}^{(1/2)}(e_x)\big]^4\\&+\frac{j_y^{3/2}}{\sqrt{j_x j_z}}\big[\hat{s}^{(1/2)}(e_y)\big]^4+\frac{j_z^{3/2}}{\sqrt{j_x j_y}}\big[\hat{s}^{(1/2)}(e_z)\big]^4\Bigg]|j_x j_y j_z\rangle\end{aligned}$$

where

$$\hat{s}^{(k)}(e_a)|j_a\rangle = \frac{1}{2i}\Big(|j_a+k\rangle - |j_a-k\rangle\Big)$$

Effective dynamics

Effective dynamics: Dynamics on a classical phase space generated by an effective Hamiltonian function motivated by considerations from the quantum theory.

For example

$$H_{\text{eff}} = H_{\text{gr}}(p, c) + H_{\text{matter}} \qquad\qquad \frac{d}{dt}F(p, c) = \{F, H_{\text{eff}}\}$$

on the phase space of a homogeneous and isotropic universe:

$$A_a^i = c(t)\delta_a^i \qquad E_i^a = p(t)\delta_i^a \qquad \{c, p\} = 1$$

A related quantity in LQG is the expectation value of the Hamiltonian operator in a semiclassical state (peaked on a homogeneous and isotropic geometry). For the Euclidean part of the Hamiltonian one has [Dapor, Liegener 2017; Zhang, Song, Han 2020]

$$H_E^{\text{eff}} = \langle\psi_{(p,c)}|\hat{C}_E|\psi_{(p,c)}\rangle = -\frac{3}{\beta^2}\sqrt{p}\frac{\sin^2\mu c}{\mu^2} \qquad (\mu = \text{const.})$$

The classical expression $H = -(3/\beta^2)\sqrt{p}c^2$ is recovered in the limit $\mu \to 0$.

The effective Hamiltonian: Lorentzian part

There is a certain formal similarity between the effective Hamiltonian

$$H_E^{\text{eff}} = -\frac{3}{\beta^2}\sqrt{p}\frac{\sin^2 \mu c}{\mu^2}$$

and the Euclidean Hamiltonian of the one-vertex model:

$${}^R\hat{C}_E = -\frac{1}{\beta^2}\left[\sqrt{\frac{\hat{j}_x\hat{j}_y}{\hat{j}_z}}\hat{s}^{(1)}(e_x)\hat{s}^{(1)}(e_y) + \text{cycl. perm.}\right]$$

If we imagine that the same relation should hold for the Lorentzian part

$${}^R\hat{C}_L = -16\frac{1+\beta^2}{\beta^2}\left[\frac{\hat{j}_x^{3/2}}{\sqrt{\hat{j}_y\hat{j}_z}}\left[\hat{s}^{(1/2)}(e_x)\right]^4 + \text{cycl. perm.}\right]$$

we can propose a conjecture:

$$H_L^{\text{eff}} = -48\frac{1+\beta^2}{\beta^2}\sqrt{p}\frac{\sin^4(\mu c/2)}{\mu^2}$$

A possible new Hamiltonian for loop quantum cosmology?

Note: $H_L^{\text{eff}} \to 0$ as $\mu \to 0$, consistent with the interpretation as spatial curvature.

Effective dynamics: Evolution of the volume

$$H = H_{\rm gr}(p, c) + \frac{\pi_\phi^2}{2p^{3/2}}$$

Classical trajectory:

$$H_{\rm gr} = -\frac{3}{\beta^2}\sqrt{p}c^2$$

The standard Hamiltonian:

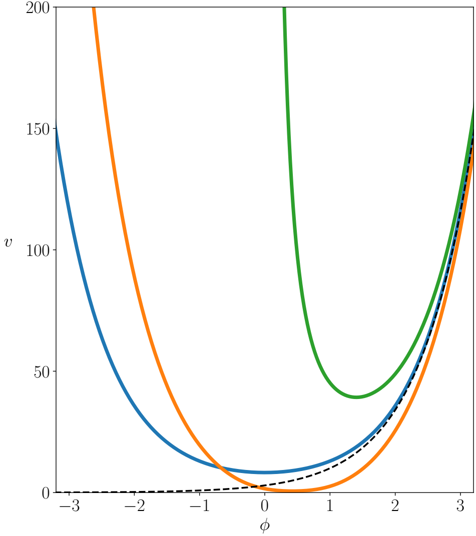
$$H_{\rm gr} = H_E = -\frac{3}{\beta^2}\sqrt{p}\frac{\sin^2 \mu c}{\mu^2}$$

The new proposal:

$$H_{\rm gr} = H_E - 48\frac{1+\beta^2}{\beta^2}\sqrt{p}\frac{\sin^4(\mu c/2)}{\mu^2}$$

Dapor–Liegener model:

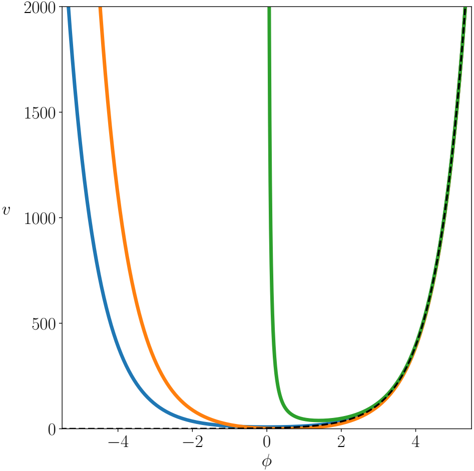
$$H_{\rm gr} = H_E + 3\frac{1+\beta^2}{\beta^2}\sqrt{p}\frac{\sin^4 \mu c}{\mu^2}$$



Effective dynamics: Classical region

The effective dynamics of the volume agrees with the classical trajectory in the far future as well as far past of the bounce (unlike in the Dapor–Liegener model).

The trajectory $v(\phi)$ is symmetric under $\phi \to \phi_0 - \phi$ (where ϕ_0 is the value of ϕ at the bounce).



Effective dynamics: Bounce

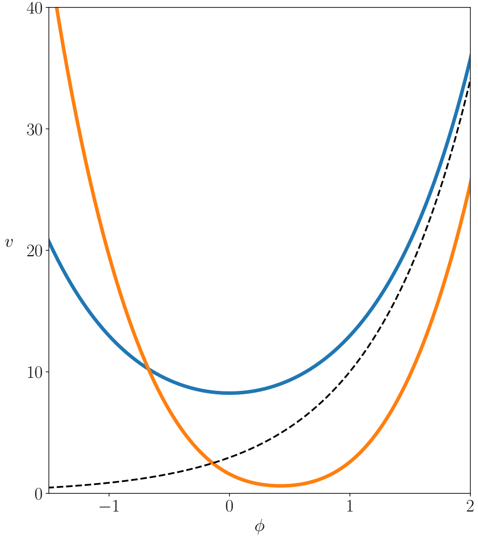
Compared to the standard scenario, the duration of the bounce is shortened, and the volume reaches a lower minimum value.

The value of volume at the bounce is given by

$$v_{\mathrm{min}}^{(0)} = \left(\frac{\beta \mu \pi_\phi}{\sqrt{6}} \right)^{3/2}$$

and

$$v_{\mathrm{min}} = \frac{v_{\mathrm{min}}^{(0)}}{8(1+\beta^2)^{3/4}}$$



Quantum dynamics of semiclassical states

In the setting of the one-vertex model, we wish to study the dynamics of the states

$$|\psi_0\rangle = |p_0, c_0\rangle_{e_x} |p_0, c_0\rangle_{e_y} |p_0, c_0\rangle_{e_z}$$

where $\quad |p_0, c_0\rangle_e = \sum_j \sqrt{2j+1} e^{-t(j-j_0)^2/2} e^{-ic_0 j} \mathcal{D}^{(j)}_{jj}(h_e)_{i_e} \qquad (p_0 = j_0 + \frac{1}{2})$

Using irrotational dust as the physical time variable, and considering only the Euclidean part of the Hamiltonian, the time evolution of the state is given by

$$|\psi(T)\rangle = e^{-i\hat{H}_{\rm phys}T}|\psi_0\rangle \qquad \text{with} \qquad \hat{H}_{\rm phys} = \hat{C}_E(1)$$

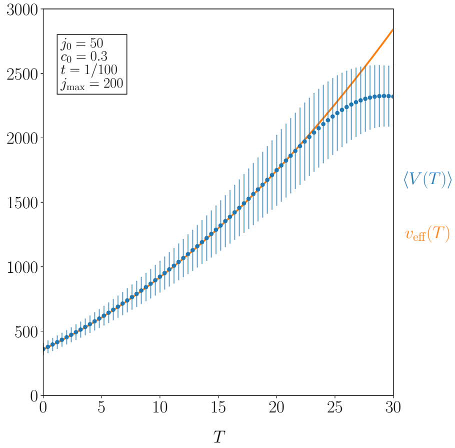
We compute this numerically (using the functions `expm_multiply` from `scipy.sparse.linalg` and `expv` from `ExponentialUtilities.jl`) under the assumption that the spins take values in the finite range

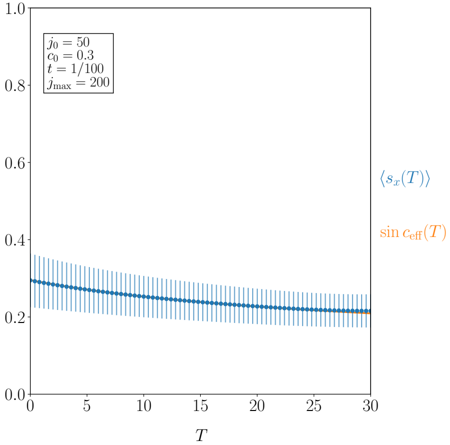
$$j_a \in \{j_{\rm min},\, j_{\rm min}+1,\, \dots,\, j_{\rm max}-1,\, j_{\rm max}\}$$

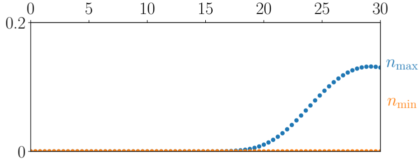
Remarks:

- The subspace of $|j_x j_y j_z\rangle$ with all $j_a \in \mathbb{N}$ is preserved under $\hat{H}_{\rm phys}$
- The lower limit $j_{\rm min} = 1$ can be achieved by suitable factor ordering of $\hat{H}_{\rm phys}$
- The upper limit $j_{\rm max}$ is imposed by hand (e.g. $j_{\rm max} = 200$; $\dim\mathcal{H} = 8\times 10^6$)

Example 1: An expanding universe



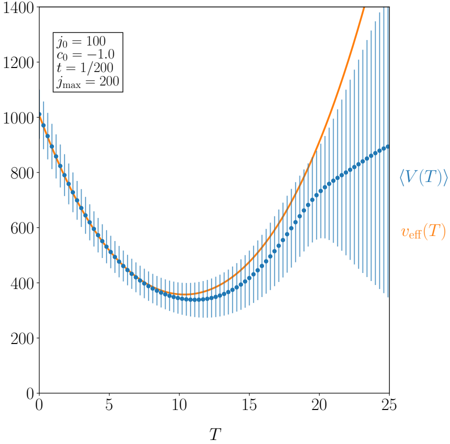


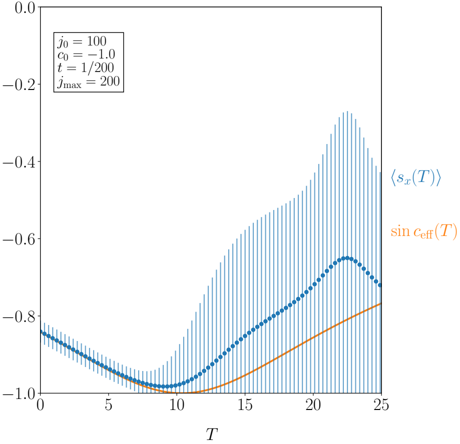


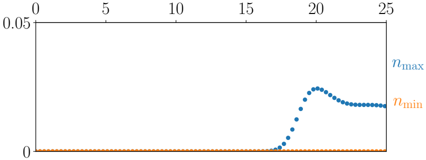
$$|\psi(T)\rangle = \sum_{j_x j_y j_z} c_{j_x j_y j_z}(T) |j_x j_y j_z\rangle$$

$$n_{\max} = \sum_{\text{any } j_a = j_{\max}} |c_{j_x j_y j_z}|^2$$

Example 2: A contracting universe



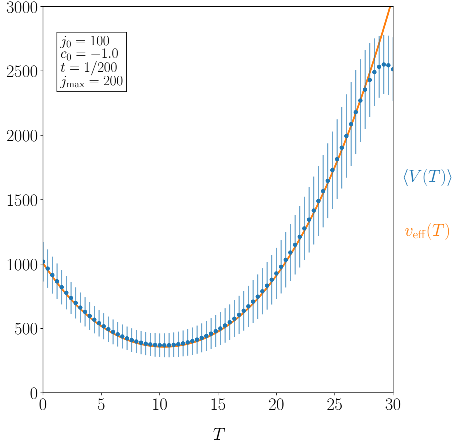


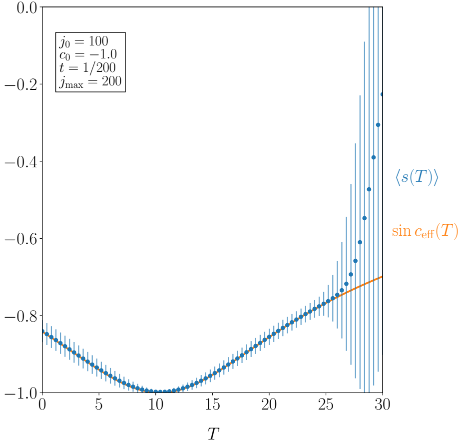


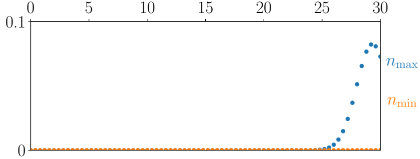
$$|\psi(T)\rangle = \sum_{j_x j_y j_z} c_{j_x j_y j_z}(T) |j_x j_y j_z\rangle$$

$$n_{\max} = \sum_{\text{any } j_a = j_{\max}} |c_{j_x j_y j_z}|^2$$

Comparison with a 1D toy example



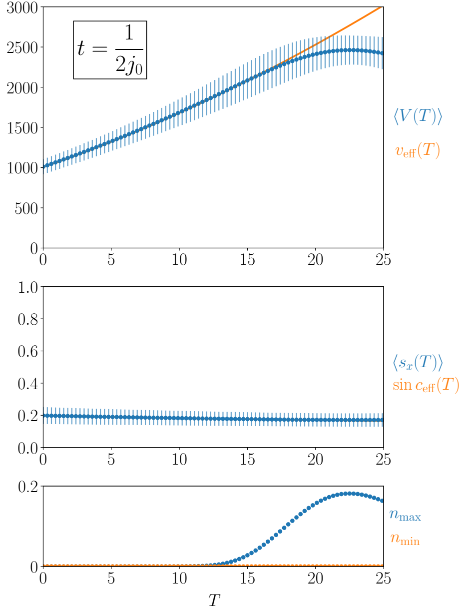


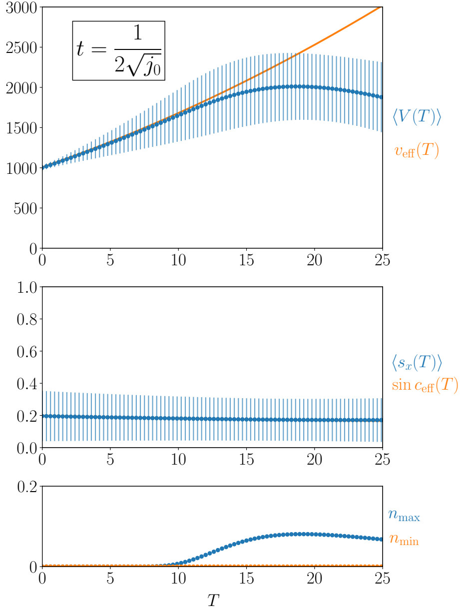


$$\mathcal{H} = \text{span}\Big(|j\rangle : j \in \{1, 2, \ldots, j_{\max}\} \Big)$$

$$\hat{H}_{\text{phys}} |j\rangle = -\frac{3}{\beta^2} \sqrt{j} \big(\hat{s}^{(1)} \big)^2 |j\rangle$$

Choice of the semiclassicality parameter t





Conclusions

We gave a concrete proposal for a Hamiltonian constraint operator for quantum-reduced loop gravity (as well as full LQG on a fixed cubical graph). The Lorentzian part of the Hamiltonian is given by an operator representing the scalar curvature of the spatial manifold.

The effective dynamics of a homogeneous and isotropic universe agrees with the classical dynamics far from the bounce, but the specific details of the bounce differ from those generated by the standard Hamiltonian.

Time evolution of quantum states in quantum-reduced loop gravity is numerically accessible (over short enough time intervals, due to the cutoff imposed on the spins) at least in the simplest possible example of the single-vertex graph.

To do:

- Include the Lorentzian term in the numerical simulations
- Study the anisotropic case (Bianchi I)
- Loop quantum cosmology with the new Lorentzian term?
- Effect of quantization ambiguities?

Thank you for your attention