Lorentzian cosmological path integrals in effective spin foams

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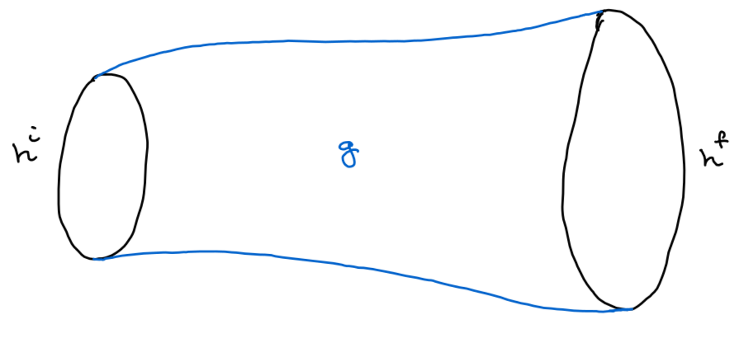
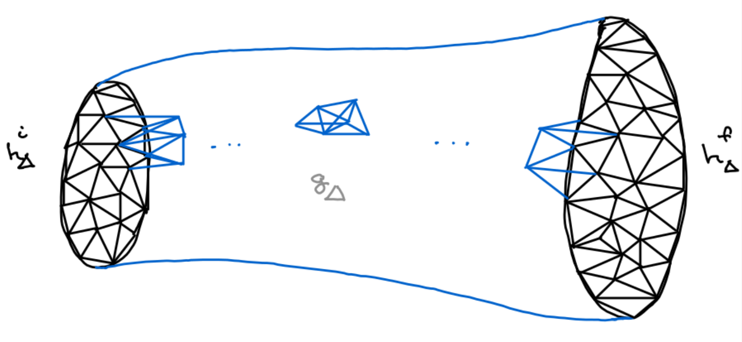
Discretize gravitational path integral



$$Z = \int_{h_i}^{h_f} \mathcal{D}g \; e^{\frac{i}{\hbar} S_{\text{EH}}[g]}$$

Formal definition: integral over all **geometries**!

Discretize gravitational path integral

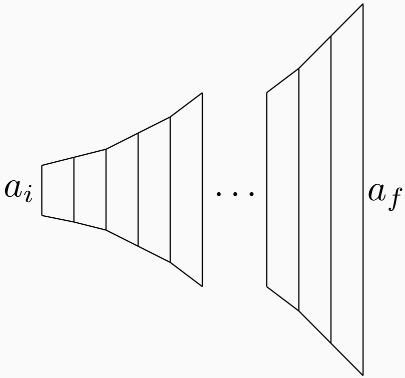


$$Z(\Delta) = \int_{h_i^\Delta}^{h_f^\Delta} \prod_e dl_e \; e^{\frac{i}{\hbar} S_{\text{Regge}}[l_e]}$$

Regularize by replacing manifold by a **triangulation**.

Replace metric by **edge lengths** in (quantum) **Regge calculus** [Regge, Hamber, Williams, Rocek,...].

Discretize gravitational path integral



$$Z(\Delta) = \sum_{\{A_n\}} \mu(\{A_n\})\, e^{\frac{i}{\hbar} S_{\text{Regge}}[\{A_n\}, a_i, a_f]}$$

Reduce discretization to **cosmological subsector** [Brewin, Williams, ...]

Tackle this in **effective spin foams** [Asante, Dittrich, Haggard, Padua-Argüelles]

Some related literature

- **Lorentzian cosmological path integrals** [Turok, Lehners, Feldbrugge ...]
  - Lefshetz thimble method for lapse integration
- Cosmological models in **Regge calculus**
  - Spherical model $\sim$ positive spatial curvature [Collins, Williams '73, Hartle '85, Liu, Williams '15]
  - Spatially flat / "frusta" model [Bahr, Klöser, Rabuffo '17, Bahr, Rabuffo, S.St. '18]
- **Causal Dynamical Triangulations** [Ambjørn, Loll, Jurkiewicz, Görlich, Gizbert-Studnicki,...]
- **Complex critical points for spin foams** [Han, Liu, Qu, Huang, Zhang]
  - Spatially flat cosmology with scalar field [Han, Liu, Qu, Vidotto, Zhang '24]
- **Effective spin foams** [Asante, Haggard, Dittrich '20, Asante, Dittrich, Padua-Arguëlles '21]
  - Classical (discrete) Lorentzian Cosmology [Dittrich, Gielen, Schander '21]
  - Single time-step path integral [Dittrich, Padua-Arguëlles '23]
- **No boundary proposal in spin foams** [Frisoni, Gozzini, Vidotto '22, '23]

Outlook

(Lorentzian) Regge Calculus

(Lorentzian) Cosmological Subsector in Regge calculus

Towards the path integral

Lessons from a (2+1)d spin foam model

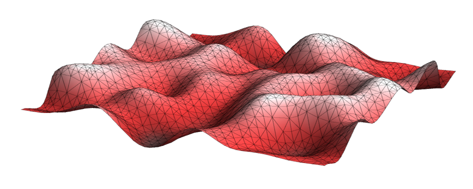
(3+1)d path integral

Summary & Outlook

(Lorentzian) Regge Calculus

Regge calculus in a nutshell [Regge '61]

- Replace manifold by a **triangulation**
  - **Simplicial manifold**
  - Locally look like a d-dim manifold
- Triangulation is built from **simplices**
- Simplex: piece of **flat space-time**
  - Embeddable into flat space-time
- **Edge lengths** encode metric
- **Curvature** distributional
  - Deficit angle
- **Regge action**: discretized Einstein-Hilbert

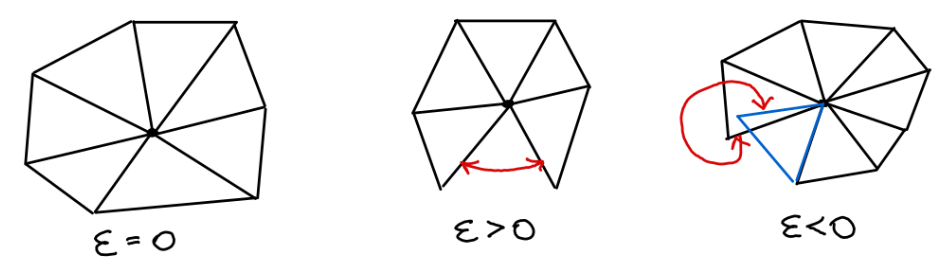


Edge lengths are degrees of freedom

Volumes, angles... are **functions** of edge lengths

Curvature as deficit angle [Regge '61]

- **Curvature**: **Difference** to flat space



Deficit angle

$$\epsilon_h(l_e) := 2\pi - \sum_{\sigma \supset h} \theta_h^{(\sigma)}$$

Deficit angle located at **hinge** ($(d-2)$-simplex)

Regge action in 4d

Regge action

$$S_{\text{Regge}}(l_e) := \sum_{h \in \Delta^\circ} A_h(l_e)\, \epsilon_h(l_e) + \text{boundary terms} \quad .$$

- A_h: area of triangle h
- ϵ_h: deficit angle at triangle h

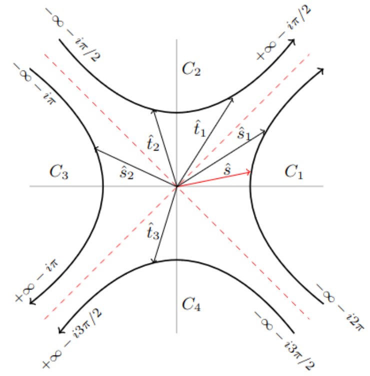
Equations of motion

$$\frac{\partial S_{\text{Regge}}}{\partial l_e} := \sum_{h \in \Delta} \frac{\partial A_h}{\partial l_e}\, \epsilon_h \stackrel{!}{=} 0 \quad .$$

- Schläfli identity: **derivative** of ϵ **vanishes**

Equations look innocent, but they are **transcendental**.

- **Lorentzian theory**: angles in $\mathbb{R}^{1,1}$
  - **Analytic continuation** of angles



Lorentzian Regge action can be complex $\rightarrow$ (hinge) **causality violations**.

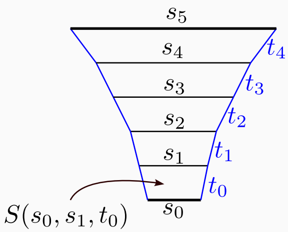
Can be **suppressed** or **enhanced** in path integral!

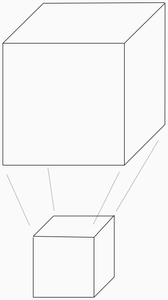
(Lorentzian) Cosmological Subsector in Regge calculus

Constructing a cosmological subsector [Jercher, S.St. '23]

- **Symmetry reduce triangulation**
- **Spatial** equilateral **hypersurfaces** of **fixed topology**
  - Positive curvature: Spherical topology [Brewin '87, Liu, Williams '15]
  - **Flat: Torus topology** [Bahr, Klöser, Rabuffo '17, Bahr, Rabuffo, S.St. '18]
- **Connect** hypersurfaces by edges
  - Edges can be **space-like** or **time-like**
- "Slab" of space-time: **three variables**
  - Spatial lengths: s_n, s_{n+1}
  - Strut lengths: t_n
- Regge action **additive per slab**:

$$S_{\text{Regge}} = \sum_{n=0}^{N} S^{(n)}(s_n, s_{n+1}, t_n)$$



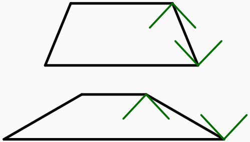


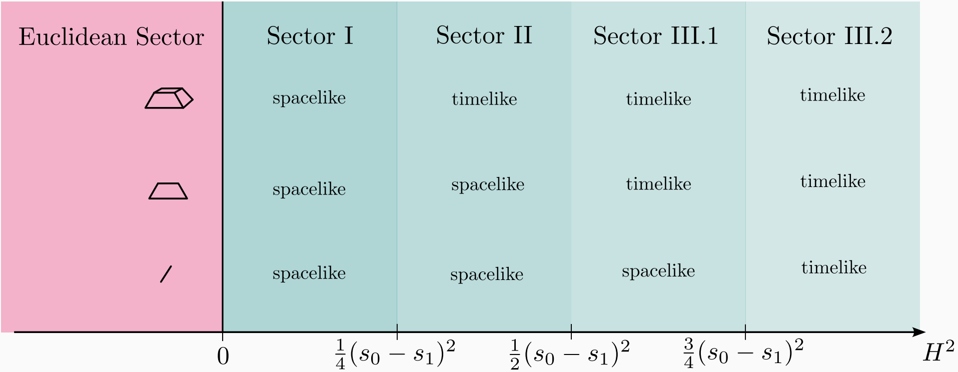
Should we sum over **space-** and **time-like building blocks**?

What does this imply for the **causal structure**?

Causality violations [Asante, Dittrich, Padua-Agüelles '21, Dittrich, Padua-Agüelles '23, Jercher, S.St. '23]

- **Causal irregularities** in Regge calculus
  - More or less than two lightcones at a vertex
  - Topology change
- **Causally regular** if all building blocks **time-like**!

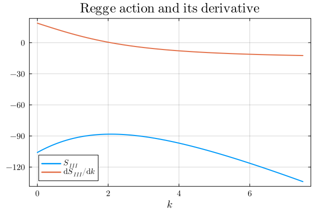




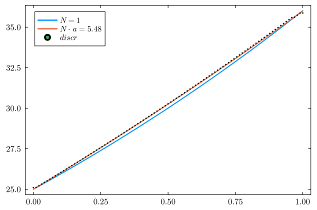
Classical, discrete dynamics [Jercher, S.St. '23]

- **Matter content**
  - Massless / massive scalar field
  - Cosmological constant $\Lambda > 0$
- Equations of motion are **transcendental**

$$\frac{\partial S_{\text{Regge}}}{\partial H_n} = \frac{6H_n(l_n + l_{n+1})}{\sqrt{H_n^2 - \frac{1}{2}(l_n - l_{n+1})^2}} \left[\frac{\pi}{2} - \theta(l_n, l_{n+1}, H_n)\right] - \Lambda \frac{\partial V}{\partial H_n} = 0$$



  - **Numerical methods** necessary, e.g. NSolve
- **Solutions** only exist for **causally regular** configurations!
  - Vacuum: static universe
  - Scalar field: exponential expansion
- **Continuous time limit** (small deficit angle)



Discrete dynamics **qualitatively** reflects **continuum solutions**.

Curved solutions break **time-reparametrization invariance**.

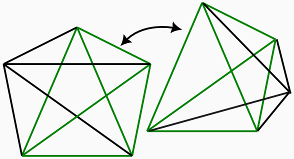
Towards the path integral

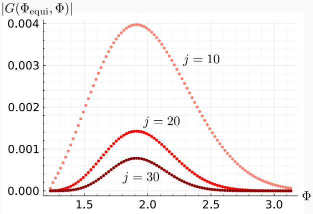
Effective spin foams in a nutshell [Asante, Haggard, Dittrich '20]

- **Key ingredients** of **spin foam models**
- **Area** and **angle** variables
    - **Discrete area spectrum** (of EPRL-CH model) [Conrady, Hnybida '08]
- **Lorentzian** Area Regge action
- **Gluing constraints** (per tetrahedron)
    - Computed in full theory [Asante, Simão, S.St. '22]

$$Z = \sum_{\{A_n\}} \mu(\{A_n\}) \prod_{\tau} G_\tau(\theta_\tau^\sigma - \theta_\tau^{\sigma'}) e^{iS_{\text{Regge}}(\{A_n\})}$$

- **Not Wick-rotated**!
- Causality violations **exponentially suppressed**
    - Depends on choice in branch cut of Regge action.





These sums are **highly oscillatory** and potentially divergent.

How to compute? Do they converge?

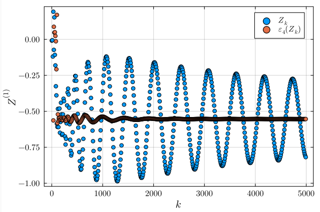
- Compute $A = \sum_{m=0}^{\infty} a_m$ for a sequence $\{a_m\}$.
- **Convergence** of series of partial sums $A_n = \sum_{m=0}^{n} a_m$.
- **Shank's transform** $\to$ new, faster converging series:

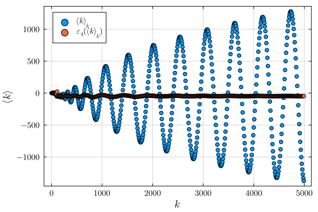
$$S(A_n) = \frac{A_{n+1}A_{n-1} - A_n^2}{A_{n+1} - 2A_n + A_{n-1}}.$$

  - Repeating transform improves convergence
  - For large n: $A_n = A + \alpha q^n$ for $|q| < 1$.
- Convenient to work with **Wynn's ϵ algorithm**

$$\epsilon_{r+1}(A_n) = \epsilon_{r-1}(A_{n+1}) + \frac{1}{\epsilon_r(A_{n+1}) - \epsilon_r(A_n)}$$

  - $\epsilon_{2r}(A_n) \sim r$th Shank's transform of A_n.



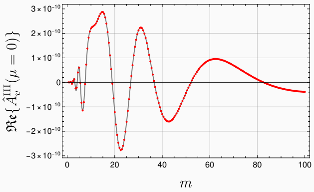


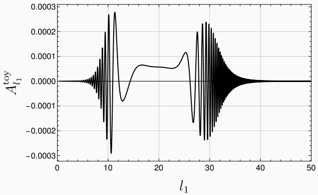
Drastically improves **convergence** of **oscillating sums**.

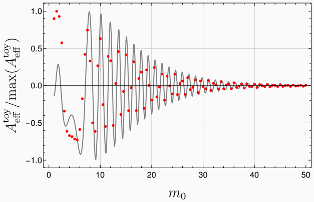
Lessons from a (2+1)d spin foam model

Key lessons from the $(2+1)$ model [Jercher, Simão, S.St. '24]

- **"Freezing oscillations"**: asymptotically stationary action
  - Height ($\sim$ lapse N) $\to \infty$
  - Cured by scalar field mass μ, $\Lambda \neq 0$
- **Causality violations** must be **suppressed** for classical results
  - No mechanism from asymptotic formula of 1 building block
- **Measure crucial** for classical physics
  - Measure may suppress critical points
- **Acceleration operators** work well for **two time-steps**
  - Series in two parameters t_1, t_2.
- Explicit integration over spatial distance (peculiar to 3d)







Explicit calculations of oscillatory **Lorentzian path integral** for multiple time steps possible!

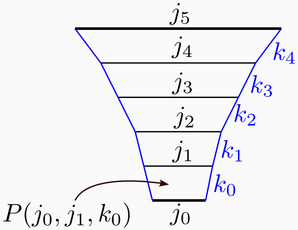
Identified **potential obstacles** for semi-classical physics

(3+1)d path integral

The setup - "transfer matrix" [Jercher, S.St. w.i.p.]

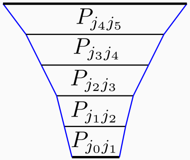
- **(3+1)d (spatially flat)** path integral in effective spin foams
  - Lengths $s_n, t_n \to$ **areas** j_n, k_n
  - "Frusta", no matter, $\Lambda > 0$.

$$Z = \sum_{\{j_n\},\{k_n\}} \prod_n \mu_n(j_n, k_n, j_{n+1})\; e^{i \sum_n S^{(n)}_{\mathrm{Regge}}[j_n,k_n,j_{n+1};\Lambda]}$$



- Write it as a **product of propagators**:

$$Z = \sum_{\{j_n\}} \prod_n \left(\sum_{k_n} P^{(n)}(j_{n-1}, j_n, k_n) \right).$$



- Since j_n discrete, write $P^{(n)}$ as a **"matrix"** P_{j_{n-1}, j_n}:
  - Matrix elements: complex (transition) amplitude

$$Z = P_{j_0,j_1} P_{j_1,j_2} \ldots P_{j_{n-1},j_n} = P^n_{j_0,j_n}$$

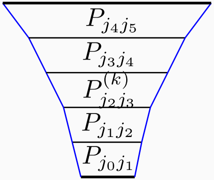
Assumptions on measure! (factorizes per "slab" and total convergence of sums)

Greatly **simplifies calculation** (time-like sums and observables)!

Computing observables

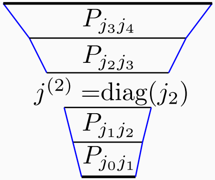
- **(Simple) observables** in "transfer matrix" formalism
- **"Time-like" observables**
  - Define new matrix $P^{(k)}_{j_i,j_{i+1}} = \sum_{k_n} \mu(\dots)\, k_i\, e^{iS^{(i)}_{\text{Regge}}(j_i,k_i,j_{i+1})}$

$$\rightarrow \langle k_i \rangle = \frac{P^{i-1}_{j_0,j_i} P^{(k)}_{j_i,j_{i+1}} P^{n-i}_{j_{i+1},j_n}}{P^n_{j_0,j_n}}$$



- **"Space-like" observables**
  - Insert operator between matrices: $j^{(i)} = \text{diag}(j_i)$

$$\rightarrow \langle j_i \rangle = \frac{P^{i}_{j_0,j_i} j^{(i)} P^{n-i}_{j_i,j_n}}{P^n_{j_0,j_n}}$$

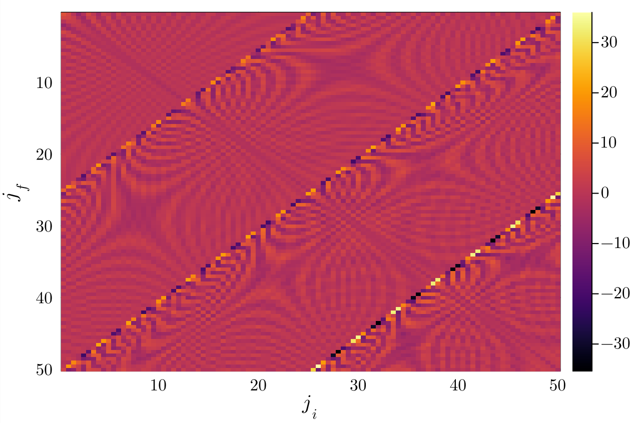


- More intricate for more non-local observables

Efficient and straightforward to compute **expectation values**.

A single time step

- Let us consider a **single time step**

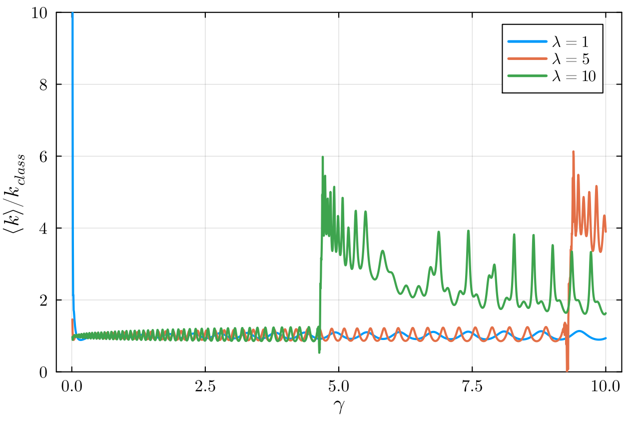


Divergences of $Z^{(1)}$ depending on boundary data

Mark **boundary of classicality**: $j_0 + j_1 \lesssim \frac{8\pi}{\gamma\Lambda}$

A single time step

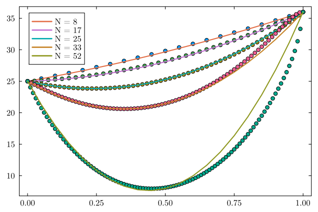
- Let us consider a **single time step**



Divergences of $Z^{(1)}$ depending on boundary data

Mark **boundary of classicality**: $j_0 + j_1 \lesssim \frac{8\pi}{\gamma\Lambda}$: similar to [Asante, Dittrich, Padua-Argüelles]

A few preliminary results



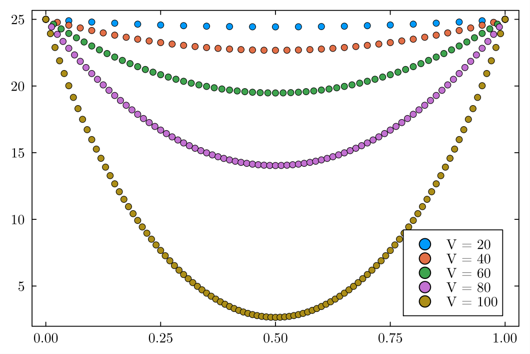
- Boundary data: **allow for classical solution**
  - Expanding: $s_0 < s_n$
  - Contracting: $s_0 > s_n$
- **Few time steps**: **match well** with discrete solutions
  - Expectation values agree well
- **Many time steps**: **deviations**
  - **Area gap** for **time-like areas**
  - Smallest time-like area larger than classical solution
  - Proper time grows with number of time steps
- Plots in **'coordinate time'**
  - System **appears to "bounce"**
  - Classical continuum: gauge artifact for special lapse
- Interpret "height" as **proper time** (see also CDT)

What is physical? What is gauge? $\rightarrow$ **Relational evolution**

Discrete area spectra may fail to resolve critical point [Dittrich, Padua-Arguëlles '23]

There is a bounce...

- **Boundary data** $s_0 = s_n$ has **no classical solution**
  - **No classical interpretation** available
  - So far no complex classical solution found
  - **Extrinsic curvature** via "squeezed coherent states" (complexifier coherent states [Thiemann '00])



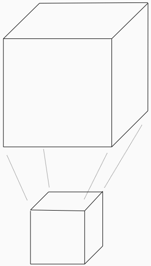
Bounce is not a **gauge-artifact**, but **better understanding** necessary.

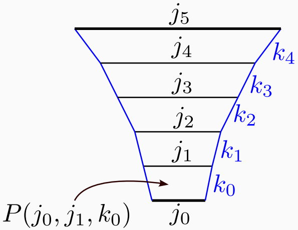
Again, **relational clock** might help interpretation.

Summary & Outlook

Summary

- **Cosmology** in **Lorentzian Regge calculus**
  - Classical solutions capture continuum dynamics
  - Height $\sim$ proper time
- Lorentzian **non-Wick-rotated path integral**
  - Based on effective spin foams
  - Acceleration operators improve convergence
  - **"Transfer matrix"** $\rightarrow$ efficient evaluation
- **First results**
  - Classical discrete dynamics (sometimes)
  - Bounce for classically forbidden transition
- **Many open questions** remain:
  - Better understanding of quantum model
  - Relational observables?
  - Impact of discrete area spectrum?

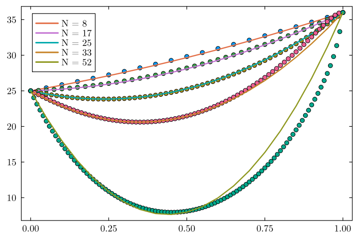


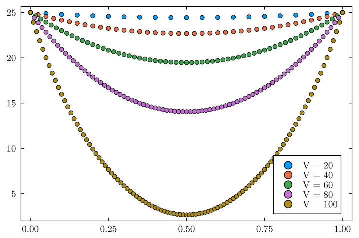


Discrete path integrals with **many time steps** can be computed!

Outlook

- Address **conceptual challenges** in spin foams
  - **Refinement limit**
  - Comparison to classical dynamics
  - **Relational observables** / clocks
  - **Truly Lorentzian** dynamics
  - Algorithms beyond transfer matrix
- Contact to **Cosmology**?
  - Spatial curvature / **no boundary proposal**?
  - Comparison to **quantum cosmology**
  - Beyond Friedmann? **Perturbations?**
- Relation **full theory** and **effective spin foams**?
  - Effective subsector via Monte Carlo [Siebert, S.St. w.i.p.]
- **Tool** to compute (simplicial) **Lorentzian path integrals**?





Thank you for your attention!