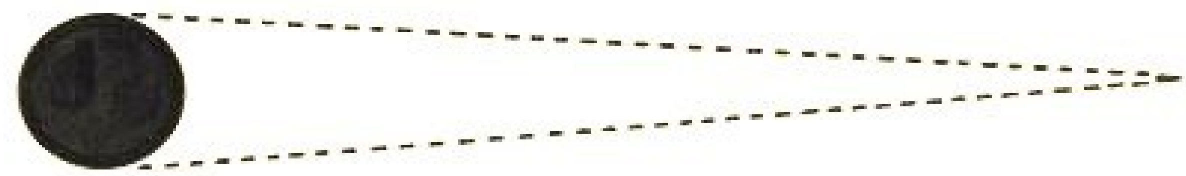
Darboux Transformations in (the Interior of) Nonrotating Black Holes

FAU^2,
22 May 2025

CSIC
CONSEJO SUPERIOR DE INVESTIGACIONES CIENTÍFICAS

Guillermo A. Mena Marugán
Instituto de Estructura de la Materia, CSIC
(with A. Mínguez-Sánchez)

Introduction



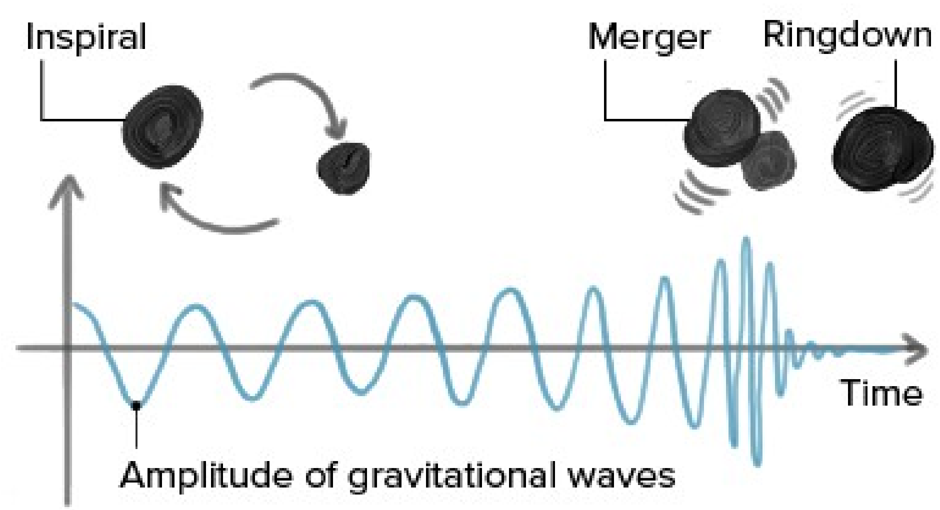
- Black hole spacetimes are a challenge for classical, semiclassical, and quantum gravity.

- **<u>PERTURBATIONS</u>** of black holes are crucial to analyze their stability.

- They also have applications in astrophysics. For instance, they describe some regimes in the evolution of a black hole merger.

- This connects with the emission of gravitational waves, a reason explaining the increasing attention paid to this topic.

- The ringdown of perturbed black hole is dominated by quasinormal modes.



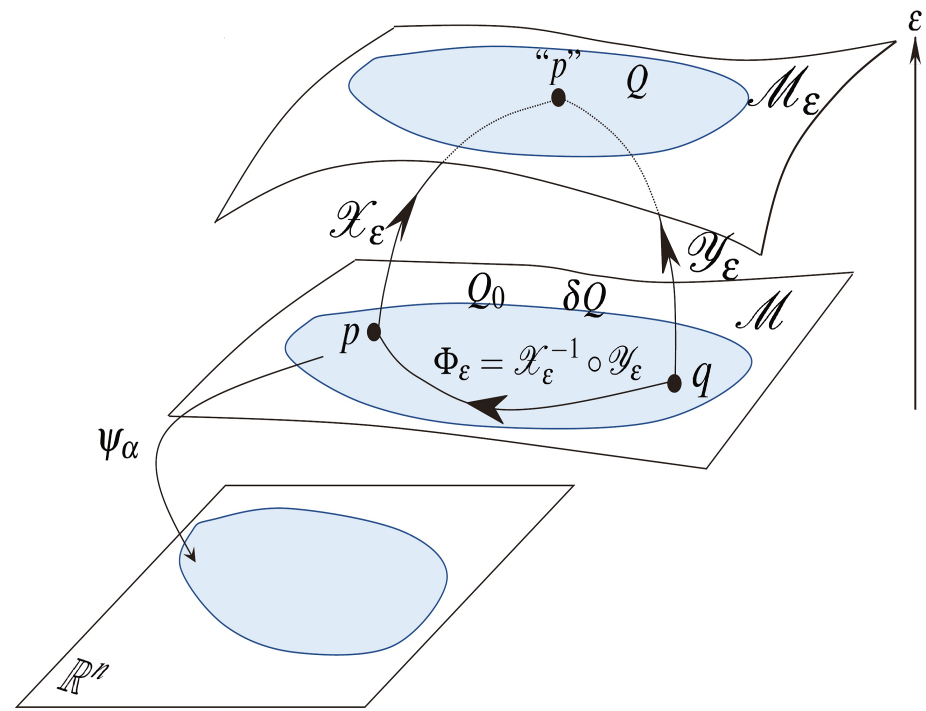
Introduction

- Identification of points of the original and the perturbed manifold introduce some gauge freedom.

- Only perturbative quantities invariant under this freedom are physical.

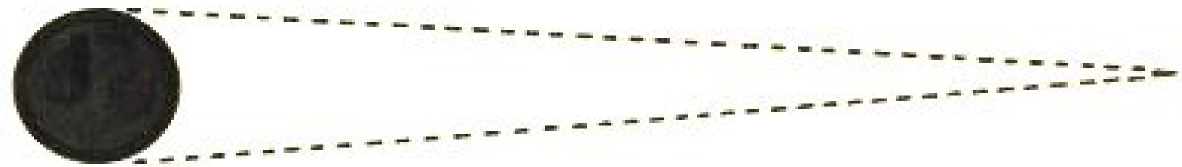
- These are the **<u>PERTURBATIVE GAUGE INVARIANTS</u>**.

- At first order, they are linear in the perturbations and can be multiplied by any background-dependent factor.



- They satisfy second-order differential equations, defined in the set of orbits of spherical symmetry. Quasinormal modes solve these equations with outgoing boundary conditions.

Introduction



- Most of the perturbative studies have been carried out in the Lagrangian formalism.
- A **HAMILTONIAN** formulation for nonrotating black holes -as well as a higher-order perturbative formalism- was developed by J.M. Martín-García & D. Brizuela (and G.A.M.M.) in the 90s.
- This formulation employs spherical symmetry as a key ingredient. It splits the 4-dimensional manifold into two 2-dimensional ones.
- Perturbative gauge invariants are easily characterized by commuting with the generators of perturbative spacetime diffeomorphisms.
- The Hamiltonian formulation is especially suitable for the transition to the quantum theory.
- However, the radial dependence highly complicates the analysis.

Introduction

- There exist an intriguing relation between different perturbative gauge invariants, given by **<u>DARBOUX TRANSFORMATIONS</u>**.

- Suppose that φ satisfies a wave equation in two dimensions for a potential v_l, which depends on the angular-momentum number l.

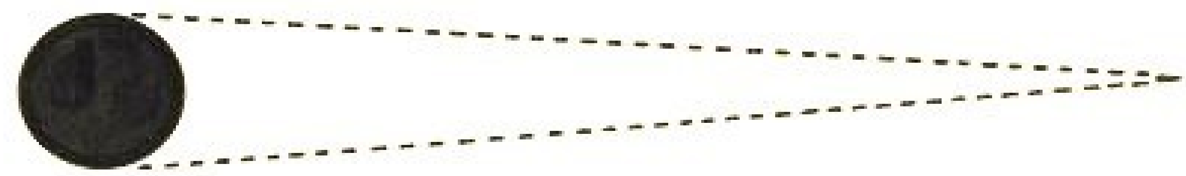
- Consider the transformation to $\Psi = \acute{\varphi} - g_l \varphi$, where the acute stands for the derivative wrt. a tortoise "radial" coordinate, and g_l satisfies the Ricatti equation $-\acute{g}_l + g_l^2 + v_l = c$, with c a constant.

- Then Ψ is a solution to the equation for the new potential $V_l = v_l + 2\,\acute{g}_l$.

- Given a solution $\acute{\acute{\varphi}}_0 - v_l \varphi_0 = -\omega_0^2 \varphi_0$, define $g_l = (\ln(\varphi_0))'$, with $c = \omega_0^2$.

  Then, the old and new potentials admit **isoespectral** solutions (with the same "frequency"), related by $\Psi = [\acute{\varphi}\varphi_0 - \varphi\acute{\varphi}_0]/\varphi_0$.

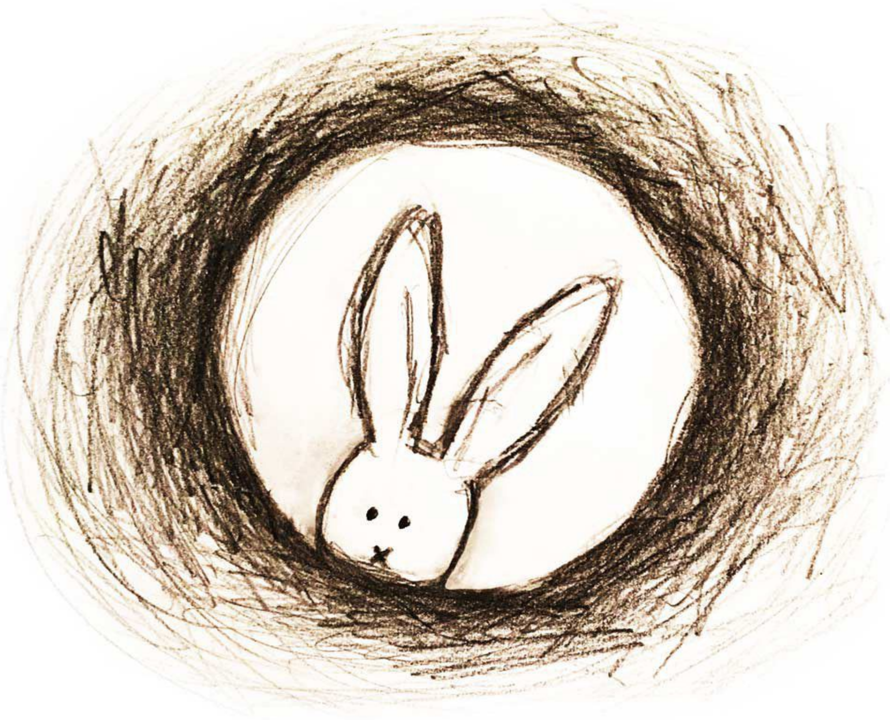
Introduction



- Darboux transformations suggest a higher symmetry. They have been related to a KdV system with a bi-Hamiltonian structure.

- The complications with the radial dependence disappear in the interior region of the black hole, where it becomes a time dependence.

- This interior is isometric to a Kantowski-Sachs cosmology.

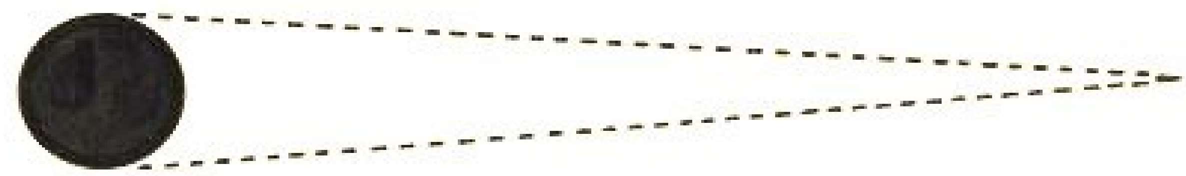


- Can the Hamiltonian formulation of the perturbations be further developed in this interior? **Yes** (A. Mínguez-Sánchez & G.A.M.M.).

- Can we use it to understand Darboux transformations?

- And quantum mechanically?.

Background



- The metric in the interior region can be written in the form

$$ds^2 = p_b^2(\tau)\left(-\underline{N}^2(\tau)\left|p_c(\tau)\right|d\tau^2 + \frac{1}{\left|p_c(\tau)\right|}dx^2\right) + \left|p_c(\tau)\right|\left(d\theta^2 + \sin^2\theta\, d\phi^2\right).$$

The chosen metric functions are in fact triad variables.

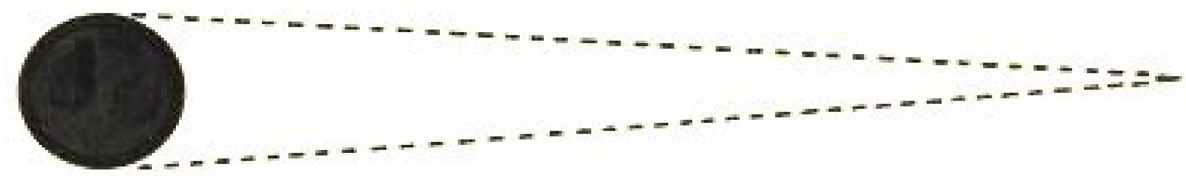
- The geometry has **two canonical pairs** of degrees of freedom, with extrinsic-curvature variables such that $\{b, p_b\} = 1, \ \{c, p_c\} = 2.$

- This KS background is subject ONLY to the **Hamiltonian** constraint

$$\underline{N}\, H_{KS} = -\frac{\underline{N}}{2}\left(\Omega_b^2 + 2\,\Omega_b\Omega_c + p_b^2\right), \quad \Omega_j = j\, p_j, \quad j = b, c.$$

The Omega-operators are generators of dilations.

Background



- We consider compact sections with the topology of $S^1 \times S^2$. Then, zero-modes are isolated and can be treated exactly.

- We expand our perturbations in REAL spherical harmonics and **Fourier modes**.

- Spherical harmonics split in polar and axial depending on their behavior under parity.

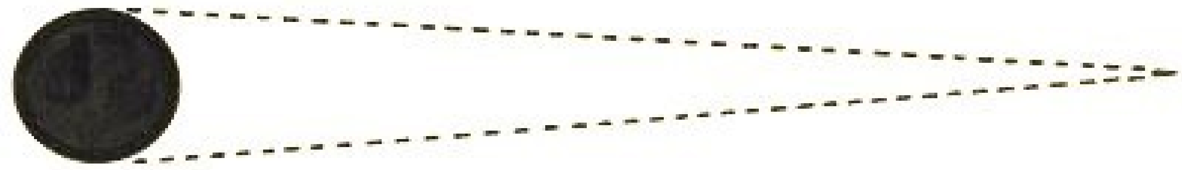
- A polar harmonic of eigenvalue $-l(l+1)$ for the Laplacian on S^2 has parity eigenvalue equal to $(-1)^l$. Scalar harmonics Y_l^m are polar.

- We use a real Regge-Wheeler-Zerilli basis of harmonics.

- Using capital Latin letters for S^2-indices, we decompose any symmetric tensor as

$$T_{ab} dx^a dx^b = T_{xx} dx^2 + 2\mathrm{T}_{xA} dx\, dx^A + T_{AB} dx^A dx^B .$$

Perturbations



- For scalars on S^2, we have $\zeta(\theta,\phi)=\sum \zeta_l^m Y_l^m$.
- For covectors, $w_A(\theta,\phi)=\sum\left(W_l^m Z_{l\ A}^m + w_l^m X_{l\ A}^m\right)$,

  where we include polar and axial contributions.

  Using the metric γ_{AB} on S^2 and its covariant derivative, we have

  $$Z_{l\ A}^m = Y_{l\ :A}^m, \quad X_{l\ A}^m = \epsilon_{AB}\gamma^{BC} Y_{l\ :C}^m, \quad l \geq 1.$$

  which are orthonormalized to $l(l+1)$.

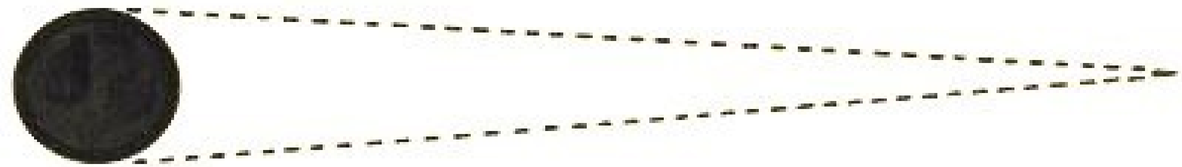
- Finally, for tensors

  $$T_{AB}(\theta,\phi)=\sum \tilde{T}_l^m \gamma_{AB} Y_l^m + \sum\left(T_l^m Z_{l\ AB}^m + t_l^m X_{l\ AB}^m\right),$$

  with $X_{l\ AB}^m = \frac{1}{2}\left(X_{l\ A:B}^m + X_{l\ B:A}^m\right), \quad Z_{l\ AB}^m = Y_{l\ :AB}^m + \frac{l(l+1)}{2}\gamma_{AB} Y_l^m, \quad l \geq 2.$

  These tensor harmonics are orthonormalized to $l(l+1)(l+2)/2$.

Perturbations



- We choose real spherical harmonics,

$$Y_l^m \rightarrow \left\{ Y_l^m,\ m=0;\ \ \frac{(-1)^m}{\sqrt{2}}\left(Y_l^m + Y_l^{m\star}\right),\ m>0;\ \ \frac{(-1)^m}{i\sqrt{2}}\left(Y_l^{|m|} - Y_l^{|m|\star}\right),\ m<0 \right\}.$$

- Similarly, for the Fourier expansion on S^1, we employ real modes,

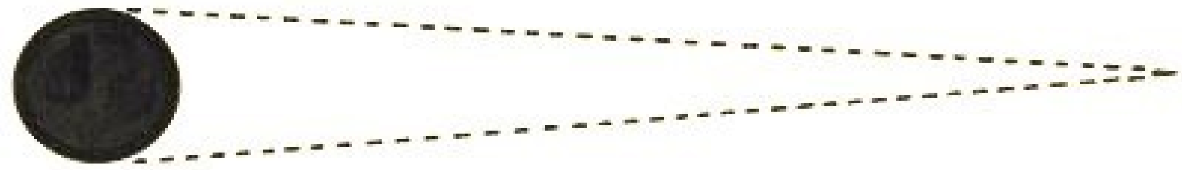
$$W_{n,\lambda} \rightarrow \left\{ W_0 = 1;\ \ W_{n,+} = \sqrt{2}\cos\omega_n x,\ \ W_{n,-} = \sqrt{2}\sin\omega_n x,\ \ \omega_n = 2\pi n,\ n \geq 1 \right\}.$$

- For simplicity, we will restrict ourselves to AXIAL perturbations with $l \geq 2$. The study of polar perturbations can be carried out along similar lines.

- There are no scalar axial perturbations. And we will see that axial vector pertubations are pure gauge in our system.

- We might include a perturbative **scalar field** in the analysis (with no zero mode). But it would only contribute with polar perturbations.

Perturbations



- Calling $\{\nu\}=\{n,\lambda,l,m\}$, we can expand the axial pertubations of the spatial metric, its momentum, and the shift vector as

$$\Delta h_{ab}dx^a dx^b=-2\sum h_1^\nu(t)X^m_{l\ A}(\theta,\phi)W_{n,\lambda}(x)dx\,dx^A+\sum h_2^\nu(t)X^m_{l\ AB}(\theta,\phi)W_{n,\lambda}(x)dx^A dx^B,$$

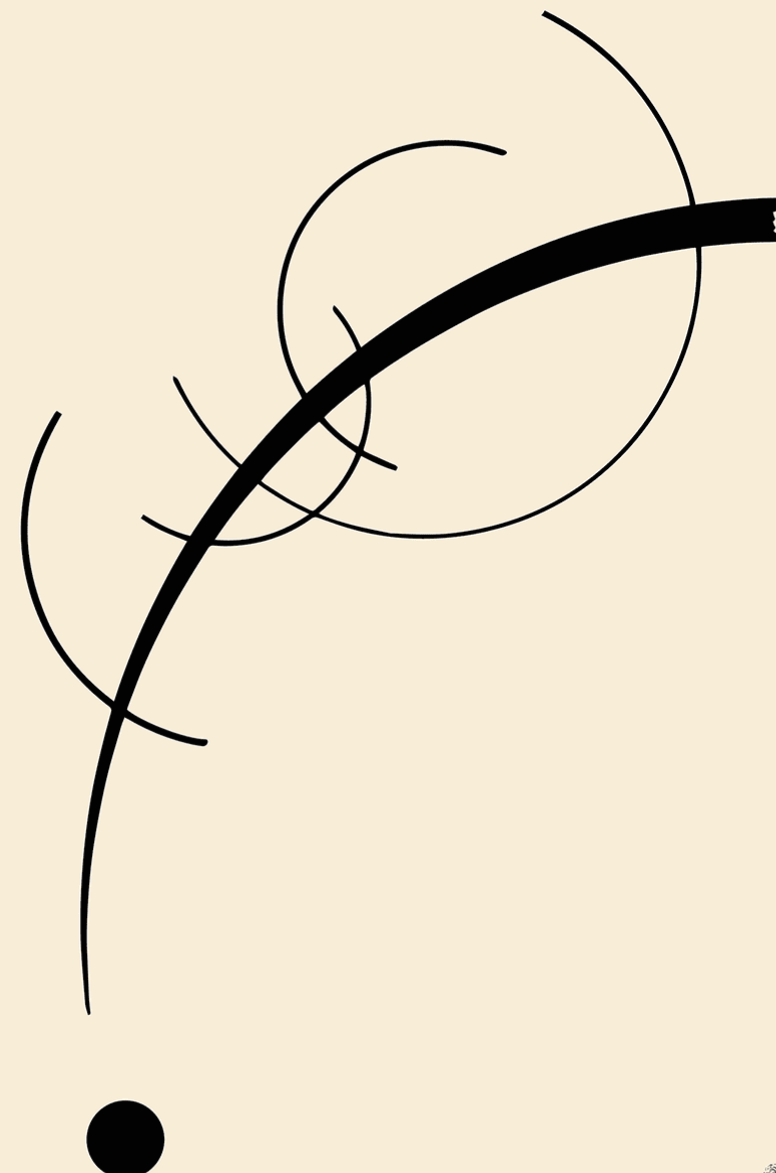
$$\Delta\left[\frac{p_{ab}}{\sqrt{h}}dx^a dx^b\right]=-\frac{4\pi p_b^2}{V}\sum\frac{p_1^\nu(t)}{l(l+1)}X^m_{l\ A}(\theta,\phi)W_{n,\lambda}(x)dx\,dx^A$$
$$+\frac{8\pi p_c^2}{V}\sum\frac{p_2^\nu(t)}{l(l+1)(l+2)}X^m_{l\ AB}(\theta,\phi)W_{n,\lambda}(x)dx^A dx^B,$$

$$N_a dx^a=-16\pi\sum h_0^\nu(t)X^m_{l\ A}(\theta,\phi)W_{n,\lambda}(x)dx^A.$$

- At second order, the contribution of the perturbations to the action has the form

$$\frac{1}{16\pi}\int d\tau\sum\left(\dot{h}_1^\nu p_1^\nu+\dot{h}_2^\nu p_2^\nu-h_0^\nu C_\nu^{ax}-\underline{N}\,\underline{H}_\nu^{ax}\right).$$

Perturbative diff. constraints   Hamiltonian constraint



Free Curve to the Point • Accompanying Sound of Geometric Curves • 1925 • Wassily Kandinsky

Gauge invariants

- Considering the background as fixed, we can perform a linear canonical transformation in the perturbations so that they are described by gauge invariant canonical pairs, and by the perturbative constraints and variables canonically conjugated to them,

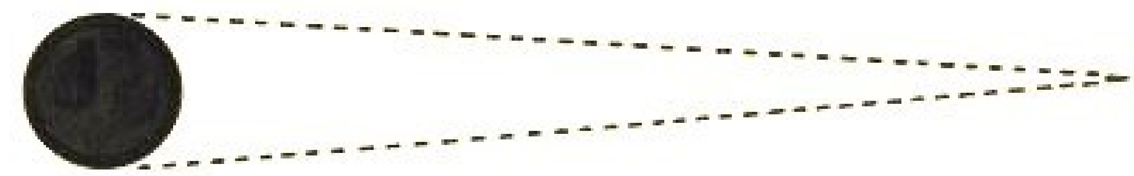
$$\{h_1^{\nu}, p_1^{\nu}, h_2^{\nu}, p_2^{\nu}\} \rightarrow \left\{\tilde{Q}_1^{\nu}, \tilde{P}_1^{\nu}, \tilde{Q}_2^{\nu}, \tilde{P}_2^{\nu} = -\frac{1}{2} C_{\nu}^{ax}\right\},$$

  with generating function

$$F^{\nu} = h_1^{\nu}\tilde{Q}_1^{\nu} + h_2^{\nu}\tilde{P}_2^{\nu} - \frac{\lambda\omega_n}{2} h_2^{\nu}\tilde{Q}_1^{\nu} + \frac{(l+2)!}{4(l-2)!\,p_c^2}(\Omega_b + \Omega_c)(h_2^{\nu})^2$$
$$- \frac{2l(l+1)}{p_b^2}\Omega_b\left(\frac{\omega_n^2}{4}(h_2^{\nu})^2 + \lambda\omega_n h_1^{\nu} h_2^{\nu}\right).$$

- The perturbative term in the Hamiltonian changes by the background evolution of this generating function, given by its Poisson bracket with the background Hamiltonian.

Gauge invariants



- The perturbative contribution to the action can be written

$$\int d\tau \left\{ (\underline{N}-\underline{\tilde{N}})\, H_{KS} + \frac{1}{16\pi}\sum \left(\dot{\tilde{Q}}_1^\nu \tilde{P}_1^\nu + \dot{\tilde{Q}}_2^\nu \tilde{P}_2^\nu\right) + \frac{1}{8\pi}\sum \tilde{h}_0^\nu \tilde{P}_2^\nu - \underline{\tilde{N}} \sum \underline{\tilde{H}}_\nu^{ax} \right\},$$

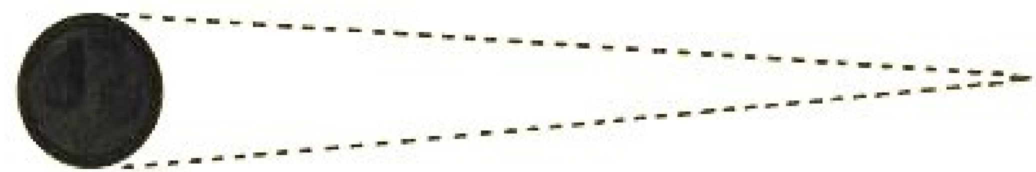
  where the new lapse includes quadratic perturbative terms and

$$\underline{\tilde{H}}_\nu^{ax} = \frac{p_b^2\, (\tilde{Q}_1^\nu)^2}{2l(l+1)} + \frac{l(l+1)}{2\, p_b^2}\left[8\,\Omega_b^2 + 8\,\Omega_b \Omega_c + 4\, p_b^2 + (l+2)(l-1)\, p_b^2\right](\tilde{P}_1^\nu)^2$$

$$+ \frac{(l-2)!}{2(l+2)!}\,\omega_n^2 p_c^2 \left[\tilde{Q}_1^\nu + \frac{4l(l+1)}{p_b^2}\,\Omega_b \tilde{P}_1^\nu\right]^2 + 2\,\Omega_b \tilde{Q}_1^\nu \tilde{P}_1^\nu .$$

- We can now eliminate the cross-terms in the perturbative contribution to the Hamiltonian and introduce the so-called Gerlach-Sengupta gauge invariant, generalized to any background and evaluated in the interior.

Gerlach-Sengupta



- This gauge invariant and its momentum are given by

$$Q_{GS}^{\nu}=-\sqrt{\frac{(l-2)!}{(l+2)!}}\left[\tilde{Q}_1^{\nu}+\frac{4l(l+1)}{p_b^2}\Omega_b\tilde{P}_1^{\nu}\right],$$

$$P_{GS}^{\nu}=-\sqrt{\frac{(l+2)!}{(l-2)!}}\tilde{P}_1^{\nu}+2\Omega_b\sqrt{\frac{(l-2)!}{(l+2)!}}\left[\tilde{Q}_1^{\nu}+4l(l+1)\frac{1}{p_b^2}\Omega_b\tilde{P}_1^{\nu}\right].$$

- After this canonical transformation, the perturbative term of the Hamiltonian adopts a simple form (easy to quantize!),

$$H_{\nu}^{ax,(GS)}=\frac{1}{2}(P_{GS}^{\nu})^2+\frac{\tilde{V}}{2}(Q_{GS}^{\nu})^2, \qquad \tilde{V}=\omega_n^2p_c^2+l(l+1)p_b^2-4(\Omega_b^2+p_b^2).$$

- In the **2-dimensional** set of spherical orbits, the generalized Gerlach-Sengupta master variable for **any background** is

$$Q_{GS}^{lm}(\tau,x)=\sum_{n,\lambda}Q_{GS}^{\nu}(\tau)W_{n,\lambda}(x).$$

Master equation

- The Gerlach-Sengupta modes satisfy

$$\left[\left((N^{-1}\partial_\tau)^2+\omega_n^2 p_c^2\right)+\left(l(l+1)\,p_b^2-4\,\Omega_b^2-4\,p_b^2\right)\right]Q_{GS}^{\nu}=0.$$

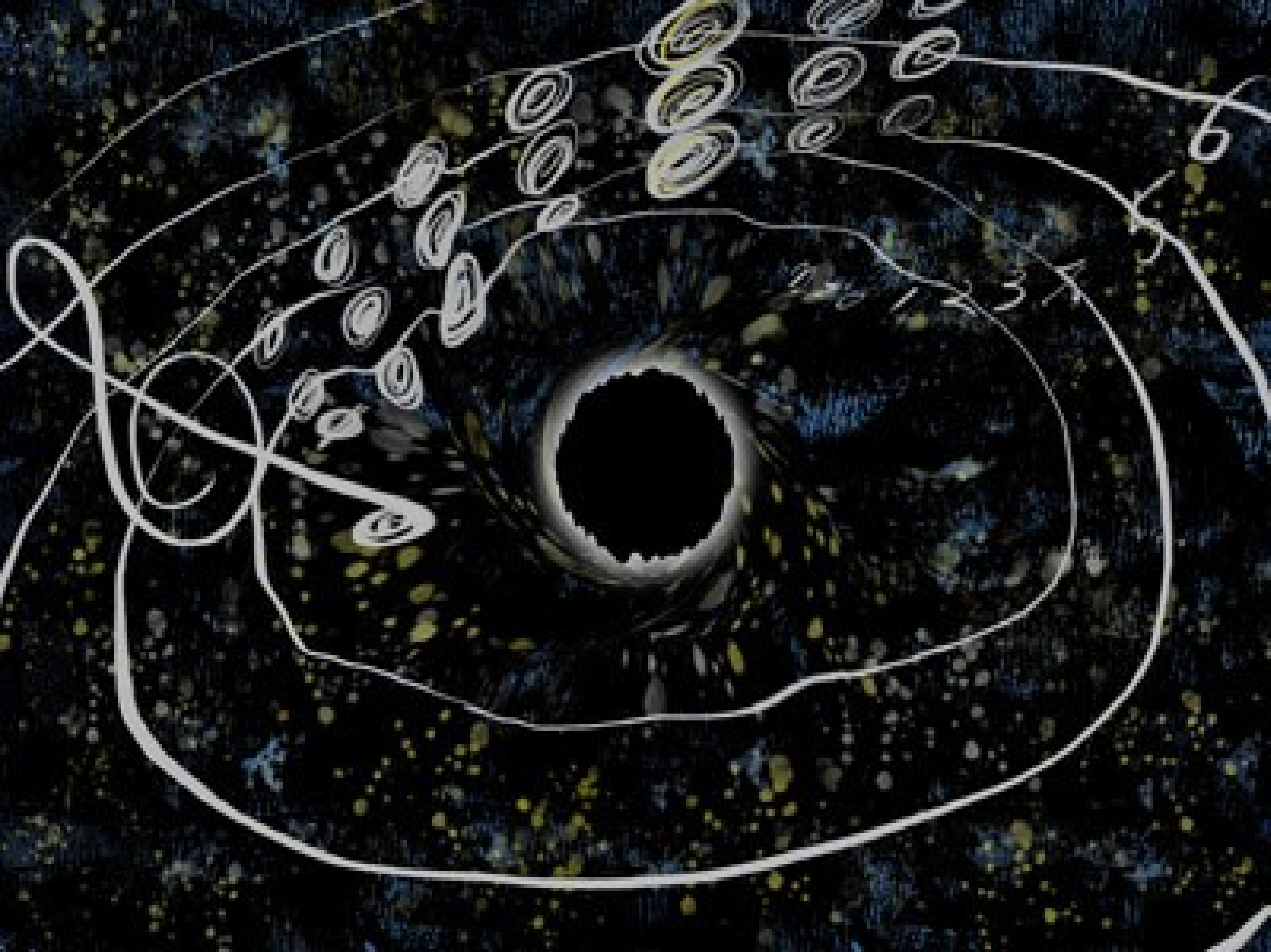
In terms of the Laplacian of the 2-dimensional metric induced on the set of spherical orbits and using the Gerlach-Sengupta master variable, this equation can be rewritten as

$$\boxed{\left[\Box_2+\frac{|p_c|}{p_b^2}\left(l(l+1)\,p_b^2-4\,\Omega_b^2-4\,p_b^2\right)\right]Q_{GS}^{lm}(\tau,x)=0.}$$

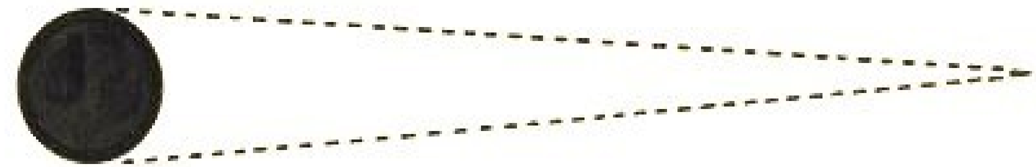
- Another useful gauge invariant is the **Cunningham-Price-Moncrief** invariant,

$$Q_{CPM}^{\nu}=2\sqrt{|p_c|}\,\sqrt{\frac{(l-2)!}{(l+2)!}}\,Q_{GS}^{\nu},$$

or, in two dimensions, the corresponding master variable.



Master equation



- In the Gerlach-Sengupta Hamiltonian, the contribution of ω_n^2 to the potential $\tilde{V}$ is not constant, but appears multiplied by p_c^2.

- We can render it constant by the **canonical** transformation

$$Q_\nu = \sqrt{|p_c|} Q_{GS}^\nu , \qquad P_\nu = \sqrt{|p_c|} P_{GS}^\nu - \frac{1}{2} \frac{\{|p_c|, H_{KS}\}}{|p_c|} Q_{GS}^\nu ,$$

where H_{KS} is the background Hamiltonian.

- The new perturbative contribution to the Hamiltonian constraint is

$$H_\nu^{ax}[N] = \frac{N|p_c|}{2} \left[(P_\nu)^2 + \left(\omega_n^2 - \frac{1}{p_c^2} \left[3\left(\Omega_b^2 + p_b^2\right) - l(l+1)\, p_b^2 \right] \right) (Q_\nu)^2 \right].$$

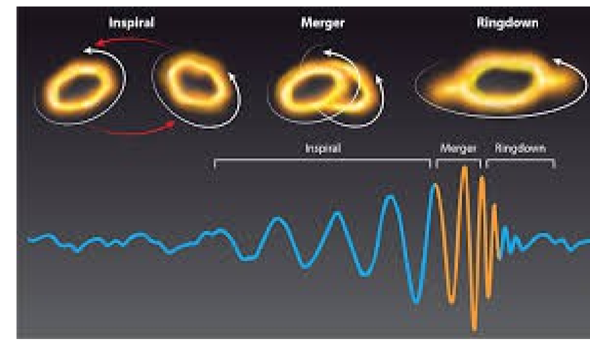
- This leads to $\left[\Box_2 - \left(\frac{l(l+1)}{|p_c|} - 3 \frac{\Omega_b^2 + p_b^2}{p_b^2 |p_c|} \right) \right] Q_\nu = 0.$

For Schwarzschild, this is the Regge-Wheeler master equation.

Darboux

- Defining $d\bar{\tau}=N\left|p_c\right|d\tau,$ the equation can be written

$$\left(\partial_{\bar{\tau}}^2+\omega_n^2-\frac{1}{p_c^2}\left[3\left(\Omega_b^2+p_b^2\right)-l\left(l+1\right)p_b^2\right]\right)Q_\nu=0.$$



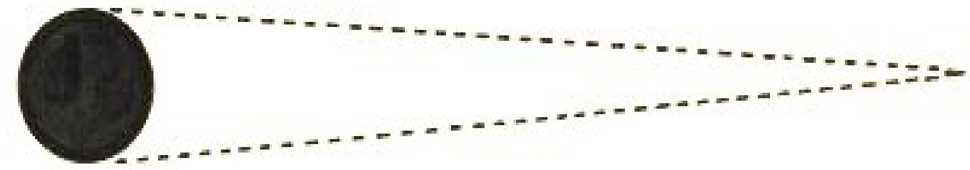
- For the time $\bar{\tau},$ the perturbative Hamiltonian has the form

$$\bar{H}_\nu^{ax}=\frac{1}{2}P_\nu^2+\frac{1}{2}(\omega_n^2+v_l)Q_\nu^2,$$

and the gauge invariant satisfies $Q''_\nu+(\omega_n^2+v_l)Q_\nu=0,$ where $f'=\partial_{\bar{\tau}}f.$

- <u>RECALL</u>:
A **Darboux** transformation $Q_\nu=\bar{Q}'_\nu+g_l\bar{Q}_\nu$ leads to the new equation
$\bar{Q}''_\nu+(\omega_n^2+V_l)\bar{Q}_\nu=0,$ with $V_l=v_l+2g'_l,$ if $g'_l+g_l^2+v_l=c_l.$

Canonical Darboux



- We want to show that Darboux transformations are just **canonical transformations** that respect the structure of the Hamiltonian!

- Consider a generic canonical transformation,

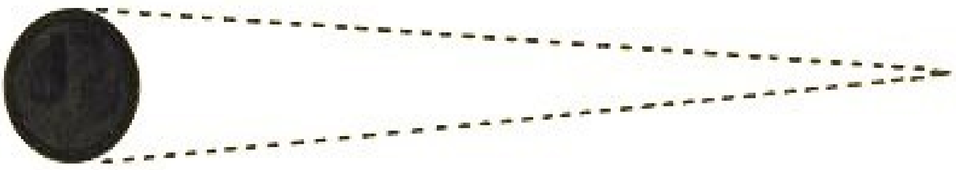
$$Q_\nu = A\bar{Q}_\nu + B\bar{P}_\nu, \qquad P_\nu = C\bar{Q}_\nu + D\bar{P}_\nu,$$

with $AD - BC = 1.$ (canonical!)

- The coefficients of the transformation are background (and thus time-) dependent. We assume $B \neq 0$ so that the transformation is not just a simple redefinition of the gauge invariant.

- With this transformation, we get a new perturbative contribution to the Hamiltonian. We ask that the momentum-configuration term vanish and the coefficient of the squared momentum be one half, **as before the transformation**!

Canonical Darboux



- Canonical transformation:

  $$Q_\nu = A\bar{Q}_\nu + B\bar{P}_\nu, \qquad P_\nu = C\bar{Q}_\nu + D\bar{P}_\nu,$$

  with $C = \dfrac{AD-1}{B}$. (canonical!)

- Hamiltonian: $\bar{H}_\nu^{ax} = \dfrac{1}{2}\bar{P}_\nu^2 + \dfrac{1}{2}(\omega_n^2 + V_l)\bar{Q}_\nu^2.$

  It is not difficult to see that this requires

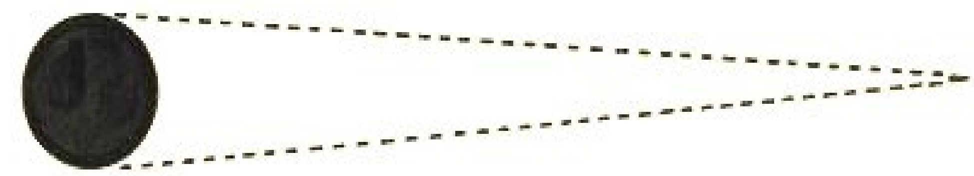
  $A = D - \acute{B}.$ (No cross term!)

  $\acute{g} + g^2 + (\omega_n^2 + v_l) = \dfrac{1}{B^2},$ with $g = \dfrac{D}{B}.$ (Momentum coeff.)

- If we do not want that the new potential depends on ω_n, we must take $B^{-2} = \omega_n^2 + c_l.$ Then, $A = D.$

  The canonical transformation is totally **fixed**, given c_l and a solution to the Ricatti equation!

Canonical Darboux



- Thus, Darboux transformations are **canonical transformations** (in the interior of the black hole) that respect the canonical form of the Hamiltonian for the perturbations. They are given by

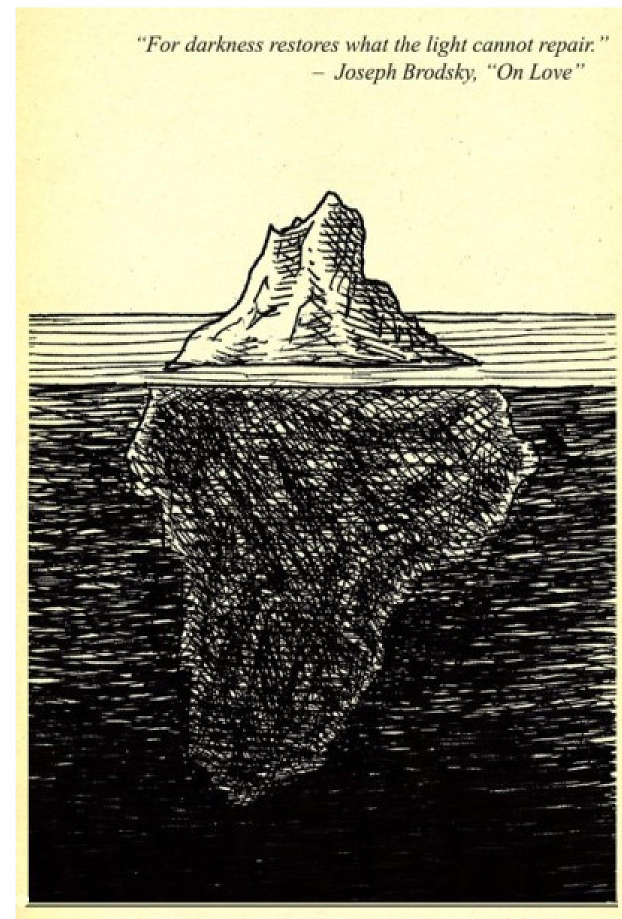
$$Q_\nu = \frac{g_l}{\sqrt{\omega_n^2 + c_l}} \bar{Q}_\nu + \frac{1}{\sqrt{\omega_n^2 + c_l}} \bar{P}_\nu ,$$

$$P_\nu = \left(\frac{g_l^2}{\sqrt{\omega_n^2 + c_l}} - \sqrt{\omega_n^2 + c_l} \right) \bar{Q}_\nu + \frac{g_l}{\sqrt{\omega_n^2 + c_l}} \bar{P}_\nu ,$$

where $g_l' + g_l^2 + v_l = c_l .$

- The new gauge invariant satisfies

$$\bar{Q}_\nu'' + (\omega_n^2 + v_l + 2 g_l') \bar{Q}_\nu = 0 .$$



Conclusions

- We have developed a **Hamiltonian formalism** for perturbations of non-rotating black holes, adapted to their interior.
- We have identified standard (axial) gauge invariants/master variables. Their dynamics involve quasinormal modes, relevant during **ringdown**.
- Although derived from General Relativity, all relations are expressed in terms of the background, and can be extended to **effective** ones.
- The formalism allows for an almost direct hybrid **LQC quantization**!
- Darboux transformations become **canonical transformations**!
- We can now investigate whether isospectroscopy is realized as a true unitary transformation in quantum field theory (including all l's).