Dynamical models of RBH

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The fate of a collapsing star

- In Newtonian theory, the total energy $E = Mc^2 - \frac{GM^2}{R}$ must eventually become negative.
- Thermal pressure cannot halt the collapse, as the star continues to radiate energy.
- Ultimately, an unlimited amount of gravitational energy will be released into the environment.
- The strongest evidence for the existence of a "cosmic censor" is the anthropic principle

The fate of a collapsing star: GR

- In Einstein's theory the total energy is always positive but the evolution of the star becomes causally disconnected from our universe *provided* an EH is formed.
- Event horizon conjecture (EHC): "An event horizon will form whenever a matter distribution whose stress energy tensor satisfies appropriate physical conditions, collapse until a trapped surface is formed" (Israel, Thorne hoop conjecture)
- Singularity theorems (Penrose, 1965)

Trapped surface

A trapped surface is a closed 2-space with the property that narrow beams of light orthogonal to it at any point decrease in area, at least initially, even when propagating outward

Dynamical Collapse

- Energy Momentum

$$T_{\mu\nu} = (\epsilon + p) u_\mu u_\nu + p g_{\mu\nu}$$

- Spherical symmetry

$$ds^2 = g_{ab} dx^a dx^b + r^2 d\Omega^2$$

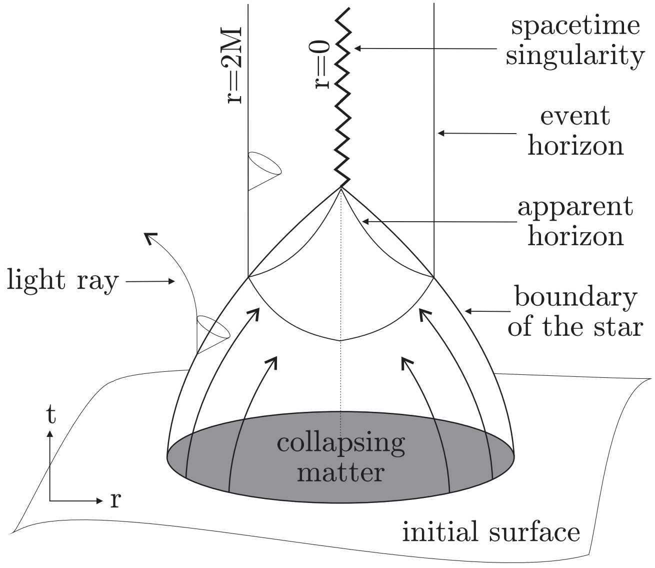


Figure 1: [Credit: Joshi and Malafarina]

Field Equations

$$ds^2 = -e^{2\nu}dt^2 + e^{2\psi}dr^2 + R^2 d\Omega^2$$

- Einstein's equations

$$\epsilon = \frac{F'}{R^2 R'}$$

$$p = -\frac{\dot{F}}{R^2 \dot{R}}$$

$$\dot{R}' = R'\dot{\psi} + \dot{R}\nu'$$

- Bianchi Identities

$$\nu' = -\frac{p'}{\epsilon + p}$$

- Scaling

$$\begin{cases} R = ra, \\ F = r^3 m, \\ e^{-2\psi} = \frac{R'^2}{1-kr^2} \end{cases}$$

Misner-Sharp Mass

$$F = R\left(1 - e^{-2\psi}R'^2 + e^{-2\nu}\dot{R}^2\right)$$

Homogeneous dust

$$ds^2 = -dt^2 + \frac{a^2}{1-kr^2}dr^2 + (ra)^2 d\Omega^2$$

- Einstein's equations

$$\epsilon = \frac{F'}{R^2 R'} = \frac{3m}{a^3}$$

$$p = -\frac{\dot{F}}{R^2 \dot{R}} = 0$$

$$\dot{R}' = R'\dot{\psi} + \dot{R}\nu' = \dot{a}$$

- Bianchi Identities

$$\nu' = 0$$

- Scaling

$$\begin{cases} R = ra, \\ F = r^3 m_0, \\ e^{-2\psi} = \frac{a^2}{1-kr^2} \end{cases}$$

Misner-Sharp Mass

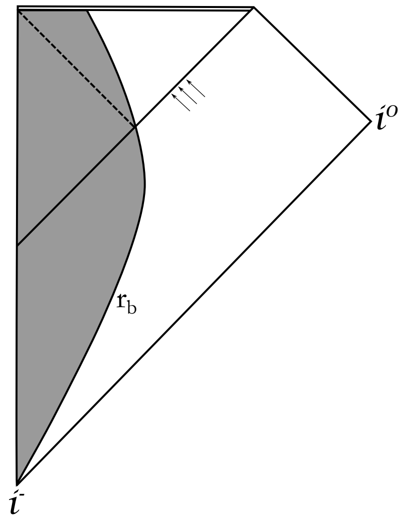
$$F = ra\left(kr^2 + r^2\dot{a}^2\right)$$

Oppenheimer-Snyder-Datt

Homogeneous marginally bound dust

$$ds^2 = -dt^2 + a^2 dr^2 + (ra)^2 d\Omega^2$$

- density $\epsilon(t) = \frac{m_0}{a^3}$
- pressure $p = 0$
- equation of motion $\dot{a} = -\sqrt{\frac{m_0}{a}}$
- solution $a(t) = \left(1 - \frac{3}{2}\sqrt{m_0 t}\right)^{2/3}$



Trapped surfaces

The condition for the formation of trapped surfaces is

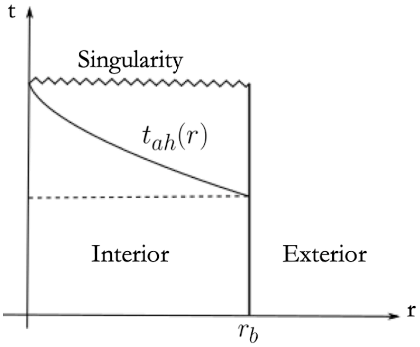
$$0 = 1 - \frac{F}{R}$$

For the exterior Schwarzchild metric this corresponds to $R = 2M$.
For the interior OSD the apparent horizon curve is given by

$$r_{\rm ah}(t) = \sqrt{\frac{a(t)}{m_0}} = \frac{1}{\dot{a}(t)}$$

which implies

$$t_{ah}(r) = \frac{2}{3m_0} - \frac{2}{3} m_0 r^3$$



fluid description

- Let us consider a self-gravitating fluid of matter described in the rest frame of the fluid by a four-velocity vector field u_μ with $u_\mu u^\mu = -1$.
- The fluid is characterized by an energy density ϵ, a pressure P and a rest mass density $\rho = m_0 n$ where n is the number density of particles.
- We assume mass conservation $(\rho u^\mu)_{;\mu} = 0$.

The local form of the first principle of thermodynamics reads:

$$d\left(\frac{\epsilon}{\rho}\right) = -Pd\left(\frac{1}{\rho}\right) + Td\left(\frac{s}{\rho}\right) \tag{1}$$

being s the entropy density. For a non-dissipative fluid $d(s/\rho) = 0$ and the above relation becomes

$$d\epsilon = \frac{P+\epsilon}{\rho} d\rho \tag{2}$$

One can then consider $\mathcal{L} = -\epsilon$ as lagrangian for this system and compute the variation of $\mathcal{L}$ under $\delta g^{\mu\nu}$ with the constraint $\delta n^{\mu} = 0$ being $n^{\mu} = n\sqrt{-g}u^{\mu}$ (no variation of baryon number flux). In this case, it is not difficult to show that

$$\delta n = \frac{n}{2}(u_{\mu}u_{\nu} + g_{\mu\nu})\delta g^{\mu\nu} \tag{3}$$

and, in particular, using (2) we deduce that

$$\delta\epsilon = \frac{(P+\epsilon)}{2}(u_{\mu}u_{\nu} + g_{\mu\nu})\delta g^{\mu\nu}. \tag{4}$$

Let us now consider a system described by the following action:

$$S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} \left(R + 2\chi(\varepsilon) \mathcal{L}(\epsilon) \right), \tag{5}$$

here $\chi(\epsilon)$ is a energy-dependent multiplicative coupling. Note that for $\chi = 8\pi G_N = const$ classical GR is recovered. In this case in fact variation of (16) yields

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu}, \tag{6}$$

with

$$T_{\mu\nu} = \epsilon u_\mu u_\nu + P h_{\mu\nu} \tag{7}$$

where $h_{\mu\nu}$ is the projection tensor on the 3-hypersurfaces orthogonal to u_μ, $h_{\mu\nu} = u_\mu u_\nu + g_{\mu\nu}$.

For $\chi = \chi(\epsilon)$ we finds

$$\frac{1}{\sqrt{-g}}\delta(2\sqrt{-g}\chi\varepsilon) = 2(\chi\varepsilon)_{,\varepsilon}\,\delta\varepsilon - \chi\varepsilon g_{\mu\nu}\delta g^{\mu\nu}, \tag{8}$$

and in this case, the total variation of the action leads to the modified Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \widetilde{T}_{\mu\nu}, \tag{9}$$

where

$$\widetilde{T}_{\mu\nu} = (\varepsilon\chi)_{,\varepsilon}\,T_{\mu\nu} + (\varepsilon^2\chi_{,\varepsilon})g_{\mu\nu}, \tag{10}$$

is an effective energy-momentum tensor characterized by an effective energy density $\widetilde{\epsilon}$ and $\widetilde{P}$

$$\tilde{\varepsilon} = (\chi\varepsilon)_{,\varepsilon}\varepsilon - \varepsilon^2\chi_{,\varepsilon} = \chi(\varepsilon)\varepsilon, \quad (11)$$

$$\tilde{P} = (\chi\varepsilon)_{,\varepsilon}P + \varepsilon^2\chi_{,\varepsilon}. \quad (12)$$

Looking at the effective field equations (10) we can identify the terms

$$G(\varepsilon) = (\chi\varepsilon)_{,\varepsilon} \quad (13)$$

$$\Lambda(\varepsilon) = -\varepsilon^2\chi_{,\varepsilon} \quad (14)$$

as a density dependent gravitational constant $G(\varepsilon)$ and a cosmological constant $\Lambda(\varepsilon)$.

running G_N

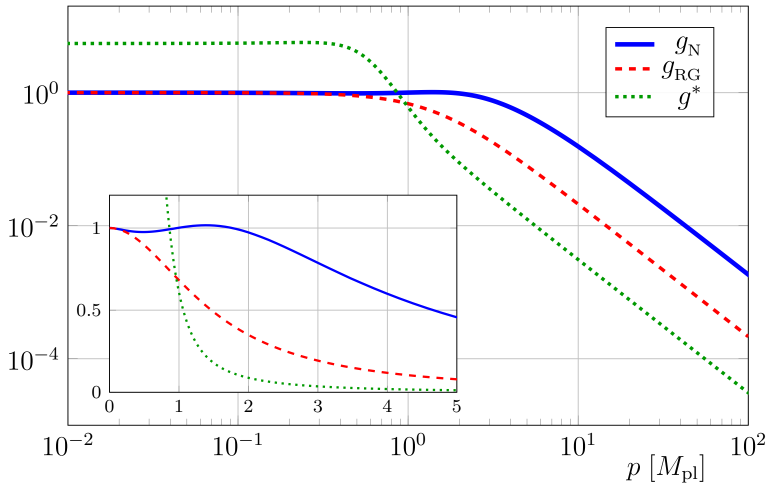


Figure 2: [AB, T. Denz, J. Pawlowski, M. Reichert, *SciPost, 2022*]

UV scaling

At large external momenta $G_N(p, k=0) \sim G(p=0, k)$
$G \propto 1/p^2 \propto 1/k^2 \propto 1/\epsilon$

- We choose

$$G(\epsilon) = \frac{G_N}{1 + \xi\epsilon}$$

- ξ is a mass-scale of the order $\sim 1/m_{\rm pl}^4$ whose precise value is not important
- We find

$$\chi(\epsilon) = \frac{\log(1+\xi\epsilon)}{\xi\epsilon}, \quad \Lambda(\epsilon) = \frac{\log(1+\xi\epsilon)}{\xi} - \frac{\epsilon}{1+\xi\epsilon}. \quad (15)$$

Equation of motion

$$\dot{a} = -\sqrt{\frac{\log(1 + 3m_0\xi/a^3)}{3\xi}a^2 - K}.$$

At large times, the scale factor behaves as:

$$a(t) \sim e^{-t^2/4\xi}, \quad t \to \infty,$$

indicating that $a = 0$ is never reached at any finite time, and the spacetime is geodesically complete

Israel matching conditions

- We consider the matching across a comoving boundary $r = r_b$ in the interior, which corresponds to a collapsing boundary $C_b(t) = C(t, r_b) = r_b a(t)$.
- The induced metric on the matching surface Σ in comoving coordinates can be expressed as:

$$ds^2_\Sigma = -dt^2 + r_b^2 a^2 d\Omega^2$$

- For the exterior, we consider a generic static and spherically symmetric line element

$$ds^2 = -f(R)dT^2 + \frac{1}{f(R)}dR^2 + R^2 d\Omega^2,$$

here $f = 1 - 2M(R)/R$, and we assume a continuous matching between the two geometries.

The continuity condition uniquely determines the form of $M(R)$ in the exterior. Specifically, if the collapsing boundary is parametrized by $R = R_b(T)$, the induced metric on the boundary becomes:

$$ds^2_\Sigma = -\left[f(R_b) - f(R_b)^{-1}\left(\frac{dR_b}{dT}\right)^2\right] dT^2 + R_b^2 d\Omega^2.$$

The matching conditions for the metric functions on the boundary surface Σ immediately provide the relation between t and T on Σ and the condition $R_b(T(t)) = r_b a(t)$.

The second fundamental form for the interior metric in comoving coordinates is

$$K_{tt}^{-} = 0, \quad K_{\theta\theta}^{-} = r_b a \sqrt{1 - K r_b^2}.$$

From the extrinsic curvature on the exterior we obtain

$$K_{tt}^{+} = -\frac{1}{2}\frac{2\ddot{R}_b + f_{,R}(R_b)}{\Delta(R_b)}, \quad K_{\theta\theta}^{+} = R_b \Delta(R_b),$$

with $\Delta(R_b) = \sqrt{1 - 2M(R_b)/R_b + \dot{R}_b^2}$

The exterior spacetime

On imposing

$$[K_{tt}] = K_{tt}^{+} - K_{tt}^{-} = 0, \quad [K_{\theta\theta}] = K_{\theta\theta}^{+} - K_{\theta\theta}^{-} = 0$$

the functional form of $M(R)$ can be obtained:

$$M(R) = \frac{R^3}{6\xi} \log\left(1 + \frac{6M_0\xi}{R^3}\right)$$

with $m_0 r_b^3 = 2M_0$ from the matching conditions.

The metric function

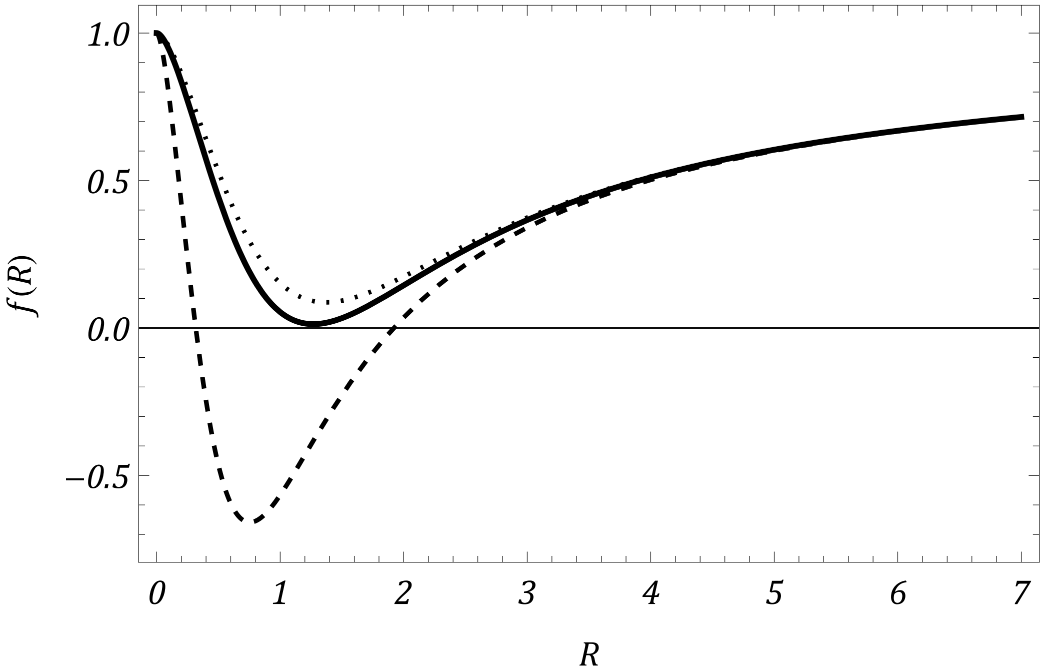


Figure 3: The behavior the metric function $f(R)$ for different values of the parameter ξ for $M_0 = 1$.

Classical Limit

The classical limit is recovered for $\xi \to 0$ (or equivalently, for $R \to \infty$), yielding:

$$M(R) = M_0 - \frac{3M_0^2\xi}{R^3} + \frac{12M_0^3\xi^2}{R^6} + O(\xi^3).$$

In the low-energy limit, this expression converges to the Schwarzschild solution. Notably, in the small-R regime, $M(R)$ behaves as:

$$M(R) = \frac{1}{6\xi}R^3 \log\left(\frac{6M_0\xi}{R^3}\right) + \frac{R^6}{36M_0\xi^2} + O(R^7).$$

As $R \geq R_b = r_b a(t)$ and with $a(t) > 0$ the singularity is never reached.

[AB, D.Malafarina, A.Panassiti, PRL 2024]

AS without AS

Known difficulties of the AS scenario

No accepted scaling solution, no definite proof the Reuter Fixed point exists, no independent evidence, no reliable computation of the effective action in non-trivial background, no clear way to restore background independence, ... still long (maybe too long for me...) way to go...

AS without RG flow

[...] Passing to the continuous theory let us discuss if we can use the Einstein action as a starting point for the quantum theory. The problem here is non-renormalizability: the quantum corrections, diverge at very high energies (and the energies of the virtual particles can be arbitrary high). A possible way out is to conjecture that the coupling constant in gravity is scale dependent and tends to zero at small distances. This "antiscreening" of the gravitational interaction is quite natural since the larger is the cloud of the virtual particles, the stronger the gravitational force is.

[A.M. Polyakov, *A few projects in String Theory*, 1992]

Let us now assume a different functional form for $G(\epsilon)$

$$G(\epsilon) = \frac{G_N}{1 + (\xi\epsilon)^2}$$

Let us now assume a different functional form for $G(\epsilon)$

$$G(\epsilon) = \frac{G_N}{1 + (\xi\epsilon)^2}$$

In this case $a(t) \sim e^{-t}$ and the external metric reads

$$f(r) = 1 - \frac{r^2}{2\xi}\mathrm{ArcTan}\left(\frac{4M_0\xi}{r^3}\right)$$

with $4M_0 = m_0 r_b^3$ from the matching conditions.

De Sitter core

$$\lim_{r\to 0} f(r) = 1 - \frac{\pi r^2}{4\xi} + \frac{r^5}{8M_0\xi^2} + O\left(r^7\right)$$

$$\lim_{r\to \infty} f(r) = \left(1 - \frac{2M_0}{r}\right) + \frac{32{M_0}^3\xi^2}{3r^7} + O\left(1/r^9)\right)$$

The metric function

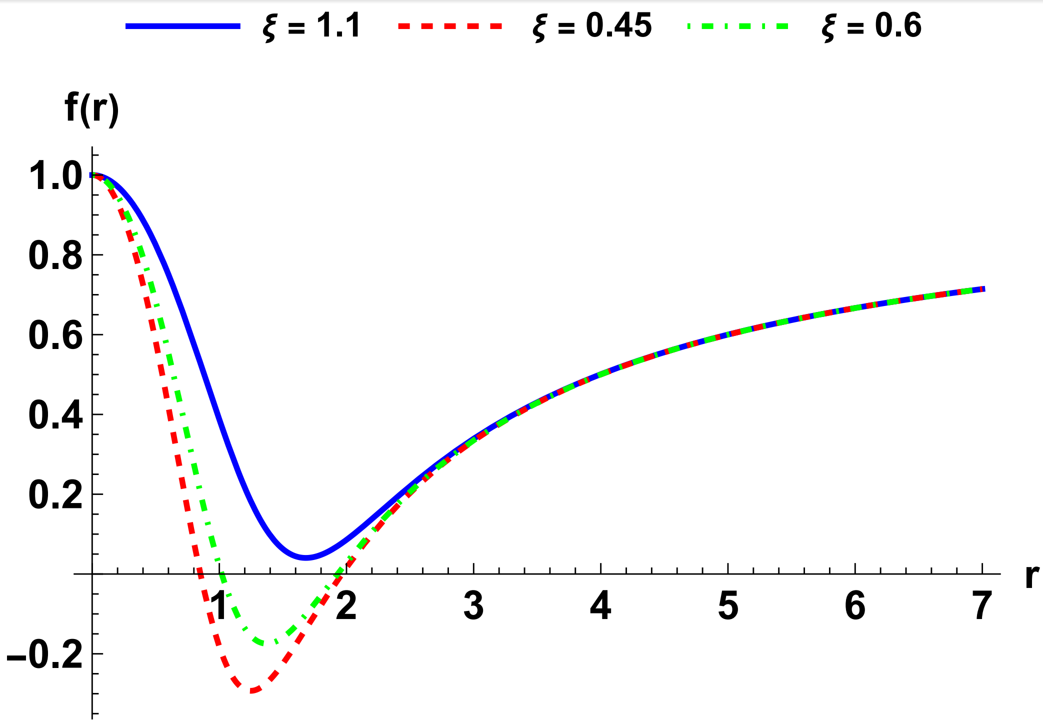


Figure 4: The behavior the metric function $f(r)$ for different values of the parameter ξ for $M_0 = 1$.

Bonanno, Spina, Konoplya, in preparation

Using the exact beta function from PT flow equation

$$f(R) = \frac{3M_0^2 + qR^4 - M_0\sqrt{9M_0 + q^2R^6}}{qR^4} + $$

$$+\frac{2}{3}qR^2 \text{arctanh}\left(\frac{\left(q - \sqrt{q^2 + \frac{9M_0^2}{R^6}}\right)R^3}{3M_0}\right)$$

Horizon structure

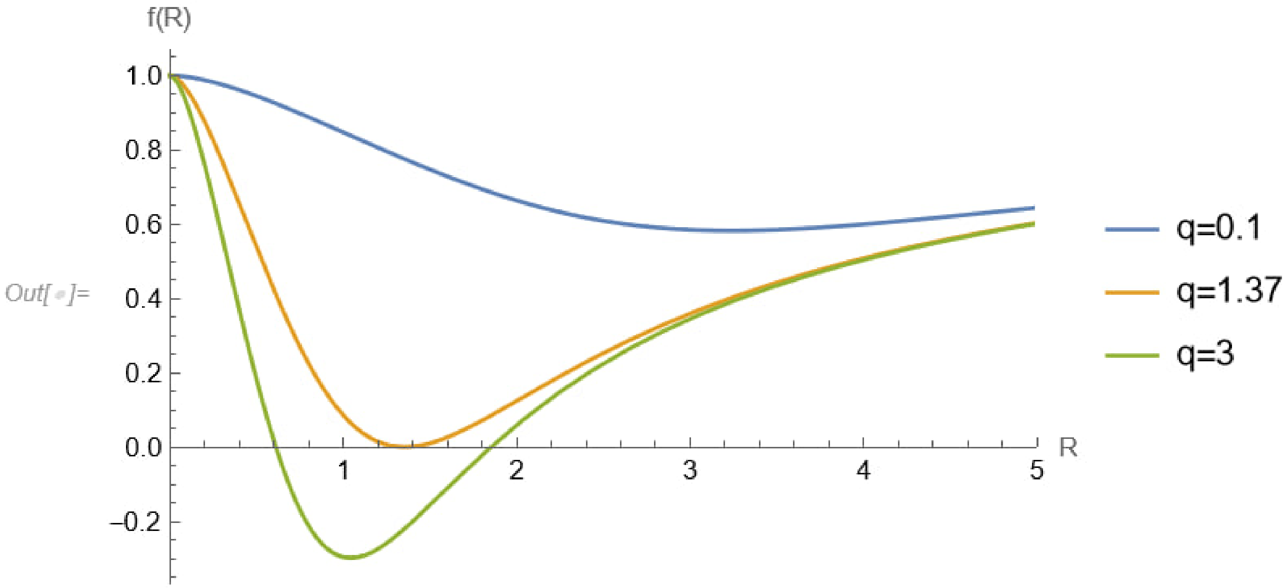


Figure 5: The behavior of the metric function $f(R)$ for different values of the parameter q for $M_0 = 1$. There are an inner and an outer horizon for $q > 1.37$, one horizon for the critical value $q = 1.37$ and no horizon for minor values of this

Bonanno, Spina, Konoplya, in preparation

p	q	Leaver Re[ωM]	WKB Re[ωM]	WKB Error [%]	Dev
80	10.57	0.37602	0.37602	0.00135	
20	3.12	0.38208	0.38213	0.01386	
10	1.93	0.38787	0.38835	0.1242	
9	1.82	0.38883	0.38909	0.0660	
8	1.71	0.38988	0.38999	0.0284	
7	1.61	0.39100	0.39101	0.0030	
6	1.52	0.39216	0.39209	0.0172	
5	1.44	0.39325	0.39317	0.0210	
4.75	1.42	0.39349	0.39345	0.0103	
4.5	1.40	0.39372	0.39375	0.0078	
4.25	1.39	0.39391	0.39390	0.0034	
4	1.38	0.39407	0.39405	0.0053	
3.75	1.37	0.39419	0.39420	0.0026	

p	q	Leaver Im[ωM]	WKB Im[ωM]	WKB Error [%]	Dev
80	10.57	-0.08789	-0.08787	0.0241	
20	3.12	-0.08468	-0.08457	0.1360	
10	1.93	-0.08079	-0.08106	0.3306	
9	1.82	-0.08003	-0.08043	0.4947	
8	1.71	-0.07916	-0.07948	0.4000	
7	1.61	-0.07816	-0.07841	0.3245	
6	1.52	-0.07704	-0.07727	0.3030	
5	1.44	-0.07590	-0.07607	0.2186	
4.75	1.42	-0.07563	-0.07574	0.1487	
4.5	1.40	-0.07539	-0.07540	0.0126	
4.25	1.39	-0.07516	-0.07523	0.0897	
4	1.38	-0.07498	-0.07505	0.0900	
3.75	1.37	-0.07484	-0.07486	0.0330	

Embedding

Let us consider the action

$$S = \int d^4x \sqrt{-g} P(\phi, X), \tag{16}$$

where $X = -\frac{1}{2}\nabla_a \phi \nabla^a \phi$ and our metric signature is $(-, +, +, +)$. A canonical scalar field is given by the choice $P = X - V(\phi)$. The equations of motion for the scalar field are found by varying the action with respect to ϕ to yield

$$\nabla_\mu (P_{,X} \nabla^\mu \phi) + P_{,\phi} = 0. \tag{17}$$

The energy-momentum tensor of the scalar field is also easily shown to be

$$T_{\mu\nu} = P_{,X} \partial_\mu \phi \partial_\nu \phi + P g_{\mu\nu}. \tag{18}$$

Notice that it takes the form of a perfect fluid with pressure $P_\phi = P$, density $\rho_\phi = 2XP_{,X} - P$ and 4-velocity $u_\mu = \partial_\mu \phi / \sqrt{2X}$.

Non-canonical kinetic terms into the theory comes at the price of potential instabilities. In particular classical stability of the theory is guaranteed if the sound speed,

$$c_s^2 = \frac{P_{,X}}{P_{,X} + 2XP_{,XX}}, \tag{19}$$

is always positive.
We write a general static spherically symmetric metric as

$$ds^2 = -A(\rho)dt^2 + \frac{d\rho^2}{A(\rho)} + r^2(\rho)d\Omega^2 \tag{20}$$

where $r(\rho)$ is the radius of the 2-sphere $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$.
The Einstein equations thus read

$$G_{\mu\nu} = 8\pi G_0(P_{,X}\partial_\mu\phi\partial_\nu\phi + Pg_{\mu\nu}) \tag{21}$$

so that $G_0 P_{,X} \equiv G_N(X,\phi)$ can be interpreted as a dynamical coupling.

Inspired by previous discussion we assume

$$P_{,X} = G_N(\phi) = \frac{\alpha}{1+\epsilon\phi} \tag{22}$$

where α and ϵ are free parameters. We immediately obtain

$$P(X,\phi) = \frac{\alpha X}{1+\epsilon\phi} - V(\phi) \tag{23}$$

so that, at least intuitively, large and positive values of ϕ will dynamically suppress or reduce the effective coupling between geometry and the scalar field.

From the tt and $\theta\theta$ component of the Einstein equations (21) we can eliminate $V(\phi)$ and obtain an equation for $A(\rho)$

$$A''(\rho) = \frac{2A(\rho)\left(r(\rho)r''(\rho) + r'(\rho)^2\right) - 2}{r(\rho)^2} \tag{24}$$

We now *assume*

$$r(\rho) = \sqrt{\rho^2 + b^2} \tag{25}$$

so that (24) can be readily integrated

$$A(\rho) = -\frac{3\pi m\rho^2}{2b^3} + \frac{3m\rho}{b^2} + \frac{3m\left(b^2+\rho^2\right)\tan^{-1}\left(\frac{\rho}{b}\right)}{b^3} - \frac{3\pi m}{2b} + 1 \tag{26}$$

and we have set an integration constant to zero in order to recover asymptotic flatness.

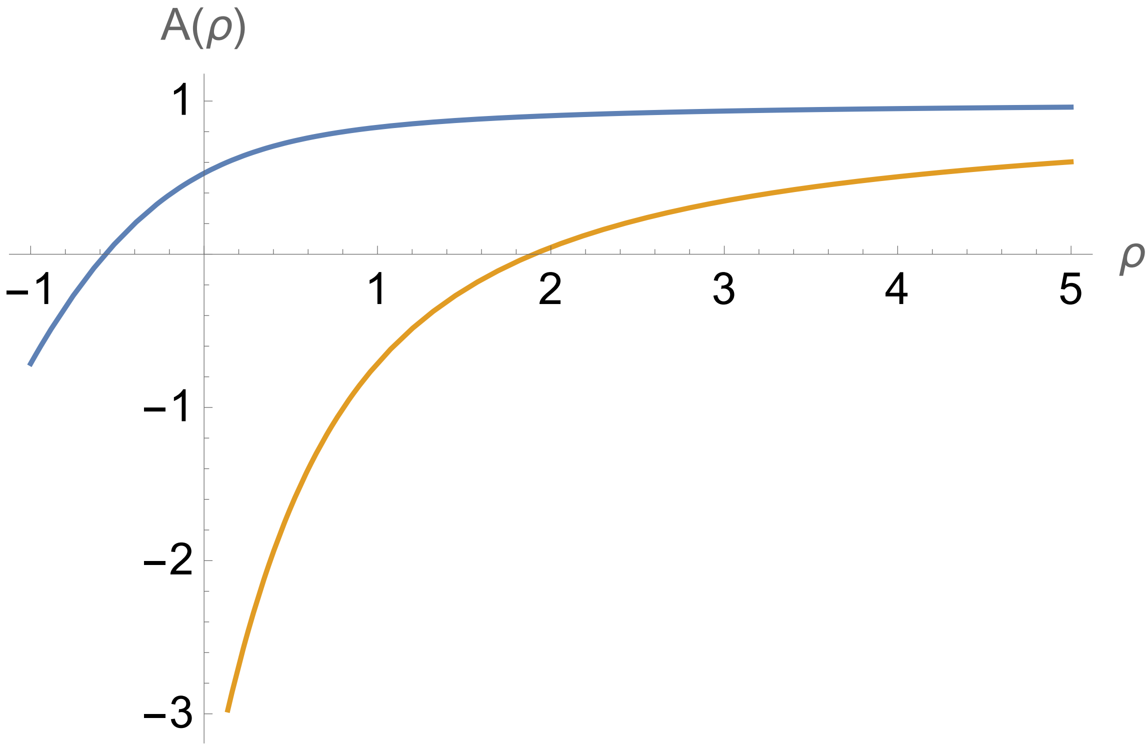


Figure 6: $A(\rho)$ for $b = 1$ and $m = 1$ (blu line) and $m = 1/10$, orange line

In particular at large $\rho \sim r$ we have

$$A(\rho) = 1 - \frac{2m}{\rho} + \frac{2b^2 m}{5\rho^3} - \frac{6\left(b^4 m\right)}{35\rho^5} + O\left(\left(\frac{1}{\rho}\right)^7\right) \tag{27}$$

Similarly, from the tt, $\rho\rho$ and $\theta\theta$ one can eliminate $V(\phi)$ and obtain an equation for the scalar field ϕ

$$2r''(\rho) + \frac{\alpha\kappa r(\rho)\phi'(\rho)^2}{\epsilon\phi(\rho)+1} = 0 \tag{28}$$

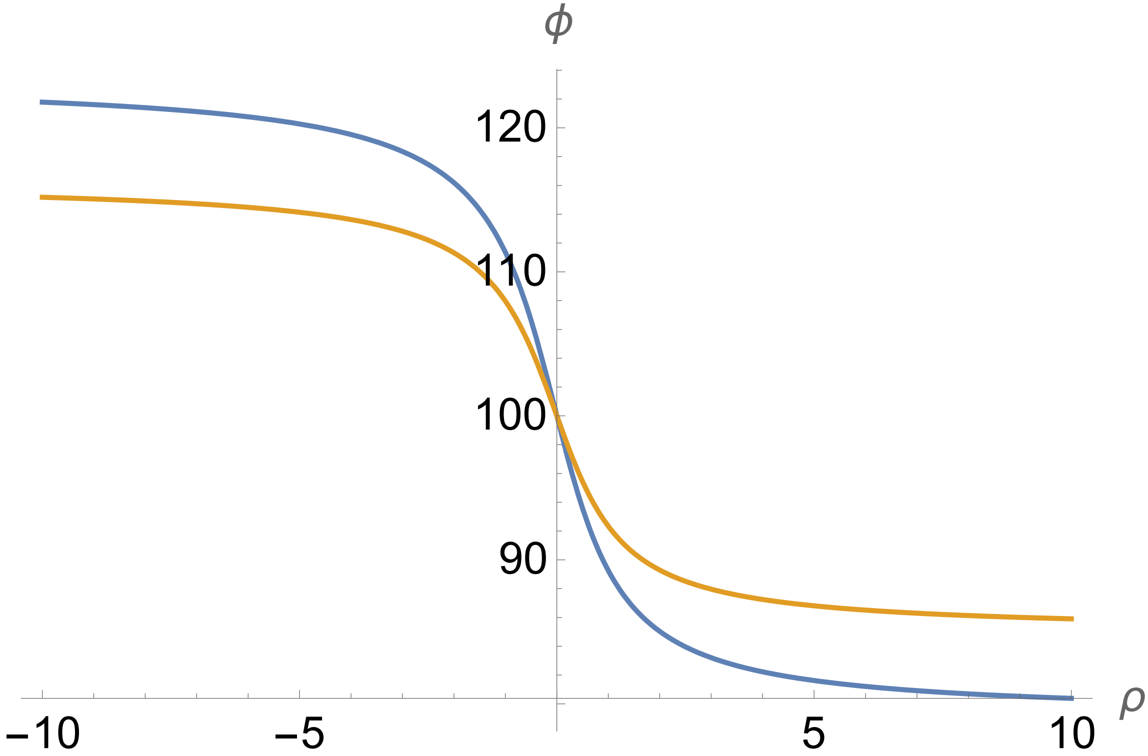


Figure 7: The evolution of ϕ as a function of ρ

By inserting the explicit form of $r(\rho)$ we arrive at two possible solutions

$$\phi_\pm = \pm\phi_0 - \frac{\sqrt{2}\sqrt{-\epsilon\phi_0 - 1}\tan^{-1}\left(\frac{\rho}{b}\right)}{\sqrt{\alpha}} - \frac{\epsilon\tan^{-1}\left(\frac{\rho}{b}\right)^2}{2\alpha} \qquad (29)$$

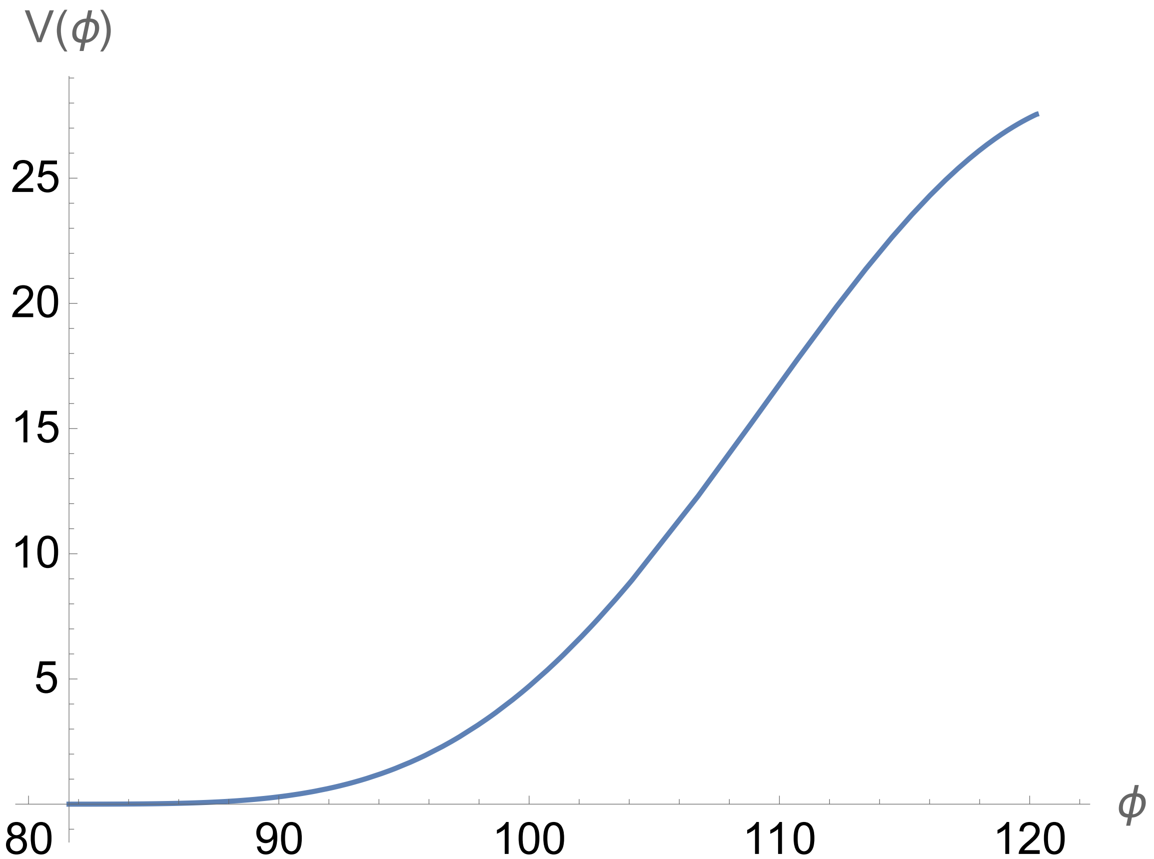
where ϕ_0 is the value of ϕ at $\rho = 0$.

We can now obtain the explicit form of the potential by eliminating $\phi'(\rho)$ and $A'(\rho)$ from the tt, $\rho\rho$ and $\theta\theta$ equations,

$$-2V(\phi) = A''(\rho) + \frac{2A'(\rho)r'(\rho)}{r(\rho)} \tag{30}$$

which gives

$$V(\rho) = \frac{3m\left(-2\left(b^2+3\rho^2\right)\tan^{-1}\left(\frac{\rho}{b}\right)+\pi b^2-6b\rho+3\pi\rho^2\right)}{2b^3\left(b^2+\rho^2\right)} \tag{31}$$


V(ϕ)
25
20
15
10
5
80
90
100
110
120
ϕ

Conclusions

- Regular BH can be the outcome of gravitational collapse if $G \to 0$ at high density
- We do not need exotic matter
- A new effective field equations are derived from a Lagrangian
- Numerical simulations are finally possible with standard Lagrangian code (eg. Musco, Miller, Rezzolla 2005)