Preserving Gauge Symmetries in Cosmology via Loop Quantum Gravity

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Fiber bundle structure of Ashtekar variables

A Yang-Mills approach

The natural language of Yang-Mills theories is the principal bundle formalism.

Principal G-bundle (P, π, M)

- P a smooth manifold called «fiber bundle»,
- M a smooth manifold called «base manifold»,
- $\pi: P \to M$ a smooth projection,

Such that:

- G has a free and transitive action on $\pi^{-1}(x), x \in M$. Hence, $\pi^{-1}(x) \cong G$,
- $P/G \cong M$.

Gauge field encoded in a connection

$\omega \in \Omega^1(P, \mathfrak{g})$

Such that:

- $\omega(X^A) = A,\ A \in \mathfrak{g}$
- $R_a^* \omega = \mathrm{Ad}_{a^{-1}} \omega,\ a \in G$

Fiber bundle structure of Ashtekar variables

Orthonormal frame bundle

A principal bundle appears from the tetrad formulation: orthonormal frame bundle

$$\begin{array}{c} P^{SO}(M) \\ \downarrow \\ M \end{array}$$

$$\boxed{g \text{ metric on } M \iff P^{SO}(M)}$$

$$P_x^{SO}(M) = \{h: \mathbb{R}^n \to T_x M \mid h \text{ oriented isometry}\} = \{\text{collection of orthonormal basis of } T_x M\}$$

A choice of the tetrad is the choice of a section $e: M \to P^{SO}(M)$

A Lorentz connection ω is a connection on $P^{SO}(M)$ (metric-compatible)

E.g. in n=4 the tetrad e_α^μ is $h(\varepsilon_\alpha) \doteq e_\alpha = e_\alpha^\mu \partial_\mu$ and so $g(e_\alpha, e_\beta) =< \varepsilon_\alpha, \varepsilon_\beta > = \eta_{\alpha\beta}$

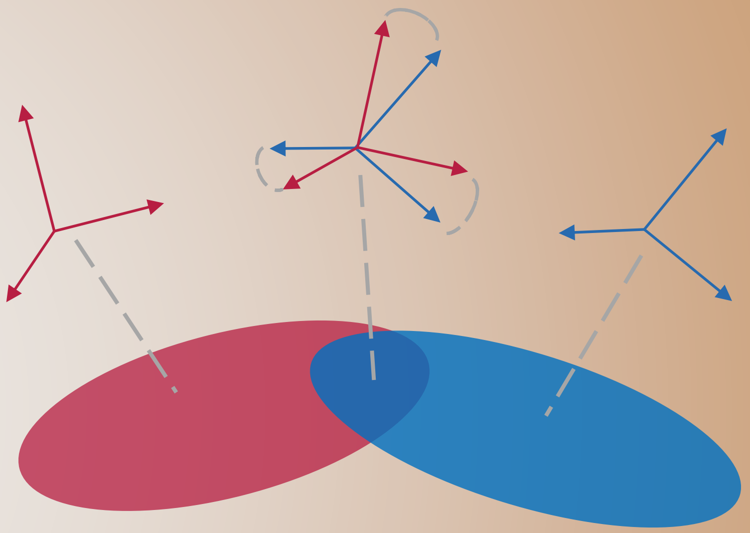
$\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ canonical basis of $\mathbb{R}^4$

Fiber bundle structure of Ashtekar variables

The dimension 3 - Dreibein

3-dimensional Riemannian manifold (Σ, q)

$$P^{SO}(\Sigma) \\ \downarrow \\ \Sigma$$



$P^{SO}(\Sigma)$ is a principal $SO(3)$-bundle $\Rightarrow$ Gauge group $SO(3)$

A choice of a triad $e_i^a(x)$ is equivalent to the choice of a section $e\colon \Sigma \to P^{SO}(\Sigma)$, i.e. $\pi(e_x) = x$

$$e_x(\varepsilon_i) = e_i^a(x)\partial_a$$

Fiber bundle structure of Ashtekar variables

The dimension 3 – SU(2) appears

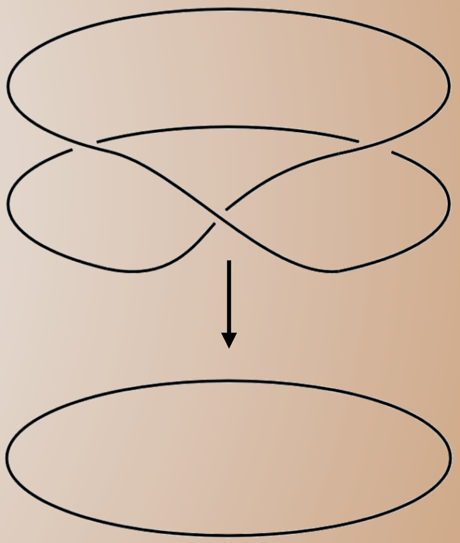
3-dimensional Riemannian manifold (Σ, q)

$$\begin{array}{c} P^{SO}(\Sigma) \\ \downarrow \\ \Sigma \end{array}$$

Spin Structure

$$\begin{array}{c} P^{Spin}(\Sigma) \\ \downarrow \\ P^{SO}(\Sigma) \\ \downarrow \\ \Sigma \end{array}$$

- Principal $SU(2)$-bundle $P^{Spin}(\Sigma)$
- Double-covering ρ: $P^{Spin}(\Sigma) \to P^{SO}(\Sigma)$



Lift of the dreibein $\bar{e}$ s.t. $e = \rho \circ \bar{e}$ (It is not unique, but it does not matter)

Fiber bundle structure of Ashtekar variables

The dimension 3 – Connection

3-dimensional spin manifold (Σ, q)

$$\begin{array}{c} P^{Spin}(\Sigma) \\ \downarrow \\ P^{SO}(\Sigma) \\ \downarrow \\ \Sigma \end{array}$$

A spin connection is represented by a 1-form ω with values in $\mathfrak{su}(2)$ on $P^{Spin}(\Sigma)$

$$\Updownarrow \quad \omega = \rho^* \varpi$$

A metric-compatible connection is represented by a 1-form ϖ with values in $\mathfrak{so}(3)$ on $P^{SO}(\Sigma)$

$\boxed{\textbf{Ashtekar connection}\text{: Local field of a spin connection } A = \bar{e}^* \omega \text{ is a 1-form with values in } \mathfrak{su}(2) \text{ on } \Sigma}$

Fiber bundle structure of Ashtekar variables

The dimension 3 – Electric field

3-dimensional spin manifold (Σ, q)

$$
\begin{array}{c}
P^{Spin}(\Sigma) \\
\downarrow \\
P^{SO}(\Sigma) \\
\downarrow \\
\Sigma
\end{array}
$$

The space of Ashtekar connection $\mathcal{A}$ is independent on the choice of spin structure and metric. Canonically it is the space of SU(2) connections

Collect the metric data in the electric field E

The **electric field** E can be interpreted as a 2-form on Σ with values in $ad^* P^{Spin}(\Sigma)$. Described locally by:

$$E = \star\, e_a^i dx^a \tau_i$$

Cosmological sector of GR in Ashtekar variables

A Yang-Mills approach

[Brodbeck '96]
[Bojowald, Kastrup '00]
[MB '24]
[MB '24]

$$P^{Spin}(\Sigma) \downarrow P^{SO}(\Sigma) \downarrow \Sigma$$

Yang-Mills variables

Connection ω is a 1-form on $P^{Spin}(\Sigma)$ with value in the Lie algebra of SU(2)

Dreibein e is a section in $P^{SO}(\Sigma)$

Ashtekar variables

Connection A is the local field $A = e^{*}\omega$

Electric field E is built from the dreibein $E = \star\, e_a^i dx^a \tau_i$

The problem is to find the cosmological sector of those variables

Cosmological sector of GR in Ashtekar variables

Cosmological hypothesis

Homogeneity: a connected Lie group S acts transitively and freely on $\Sigma \Rightarrow \Sigma \cong S$

Namely, for every two points x, y there exists a unique $g \in S$ s.t. $y = L_g x$

Homogeneous ADM variables (q, K): they are S-invariant tensors

$$L_g^* q = q$$

$$L_g^* K = K$$

Breaks diffeo-invariance
$\mathrm{Diff}(\Sigma) \to S_R \rtimes \mathrm{Aut}(S)$

Cosmological sector of GR in Ashtekar variables

Cosmological hypothesis for gauge fields

$P^{SO}(S)$ is homogeneous if it is S-invariant, i.e. there exists an action ϕ of S s.t. $\pi \circ \phi(g) = L_g$

There exists a unique(!) homogeneous spin structure on S

Homogeneity condition for connection from **Wang's theorem** $\phi(g)^*\omega = \omega, \ \forall g \in S$

(classified by linear maps $\Lambda: \mathfrak{s} \to \mathfrak{su}(2)$)

Ashtekar connections as homogeneous spin connections

The request of homogeneity for Ashtekar connection yields a homogeneous geometry for Σ

Cosmological sector of GR in Ashtekar variables

Reduced Phase Space

Configuration space $\mathcal{A}^S = \{A \mid A = e^*\omega, \omega \text{ homogeneous}\}$

Dreibein is a section in a homogeneous bundle, but it is not homogeneous itself

Phase space variables $\left(A_a^i(x), E_i^a(x)\right)$ such that, in some gauge $\left(\phi_I^i \theta^I(x), p_i^I \xi_I(x)\right)$

$\xi_I \in \mathfrak{s} \quad [\xi_I, \xi_J] = f_{IJ}^K \xi_K$

$\theta^I \in \mathfrak{s}^* \quad \theta^I + \frac{1}{2} f_{JK}^I \theta^J \theta^K$

$$\boxed{\text{The set of constraints are the same of LQG}}$$

Cylindrical functions for Cosmology
Gauge transformations

Configuration space $\mathcal{A}^S = \{A \mid A = e^*\omega, \omega \text{ homogeneous}\}$

Holonomies are SU(2)-valued **no** almost-periodic
[Brunnemann, Fleischhack '12]

Cylindrical functions $\mathrm{Cyl}(\mathcal{A}^S)$ $f_\gamma \colon \mathcal{A}^S \to SU(2)^{E(\gamma)} \to \mathbb{C}$

Infinitesimal transformations

Recover gauge symmetry
$$\begin{array}{ccc} \mathcal{G} \times \mathcal{A}^S & \to & \mathcal{A}^S \\ (u, e^*\omega) & \mapsto & e^*u^*\omega = (u \circ e)^*\omega = e'^*\omega \end{array}$$

$$A \mapsto A - D\Lambda$$

Residual diffeo symmetry
$$\begin{array}{ccc} \mathrm{Aut}(S) \times \mathrm{Cyl}(\mathcal{A}^S) & \to & \mathrm{Cyl}(\mathcal{A}^S) \\ (\varphi, f_\gamma) & \mapsto & f_{\varphi(\gamma)} \end{array}$$

$$A \mapsto A + \mathcal{L}_X A$$

Cylindrical functions for Cosmology

Classical Constraints Algebra

Configuration space $\mathcal{A}^S = \{A \mid A = e^*\omega, \omega \text{ homogeneous}\}$

Smearing functions must generate the correct gauge transformations

$\Lambda \in C_c^\infty(\Sigma, \mathfrak{su}(2))$ $\quad$ $X \in Lie\big(S_R \rtimes \mathrm{Aut}(S)\big) \subset \mathfrak{X}(\Sigma)$ $\quad$ $N = const.$

$$\{G(\Lambda), G(\Lambda')\} \propto G([\Lambda, \Lambda'])$$

$$\{V(\mathrm{X}), V(X')\} \propto V(\mathcal{L}_X X')$$

$$\{V(\mathrm{X}), H(N)\} = 0$$

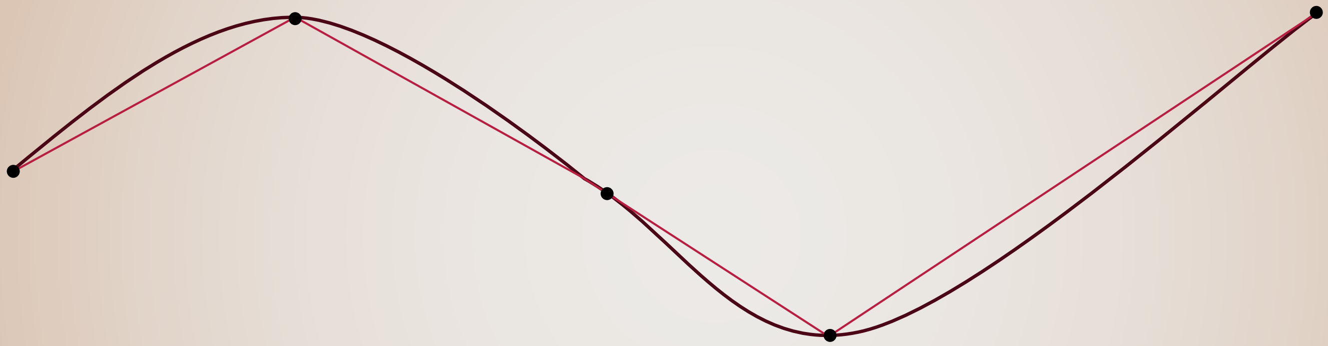
$$\{H(M), H(N)\} = 0$$

Cylindrical functions for Cosmology

Integral curves and homogeneous graphs

An important property of curves on S:

every curve can be approximated by piecewise integral curves of invariant vector field $\xi \in \mathfrak{s}$



Homogeneous graphs as suitable collections of piecewise integral curves

The homogeneous graphs are dense in the set of graphs

Distinguished subspace of cylindrical functions over homogeneous graph $Cyl_S(\mathcal{A}^S)$

Conclusions

- Ashtekar-Barbero-Immirzi formulation has a rigourous and clear geometric interpretation, in which the data are encoded in a spin connection A and a section (gauge) e.

- In this formulation we can find a cosmological sector using the Wang's theorem preserving local gauge degrees of freedom.

- We are able to provide cylindrical functions, and a distinguished invariant subspace based on homogeneous graphs.

- We have constraints associated to gauge transformations, but with a simplified algebra.

Outlook

- Implement the algebra at the quantum level. Solving the Hamiltonian constraint restricting the ambiguities by comparison with LQC.

- Recovering the LQC dynamics in some suitable limit (single-vertex model, $j \gg 1$).

Thank you for your attention

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Symmetry-reduced phase space

Configuration space $\mathcal{E} = \{\text{electric field gauge-equivalent to a homogeneous one}\}$

Fixing a point $E \in \mathcal{E}$ determine a metric on Σ, and we can build $\mathcal{A}_E^S \cong \mathcal{A}^S$

Phase space $\mathcal{P}$ as fiber bundle $\mathcal{P} = \bigsqcup_{E \in \mathcal{E}} \mathcal{A}_E^S$

$$\begin{array}{ccc} \mathcal{A}^S & \hookrightarrow & \mathcal{P} \\ & & \downarrow \\ & & \mathcal{E} \end{array}$$

On this phase space, the contraints have the same functional forms of the constraints in usual LQG

Point holonomies and invariant spin-network states

Equation of parallel transport along a homogeneous curve in a homogeneous gauge

$$\dot{u}(t) = \Lambda(v)u(t)$$

Holonomy $h_c(A) = u(1)^{-1} = \exp\big(\Lambda(v)\big)$

$$\begin{tikzcd}
P^{SO}(S) \arrow[r, "\phi(g)"] \arrow[d, "\pi"] & P^{SO}(S) \arrow[d, "\pi"'] \\
S \arrow[r, "L_g"] \arrow[u, bend left, "e"] & S \arrow[u, bend right, "e"']
\end{tikzcd}$$

$$\phi(g) \circ e = e \circ L_g$$

Homogeneous gauge

The holonomies brought by invariant homogeneous spin-network states are point holonomies

Topology of $\mathcal{A}/\mathcal{G}$

$\mathcal{A}^S/\mathcal{G}$ spin (metric-compatible) homogeneous connections modulo gauge transformations

$\mathcal{M}$ homogeneous SU(2) connections modulo gauge transformations

Relevant Homeomorphisms

	$\mathcal{A}^S/\mathcal{G}$	$\mathcal{M}$
Bianchi universes $(H = \{1\})$	$\mathbb{R}^6$	$\mathbb{R}^6$
Axial symmetry $(H = U(1))$	$\mathbb{R} \times \mathbb{R}_+$	$\bigsqcup_{n\in\mathbb{N}} \mathbb{R}$
FLRW $(H = SO(3))$	$\mathbb{R}$	$\{0\}$