Invariant observables in quantum gravity and quantum corrections to the Hubble rate and the Newtonian potential

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Perturbative quantum gravity

Perturbative quantum gravity

- Study gravity $S = S_\mathrm{G} + S_\mathrm{M}$ with $S_\mathrm{G} = \frac{1}{\kappa^2} \int (R - 2\Lambda) \sqrt{-g} \, \mathrm{d}^n x$ and S_M matter action perturbatively around a given background
- Metric decomposition $g_{\mu\nu} = g^{(0)}_{\mu\nu} + \kappa h_{\mu\nu}$, quantize $h_{\mu\nu}$
- Perturbation parameter: $\kappa = \sqrt{16\pi G_\mathrm{N}} \sim \ell_\mathrm{Pl}$
- Infinitesimal coordinate transformation (diffeomorphism) $x^\mu \to \tilde{x}^\mu = x^\mu + \kappa \xi^\mu$ $(\delta_\xi x^\mu = \kappa \xi^\mu)$ gives gauge transformation $\delta T = \kappa \mathcal{L}_\xi T$
- Gauge transformations of $h_{\mu\nu}$: $\delta_\xi h_{\mu\nu} = \nabla^{(0)}_\mu \xi_\nu + \nabla^{(0)}_\nu \xi_\mu + \kappa \mathcal{L}_\xi h_{\mu\nu}$
- Usual methods of QFTCS and gauge theories work, pQG is a well-defined theory according to the effective field theory paradigm that makes testable predictions at length scales above the fundamental scale $\kappa \sim \ell_\mathrm{Pl}$ Burgess, Living Rev. Rel. 7 (2004) 5

Perturbative quantum gravity

- Extremely efficient perturbation theory since ℓ_{Pl} is so small: only tree-level predictions can at present be experimentally verified (CMB)
- Main issue: construction of gauge-invariant observables
- For gauge theories of Yang–Mills type, gauge symmetry is internal (acts on field variables at given point), such that e.g. $(\operatorname{tr} F^2)(x) = F^a_{\mu\nu}(x) F_a^{\mu\nu}(x)$ is gauge-invariant
- In (p)QG, gauge symmetry are (infinitesimal) diffeomorphisms, which move points (external symmetry), such that fields at fixed point cannot be gauge-invariant
- $\Rightarrow$ there are no local observables in (p)QG (with small exceptions)
- If $T^{(0)} = 0$, then at linear order $\delta_\xi T^{(1)} = \mathcal{L}_\xi T^{(0)} = 0$, but at higher orders gauge-dependent since $T^{(1)} \neq 0$ in general
- Stewart–Walker lemma: $\delta_\xi T^{(k)} = 0 \Leftrightarrow T^{(k-1)}$ is linear combination of Kronecker δ's with constant coefficients Stewart/Walker, Proc. Roy. Soc. Lond. A 341 (1974) 49

Gauge-invariant observables

Gauge-invariant observables at linear order (1/2)

- Stewart–Walker lemma shows how to construct local gauge-invariant observables at linear order: take the expansion of any tensor that vanishes on the background
- Example: Background Minkowski space has flat metric $g^{(0)}_{\mu\nu} = \eta_{\mu\nu}$ and vanishing Riemann tensor $R^{(0)}_{\mu\nu\rho\sigma} = 0 \quad \Rightarrow \quad \delta_\xi R^{(1)}_{\mu\nu\rho\sigma} = 0$
- Explicitly: $R^{(1)}_{\mu\nu\rho\sigma} = \partial_\mu \partial_\rho h_{\nu\sigma} - \partial_\mu \partial_\sigma h_{\nu\rho} - \partial_\nu \partial_\rho h_{\mu\sigma} + \partial_\nu \partial_\sigma h_{\mu\rho}$, $\delta_\xi h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + \mathcal{O}(\kappa)$
- In general: use IDEAL characterization of background spacetime = a set of tensors that vanish iff the spacetime under characterization is locally isometric to reference one
- IDEAL set for Minkowski: Riemann curvature tensor $R_{\mu\nu\rho\sigma}$

Gauge-invariant observables at linear order (2/2)

- IDEAL set for (anti-)de Sitter: $R_{\mu\nu\rho\sigma} - K(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$ for positive (negative) constant $K = R/[n(n-1)]$
- First-order local observable in (anti-)de Sitter:
  $R^{(1)}_{\mu\nu\rho\sigma} - K\left(g^{(0)}_{\mu\rho}h_{\nu\sigma} + h_{\mu\rho}g^{(0)}_{\nu\sigma} - g^{(0)}_{\mu\sigma}h_{\nu\rho} - h_{\mu\sigma}g^{(0)}_{\nu\rho}\right) = g^{(0)}_{\mu\alpha}g^{(0)}_{\rho\beta}R^{\alpha}{}_{\nu}{}^{\beta}{}_{\sigma}{}^{(1)}$
- IDEAL set for FLRW = conformally flat metric $g^{(0)}_{\mu\nu} = a^2\eta_{\mu\nu}$ with scale factor $a(\eta)$, conformal time η, sourced by inflaton ϕ given in Canepa/Dappiaggi/Khavkine 1704.05542
- Observables: $W^{(1)}_{\mu\nu\rho\sigma}$ (linearized Weyl tensor), $E^{(1)}_{\mu\nu}$ (linearized Einstein equation), $C^{(1)}_{\mu\nu}$ constructed from extrinsic curvature of hypersurfaces $\phi = \text{const}$ and relation between Hubble rate $H = a^{-2}a'$ and potential $V(\phi)$ Fröb/Hack/Khavkine 1801.02632
- IDEAL construction gives complete set of local observables, also used for black holes Aksteiner/Andersson/Bäckdahl/Khavkine/Whiting 1910.08756

Gauge-invariant observables at higher orders

- Clearly if $T^{(1)}$ is non-trivial, $\delta_\xi T^{(2)} = \mathcal{L}_\xi T^{(1)} \neq 0 \;\Rightarrow$ no local observables beyond linear order
- Straightforward non-local observables: manifold averages such as $\int R\sqrt{-g}\, \mathrm{d}^n x$
- Less straightforward non-local observables: averaged correlation functions at fixed geodesic distance such as $\iint R(x)R(y)\,\delta\left(d_g(x,y)^2 - D^2\right)\sqrt{-g(x)}\sqrt{-g(y)}\,\mathrm{d}^n x\,\mathrm{d}^n y$
- Other well-known non-local observable: S-matrix in flat space
- Q: Nevertheless, we make local measurements including gravity — how can this be reconciled with the non-locality of observables?
- A: We actually make relational measurements, namely the state of one field (gravity) with respect to another (matter)

Relational observables

- Four scalar fields serve as dynamical reference frame / configuration-dependent coordinates, invariant observables are dynamical fields evaluated in this system
- Dynamical coordinates: four fields $X^{(\mu)}[g, \phi, \ldots]$ transforming under diffeos as scalars: $\delta_\xi X^{(\mu)} = \xi^\rho \partial_\rho X^{(\mu)}$, and their background value $X^{(\mu)}_{(0)}$
- Expand $X^{(\mu)} = X^{(\mu)}_{(0)} + \kappa X^{(\mu)}_{(1)} + \ldots$ in perturbation theory and invert to obtain $X^{(\mu)}_{(0)}[X]$ $\Rightarrow$ transforms inversely to a scalar
- Invariant observable $\mathcal{A}(\chi)$ is given by evaluating a field A at the position $X_0^{(\mu)}$, holding $X^{(\mu)} = \chi^\mu$ fixed
- $\Rightarrow$ Relational observables: $\mathcal{A}(\chi)$ is the value of A provided that $\chi^\mu = X^{(\mu)}$, and by evaluating at $X_0^{(\mu)}$ we interpret $\mathcal{A}$ as field on background
- Differential geometry formulation: $\mathcal{A} = A \circ X^{-1} \circ X_0$, change under diffeomorphism $x \to f(x)$ given by $\mathcal{A} \to A \circ f^{-1} \circ (X \circ f^{-1})^{-1} \circ X_0 = A \circ X^{-1} \circ X_0 = \mathcal{A}$

Dynamical coordinates (1/3)

- Choices for $X^{(\mu)}$: curvature scalars, additional scalar fields (e.g., Brown–Kuchař dust)
- First choice does not discriminate points on highly symmetric background, second choice alters physical content of theory Giesel/Herold/Li/Singh 2003.13729
- FLRW: only 1 scalar field (inflaton ϕ), but we need 4
- Minkowski/de Sitter: no scalar field at all
- FLRW background: $g^{(0)}_{\mu\nu} = a^2(\eta)\eta_{\mu\nu}$, a: scale factor, η: conformal time, $H = a'/a^2$: Hubble rate, $\epsilon = -H'/(H^2 a)$: first slow-roll parameter
- Classical inflaton $\phi^{(0)}$ with potential $V(\phi^{(0)})$, classical Einstein equation gives Friedmann equations (n: dimension of spacetime)

$$16\pi G_\mathrm{N} V(\phi^{(0)}) = 2(n-2)(n-1-\epsilon)H^2 \,, \quad 16\pi G_\mathrm{N} (\phi^{(0)\prime})^2 = 2(n-2)\epsilon H^2 a^2$$

Dynamical coordinates (2/3)

- At lowest order, we have $\delta x^\mu = \xi^\mu = \xi^\alpha \partial_\alpha x^\mu$, transforming as a scalar $\Rightarrow$ generalise to higher orders
- Time coordinate: invert background solution $\phi^{(0)}(\eta)$ to obtain $\eta(\phi^{(0)})$ and set $X^{(0)} = \eta(\phi) = \eta + \kappa\phi^{(1)}/\phi^{(0)\prime} + \mathcal{O}(\kappa^2)$ (time of comoving observers)
- Since spatial coordinates are harmonic $\triangle x^i = 0$, define $X^{(i)}(x)$ as harmonic coordinates for the full Laplacian $\triangle_{g,\phi}$ on constant-inflaton hypersurfaces $\phi = \text{const}$
  Brunetti/Fredenhagen/Hack/Pinamonti/Rejzner 1605.02573
- First complete solution in cosmology, can be computed to arbitrary orders in perturbation theory
- Invariant metric perturbation $\mathcal{H}_{\mu\nu}(X) = \frac{\partial x^\alpha}{\partial X^\mu}\frac{\partial x^\beta}{\partial X^\nu} g_{\alpha\beta}(x(X)) - g^{(0)}_{\mu\nu}$, holding X fixed.
  $\mathcal{H}_{\mu\nu} = h_{\mu\nu} - 2g^{(0)}_{\rho(\mu}\nabla^{(0)}_{\nu)}X^{(\rho)}_{(1)} + \mathcal{O}(\kappa) = h_{\mu\nu} - 2Ha\eta_{\mu\nu}X^{(0)}_{(1)} - 2\eta_{\rho(\mu}\partial_{\nu)}X^{(\rho)}_{(1)} + \mathcal{O}(\kappa)$
- On-shell (constraint equations are fulfilled): $\mathcal{H}_{00} = Q'/(Ha)$ with well-known Mukhanov–Sasaki variable Q

Dynamical coordinates (3/3)

- Bardeen variables are recovered by choosing different time coordinate $X^{(0)} = \eta \left[\phi - 3/(4|\nabla\phi|K) \, \triangle^{-1}_{g,\phi} \, R^{(3)} \right]$, with K extrinsic curvature of constant-ϕ hypersurfaces, $R^{(3)}$ induced Ricci scalar Brunetti et al. 1605.02573
- Observables are always non-local, but additional drawback: non-locality is non-causal due to inverse Laplacians ("action-at-a-distance", problems with renormalisability)
- Solution: use d'Alembertian $\Box_{g,\phi}$ instead of Laplacian and use retarded propagator as inverse $\Rightarrow$ causal generalised harmonic coordinates Fröb 1710.00839
- Additional simplification: since observables are gauge-invariant, use gauge in which first-order corrections $X^{(\mu)}_{(1)}$ vanish
- Propagator of metric perturbations can be determined in general FLRW in terms of three scalar propagators depending on scale factor a Fröb/Lima 1711.08470

Dynamical coordinates (4/4)

- Generalisations to flat space, to geodesic lightcone coordinates, to synchronous coordinates Fröb 1710.00839, Fröb/Lima 2108.11960, 2303.16218
- Use hyperbolic equation, e.g., $\nabla^2 X^{(\mu)} = 0$ (generalized harmonic coordinates) and invert using retarded Green's function to obtain causal evolution Fröb 1710.00839, Fröb/Lima 1711.08470
- Resulting invariant observables are non-local in accordance with general result, but non-locality restricted to past light cone

Graviton loop corrections to the local Hubble rate

Graviton loop corrections to the local Hubble rate (1/3)

- General definition: $H = \frac{\nabla^\mu u_\mu}{n-1}$, $u_\mu = \frac{\nabla_\mu \phi}{\sqrt{-\nabla^\mu \phi \nabla_\mu \phi}}$
- Agrees on background with $H = a'/a^2$ if Friedmann equations are satisfied
- Invariant relational observable $\mathcal{H} = H + \mathcal{H}^{(1)} + \ldots$ determined as before using causal generalised harmonic coordinates
- $\mathcal{H}^{(1)} = \frac{(n-1)\epsilon H^2 a^2 \phi^{(1)} - \triangle \phi^{(1)}}{(n-1) a \phi'} + \frac{H}{2} h_{00} + \frac{1}{2(n-1)a} \left(\partial_\eta h^k_k - 2\partial^k h_{0k} \right)$
- $\mathcal{H}^{(1)}$ is local, since H in the background only depends on time and the invariant time coordinate is local in the inflaton, but $\mathcal{H}^{(2)}$ (and higher) are non-local
- $\mathcal{H}$ measures expansion rate as seen by a comoving observer, interpretation of spatial coordinates $X^{(i)}$ less clear

Fröb 1806.11124

Graviton loop corrections to the local Hubble rate (2/3)

- Expectation value $\langle \mathcal{H} \rangle$ can be calculated using usual QFT methods, including gauge-fixing and ghosts
- For practical reasons: restriction to $\epsilon = \text{const}$
- Can also treat matter domination ($\epsilon = (n-1)/2$) and radiation domination ($\epsilon = n/2$)
- Renormalisation: scalar potential (cosmological constant), additional counterterms to define $\mathcal{H}$ as composite operator (at one-loop level and for constant ϵ they are all degenerate with either H or H^3)
- Result in the de Sitter limit $\epsilon = 0$: $\langle \mathcal{H} \rangle = H$ (no correction at one loop), consistent with different but related observable Miao, Tsamis & Woodard 1702.05694

Graviton loop corrections to the local Hubble rate (3/3)

- Result for matter domination: $\langle \mathcal{H} \rangle = H \left[1 - \frac{229}{8\pi} \ell_{\mathrm{Pl}}^2 H^2 \ln a \right]$
- Screening of the Hubble rate by quantum effects – a secular effect
- Physical picture: expansion creates particles (gravitons) which then interact, and gravity is universally attractive
- Result for radiation domination: $\langle \mathcal{H} \rangle = H$ (no one-loop correction)
- Consistent with absence of graviton production there

Grishchuk Sov.Phys.JETP 40 (1975) 409 / Parker Phys.Rev. 183 (1969) 1057

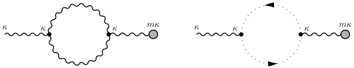
- Result for slow-roll inflation ($0 < \epsilon \ll 1$): $\langle \mathcal{H} \rangle = H(\hat{\epsilon})$ with $\hat{\epsilon} = \epsilon \left(1 - \frac{21}{16\pi} \ell_{\mathrm{Pl}}^2 H_0^2 \right)$ (closer to de Sitter space) Fröb 1806.11124
- Similar results for slow-roll inflation $\epsilon' = 2Ha\epsilon\delta \neq 0$, $|\delta| \ll 1$ Lima 2007.04995

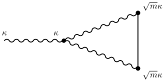
Graviton loop corrections to the Newtonian potential

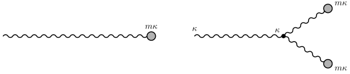
Graviton loop corrections to the Newtonian potential 1/4

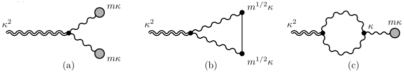
- Physical picture: vacuum polarisation (due to fluctuations of virtual gravitons) changes the gravitational potential of a point mass
- Use invariant observable $\mathcal{H}_{\mu\nu} = h_{\mu\nu} - 2\eta_{\rho(\mu}\partial_{\nu)}X^{(\rho)}_{(1)} + \mathcal{O}(\kappa)$ corresponding to metric perturbation $h_{\mu\nu}$ and define Newtonian gravitational potential $V \equiv -\frac{1}{2}\langle\mathcal{H}_{00}\rangle$
- Field-dependent coordinate system: generalised harmonic coordinates $\nabla^2 X^{(\mu)} = 0$
- First-order correction: $X^{(\mu)}_{(1)}(x) = \int G(x,x')\left[\partial_\nu h^{\mu\nu} - \frac{1}{2}\partial^\mu h\right](x')\,\mathrm{d}^4x'$ with Green's function G: $\partial^2 G(x,x') = \delta^4(x-x')$
- Gravitational action:
  $S_{\mathrm{G}} = \frac{1}{4}\int\left(-\partial^\rho h^{\mu\nu}\partial_\rho h_{\mu\nu} + 2\partial^\mu h_{\mu\rho}\partial_\nu h^{\nu\rho} - 2\partial_\mu h^{\mu\nu}\partial_\nu h + \partial^\rho h\partial_\rho h\right)\mathrm{d}^n x + \mathcal{O}(h^3)$
- Point particle action: $S_{\mathrm{PP}} = \frac{1}{2}\int\left[\frac{1}{e(\tau)}g_{\mu\nu}(z(\tau))\dot{z}^\mu(\tau)\dot{z}^\nu(\tau) - m^2 e(\tau)\right]\mathrm{d}\tau$
- e: einbein, τ: affine parameter, z^μ: particle position, m: mass
- Reparametrisation invariance: $\delta\tau = -\epsilon(\tau)$, $\delta z^\mu = \epsilon\dot{z}^\mu$, $\delta e = \dot{\epsilon}e + \epsilon\dot{e}$ with $\dot{\circ} = \mathrm{d}\circ/\mathrm{d}\tau$

Graviton loop corrections to the Newtonian potential 2/4

- Extend BRST formalism to include reparametrisations
- Large mass expansion to supress fluctuations of the particle itself: $z^\mu(\tau) = v^\mu \tau + m^{-1/2} y^\mu(\tau)$, $v^\mu = \delta^\mu_0$, $e(\tau) = m^{-1} + m^{-3/2} f(\tau)$
- Four types of diagrams to order $\kappa^4 \sim G_\mathrm{N}^2 \sim G_\mathrm{N} \ell_\mathrm{Pl}^2$:
  1. Usual graviton & ghost loops 

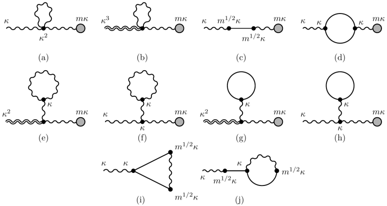
  2. Worldline corrections involving y^μ in the loop 

  3. Classical corrections (tree diagrams) 

  4. Corrections from field-dependent coordinates $X^{(\mu)}$ 

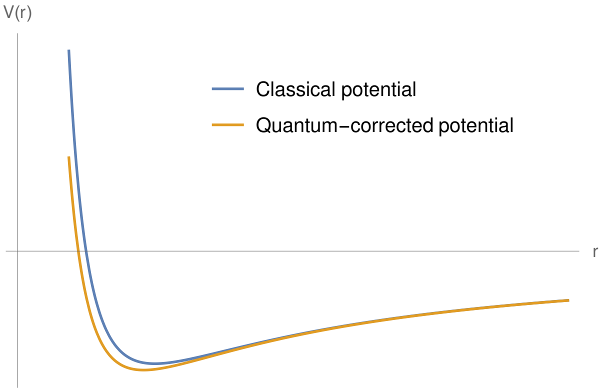
Graviton loop corrections to the Newtonian potential 3/4

- Suitable choice of gauge kills over half of diagrams



- Classical correction $\langle \mathcal{H}_{\mu\nu}(x) \rangle^{\text{class}} = \frac{G_N m}{r} (2\bar{\eta}_{\mu\nu} + 2v_\mu v_\nu) + \frac{G_N^2 m^2}{r^2} \left(\frac{\mathbf{x}_\mu \mathbf{x}_\nu}{r^2} + \bar{\eta}_{\mu\nu} - 2v_\mu v_\nu \right)$ agrees with expansion of Schwarzschild metric in harmonic coordinates ($\bar{\eta}_{\mu\nu} = \eta_{\mu\nu} + v_\mu v_\nu$, $r = |\mathbf{x}|$)
- Quantum-corrected potential $V(r) = -\frac{G_N m}{r} \left(1 - \frac{G_N m}{r} + \frac{41}{10\pi} \frac{\hbar G_N}{r^2} - \frac{43}{12\pi} \frac{\hbar G_N}{r^2} \right)$ has two contributions: first is universal and agrees with inverse scattering method, second is remnant of particle fluctuations (“Zitterbewegung”)

Graviton loop corrections to the Newtonian potential 4/4



- Gravitational force strengthened at small distances
- Decrease of black hole horizon radius: $r_\mathrm{H} = r_\mathrm{S} - \left(\frac{41}{10\pi} - \frac{43}{12\pi} \right) \frac{\hbar}{m}$ with $r_\mathrm{S} = 2 G_\mathrm{N} m$

Conclusions

- Issue of gauge-invariant observables differentiates pQG from QFTCS and connects it to fundamental quantum gravity
- Observables are non-local, relational causal observables can be constructed to all orders
- Results: Secular screening of the Hubble rate and strengthening of gravitational force at small distances
- Stay tuned for the review paper Fröb/Khavkine 25XX.XXXXX

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Outlook

Outlook: de Sitter background

- Compute corrections to the Newtonian potential due to (possibly self-interacting) conformal matter Fröb/Verdaguer 1601.0356
- Generalisation to spinning particles gives corrections to gravitomagnetic potential $V_i = h_{0i}$ Fröb/Verdaguer 1701.06576
- $V(r) = -\frac{G_\mathrm{N} m}{\hat{r}} \left[1 - \frac{128\pi b}{3} \frac{\ell_\mathrm{Pl}^2}{\hat{r}^2} - 32\pi \ell_\mathrm{Pl}^2 H^2 \left(\beta - 4b - 3b' + 2b \ln\left(\bar{\mu} \hat{r} \right) \right) \right]$
- $\mathbf{V} = -2G \frac{\mathbf{S} \times \hat{\mathbf{r}}}{\hat{r}^3} \left[1 + 96\pi b \frac{\ell_\mathrm{Pl}^2}{\hat{r}^2} + 32\pi \ell_\mathrm{Pl}^2 H^2 \left(\beta - 5b - b' + 2b \ln\left(\bar{\mu} \hat{r} \right) \right) \right]$
- H: Hubble constant, $\hat{r} = ar$: physical distance on equal-time hypersurface, b, b': conformal central charges, β: R^2 coupling constant, $\bar{\mu}$: renormalisation scale
- Future goal: Compute graviton loop corrections to Newtonian potential in de Sitter using invariant observables

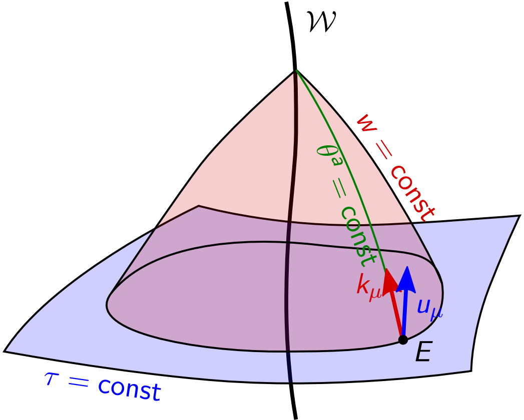
Outlook: Black hole scattering

- Compute effective action for world lines (including spin), completely integrating out gravitons
- Invariant world line observable also fixes reparametrisation invariance, for example by fixing τ to proper time (including graviton corrections!)
- Solve effective field equations to obtain quantum corrections to scattering angles etc.
- Work in progress with Jan Steinhoff (AEI Golm) and student Tim Luis Borck (TU Berlin)

Geodesic light cone coordinates

Geodesic light cone coordinates (1/4)

- Observations made in cosmology using light, which travels along the light cone (Picture from Fleury/Nugier/Fanizza 1602.04461)



- $\mathcal{W}$: observer's worldline
- E: event
- τ: observer time
  w: label for null hypersurfaces (past light cone)
  θ^a: angles
- $u_\mu = -\partial_\mu \tau$: four-velocity
- $k_\mu = \partial_\mu w$: null vector

Geodesic light cone coordinates (2/4)

- Vectors u^μ and k^μ are geodesic: $u^\mu \nabla_\mu u^\rho = 0 = k^\mu \nabla_\mu k^\rho$
- Intersection of hypersurfaces of constant τ and w isomorphic to $(n-2)$-sphere, parametrised by angles θ^a: $k^\mu \partial_\mu \theta^a = 0$
- Frequency of photon of wave vector $\mathbf{k}$: $k^\tau = -k^\mu u_\mu \equiv -\Upsilon^{-1}$
- Misalignment of photon propagation with ∂/∂_w: $U^a \equiv \Upsilon u^\mu \partial_\mu \theta^a$
- Induced metric on sphere: γ_{ab}
- Line element in GLC coordinates: $\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu =$ $-2\Upsilon\,\mathrm{d}\tau\,\mathrm{d}w + (\Upsilon^2 + U^a U_a)\,\mathrm{d}w^2 + \gamma_{ab}\Big(\mathrm{d}\theta^a - U^a\,\mathrm{d}w\Big)\Big(\mathrm{d}\theta^b - U^b\,\mathrm{d}w\Big)$
- FLRW background: $\Upsilon = a$, $U^a = 0$, $\gamma_{ab} = a^2(w-\eta(\tau))^2 s_{ab}$ with $w = r + \eta(\tau)$ and η conformal time, τ cosmic time
- Adding perturbations we construct field-dependent GLC coordinates

Geodesic light cone coordinates (3/4)

- Conditions in perturbed geometry: $\tilde{u}_\mu = -\partial_\mu \tilde{\tau}$, $\tilde{k}_\mu = \partial_\mu \tilde{w}$, normalisation $\tilde{u}^\mu \tilde{u}_\mu = -1$, $\tilde{k}^\mu \tilde{k}_\mu = 0$, orthogonality $\tilde{k}^\mu \partial_\mu \tilde{\theta}^a = 0$
- $\tau_{(1)}(\tau, w, \theta) = -\frac{1}{2} \int_{-\infty}^{\tau} \left(a^2 h_{\tau\tau} + 2a h_{\tau w} + h_{ww} \right) [s, w - \eta(\tau) + \eta(s), \theta] \, \mathrm{d}s$
- $w_{(1)}(\tau, w, \theta) = -\frac{1}{2} \int_{-\infty}^{\tau} (a h_{\tau\tau}) (s, w, \theta) \, \mathrm{d}s$
- $\theta^a_{(1)}(\tau, w, \theta) = \int_{-\infty}^{\tau} \left(a \gamma^{ab} \partial_b w_{(1)} + a^2 h^a{}_\tau \right) (s, w, \theta) \, \mathrm{d}s$
- Coordinates are non-local but causal functionals of the metric perturbation
- Define gauge-invariant relational observables as before, example: invariant metric
$$\mathcal{G}_{\mu\nu} = \begin{pmatrix} 0 & -\tilde{\Upsilon} & 0 \\ -\tilde{\Upsilon} & \tilde{\Upsilon}^2 + \tilde{\gamma}_{ab} \tilde{\mathrm{U}}^a \tilde{\mathrm{U}}^b & -\tilde{\gamma}_{bc} \tilde{\mathrm{U}}^c \\ 0 & -\tilde{\gamma}_{ac} \tilde{\mathrm{U}}^c & \tilde{\gamma}_{ab} \end{pmatrix}$$

Geodesic light cone coordinates (4/4)

- $\tilde{\Upsilon} \equiv -(k^\mu u_\mu)^{-1} = a - \kappa \left(a^2 h_{\tau w} + \frac{1}{2} a^3 h_{\tau\tau} + a \partial_\tau \tau_{(1)} + a \partial_w w_{(1)} + H a \tau_{(1)} \right) + \mathcal{O}(\kappa^2)$
- $\tilde{U}^a \equiv \tilde{\Upsilon} \tilde{u}^\mu \partial_\mu \tilde{\theta}^a = -\kappa \gamma^{ab} \left(a^2 h_{wb} - \gamma_{bc} \partial_w \theta^c_{(1)} + a \partial_b \tau_{(1)} - a^2 \partial_b w_{(1)} \right) + \mathcal{O}(\kappa^2)$
- $\tilde{\gamma}_{ab} = \kappa \left(a^2 h_{ab} - 2\gamma_{ab} \left[H - \frac{1}{a(w-\eta)} \right] \tau_{(1)} - 2\gamma_{ab} \frac{w_{(1)}}{w-\eta} - \theta^c_{(1)} \partial_c \gamma_{ab} - 2\gamma_{c(a} \partial_{b)} \theta^c_{(1)} \right) + \mathcal{O}(\kappa^2)$
- Gauge-invariant Hubble rate:
  $\mathcal{H}_u = H - \frac{\kappa}{2} \left[\frac{1}{(n-1)} \nabla_\mu \left(2k^\mu \Upsilon_{(1)} - 2a^{-1} U^\mu_{(1)} - u^\mu \gamma^{ab} \gamma^{(1)}_{ab} \right) + H a^{-1} \left(2\Upsilon^{(1)} + a \gamma^{ab} \gamma^{(1)}_{ab} \right) \right] + \mathcal{O}(\kappa^2)$ as measured by an observer whose four-velocity is equal to $\tilde{u}^\mu$
- $\mathcal{H}_\phi = \mathcal{H}_u - \frac{\kappa}{(n-1) a^2 \dot{\phi}} \nabla_\mu \left[\left(g^{\mu\nu} + u^\mu u^\nu \right) \nabla_\nu \Phi^{(1)} \right] + \mathcal{O}(\kappa^2)$ as measured by an observer co-moving with the inflaton, i.e., four-velocity $\tilde{v}^\mu \sim \tilde{\nabla}^\mu \tilde{\Phi}$
- Both agree if the inflaton perturbation is spatially homogeneous in the observer's frame

Fröb/Lima 2108.11960

Synchronous coordinates

- Geodesic light cone coordinates are very physical, but extremely awkward to work with, since translational symmetry of constant-time hypersurfaces is not manifest
- In non-relativistic situations, take (formally) limit $c \to \infty \quad \Rightarrow \quad$ light cones flatten, obtain synchronous coordinates
- Time coordinate: proper time $\tilde{\tau}$ of an observer with four-velocity $\tilde{u}_\mu$ ($\tilde{u}^\mu \tilde{u}_\mu = -1$)
- Constant-time hypersurface with fixed $\tilde{\tau}$ parametrized by coordinates $\tilde{x}^i$ such that $\tilde{u}^\mu \partial_\mu \tilde{x}^i = 0$ (orthogonal to time direction)
- FLRW background metric: $\mathrm{d}s^2 = -\,\mathrm{d}\tau^2 + a^2(\tau)\,\mathrm{d}\mathbf{x}^2$ (observer's time is cosmic time)
- First-order corrections: $\tau_{(1)}(\tau, \mathbf{x}) = -\frac{1}{2} \int_{-\infty}^{\tau} h_{00}(s, \mathbf{x})\,\mathrm{d}s$,
  $x^i_{(1)}(\tau, \mathbf{x}) = -\frac{1}{2}\partial^i \int_{-\infty}^{\tau} \int_{-\infty}^{s} h_{00}(s', \mathbf{x})\,\mathrm{d}s'\,\mathrm{d}s + \int_{-\infty}^{\tau} h_0{}^i(s, \mathbf{x})\,\mathrm{d}s$ Fröb/Lima 2303.16218
- Invariant metric perturbation at tree level is exactly the TT graviton, also in de Sitter