Particle acceleration in astrophysics

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Plan of the lecture

The problem of particle acceleration in astrophysics (from yesterday's class)

[9] Why it is a problem to understand particle acceleration in astrophysics

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[2] Way out: time varying B-fields
[3] The Hillas criterion
[4] Fermi's seminal idea

The basics

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Fermi II and Fermi I (DSA) mechanisms

[5] Second order Fermi acceleration
[6] First order Fermi acceleration
[7] Shock waves in one slide
[8] Diffusive shock acceleration

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Fermi II and Fermi I (DSA) mechanisms

[5] Second order Fermi acceleration
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[7] Shock waves in one slide
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Particle acceleration at relativistic shocks

[9] General considerations and simple estimates (I am not an expert!)

[9] Why it is a problem to understand particle acceleration in astrophysics

Charged particles and electromagnetic fields

cosmic rays are charged particles —> they are affected by electromagnetic fields

$$\vec{E}(\vec{r},t) \qquad\qquad \vec{B}(\vec{r},t)$$

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Simplifying assumption —> consider only constant fields

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A particle of charge q moving at a velocity u fill experience a force:

$$\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = q\left(\vec{E} + \frac{\vec{u}}{c} \times \vec{B}\right)$$

relativistic momentum $\vec{p} = \gamma m \vec{u}$

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Lorentz force

⊥ to velocity —>

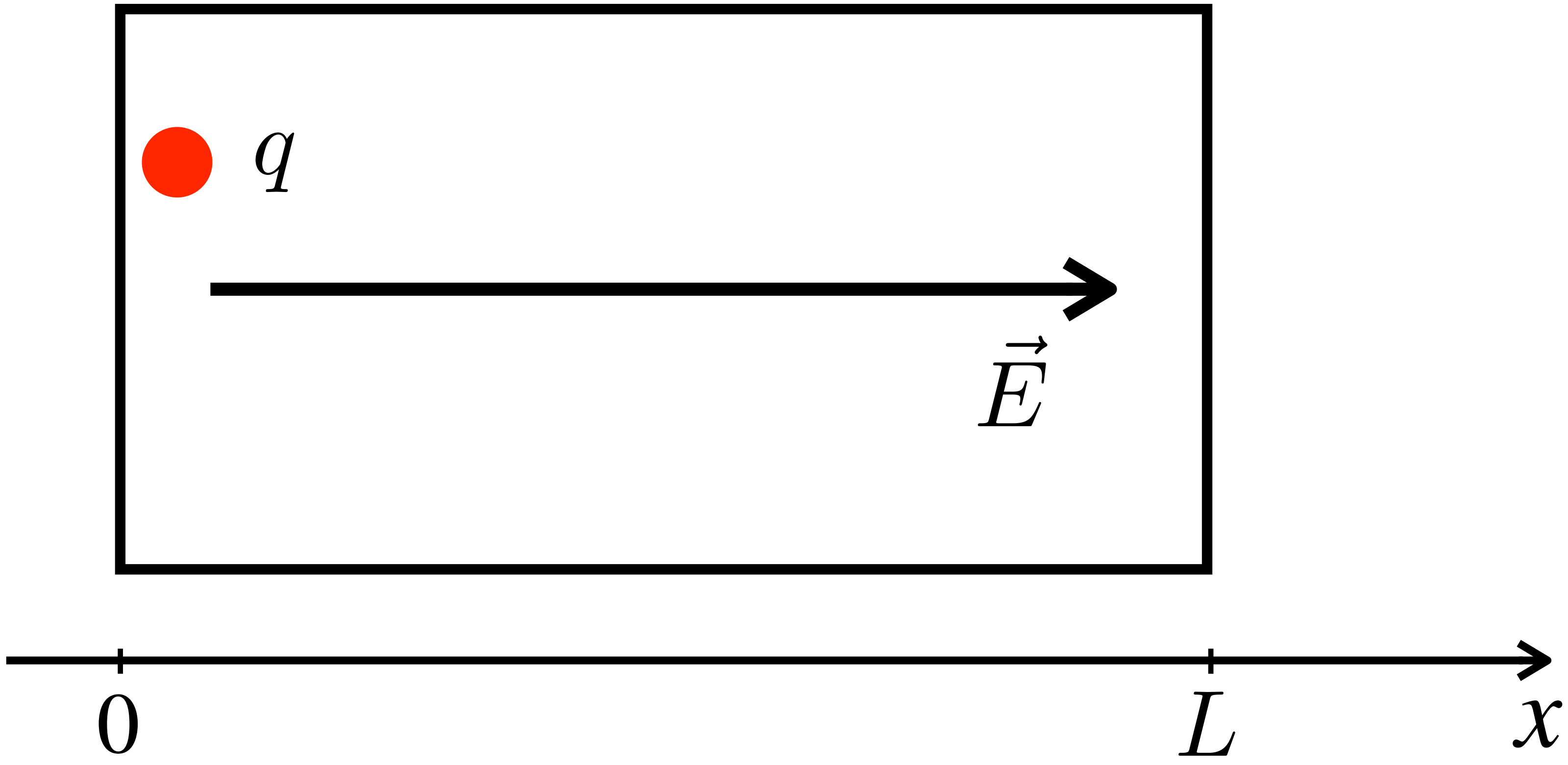
doesn't change

the particle energy!

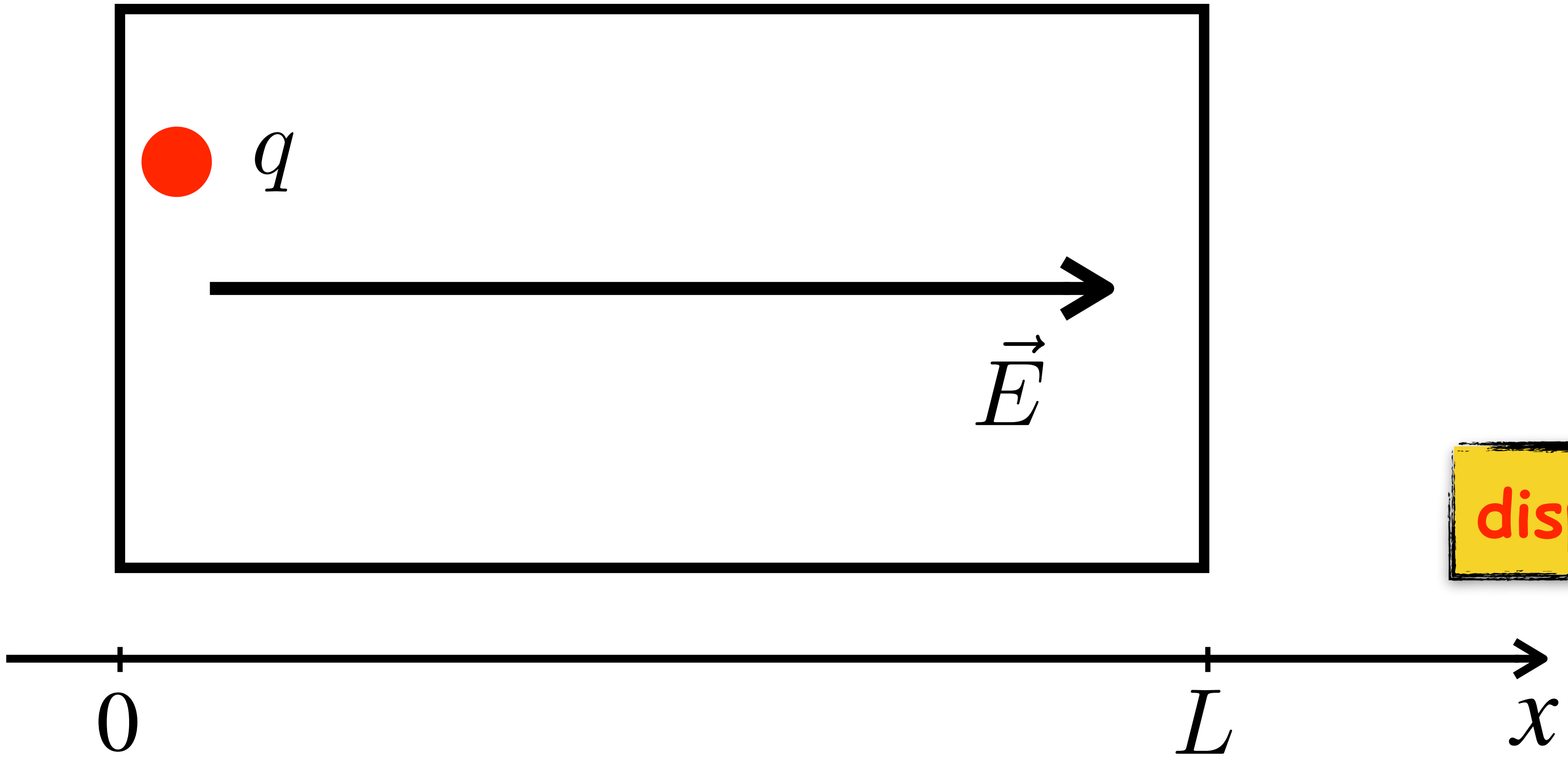
relativistic momentum $\vec{p} = \gamma m \vec{u}$

The simplest accelerator

this is an accelerator



The simplest accelerator


this is an accelerator

particle energy

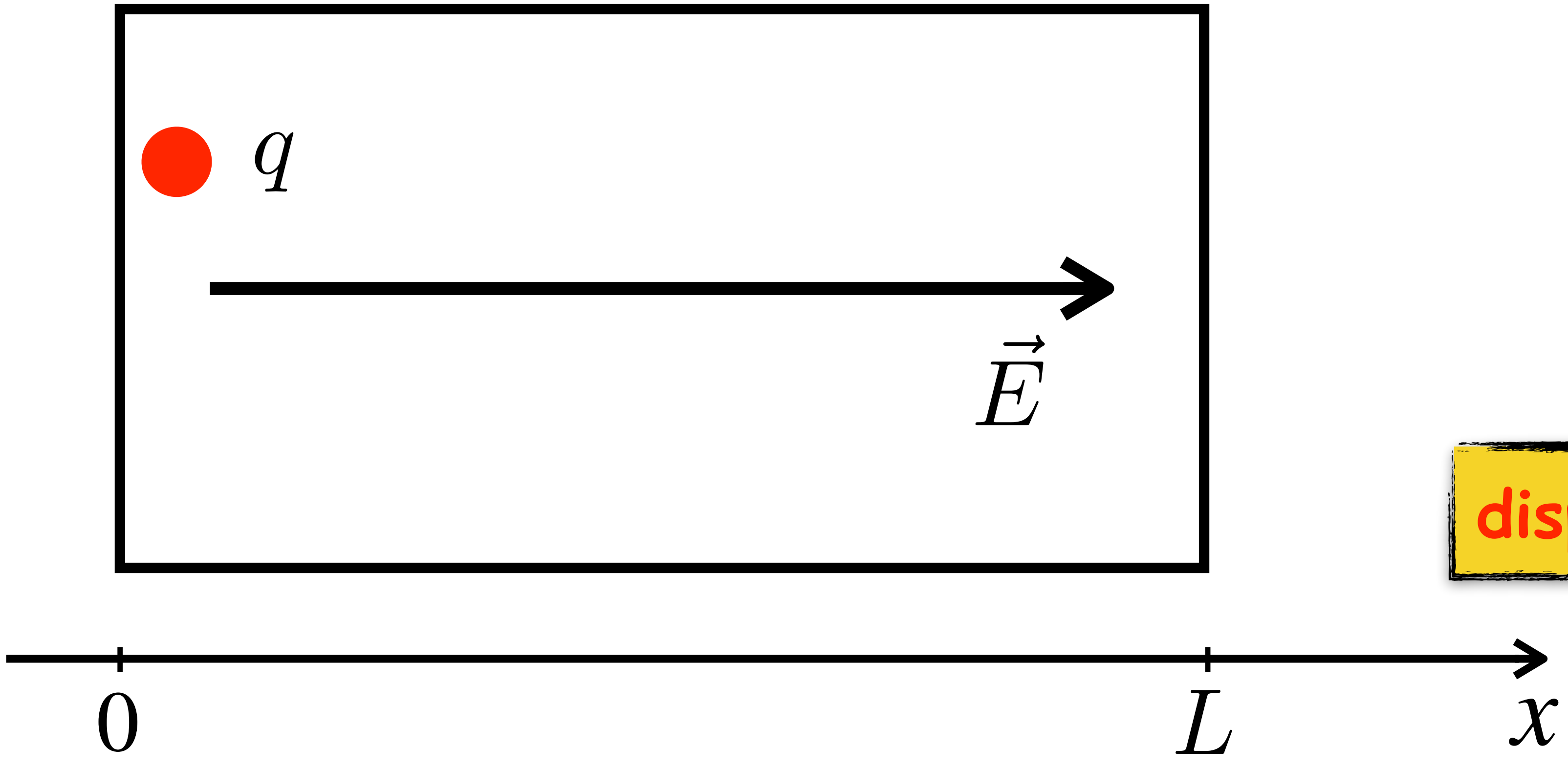
$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}x} = qE$$

displacement

electrostatic force

The simplest accelerator

this is an accelerator



particle energy

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}x} = qE$$

displacement

electrostatic force

$$\mathscr{E}_{max} = q\ E\ L$$

large accelerator

large charge

strong E field

…because we deal with plasmas

to accelerate particles, you need an electric field

…because we deal with plasmas

to accelerate particles, you need an electric field

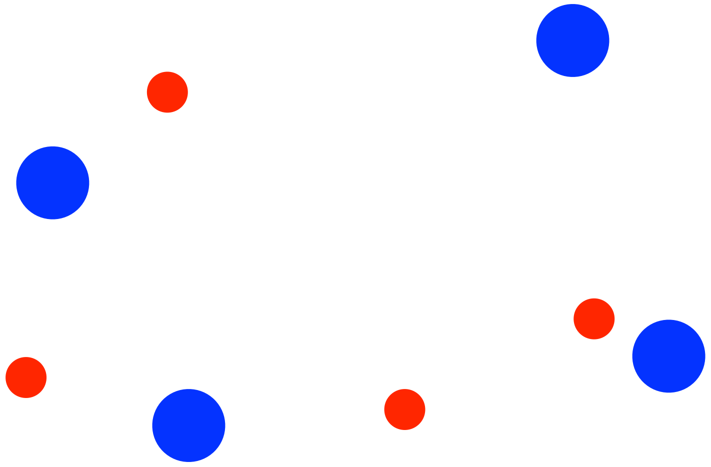
An excess of electrical charge is needed to maintain a static electric field. However we should remember…

> "…a **basic property of plasma, its tendency towards electrical neutrality**. If over a large volume the number of electrons per cubic centimeter deviates appreciably from the corresponding number of positive ions, the electrostatic forces resulting yield a potential energy per particle that is enormously greater than the mean thermal energy. Unless very special mechanisms are involved to support such large potentials, the charged particles will rapidly move in such a way as to reduce these potential difference, i.e., to restore electrical neutrality."

(Lyman Spitzer "Physics of fully ionised gases")

Quasi-neutrality

Each charge in a plasma is connected to any other charge through Coulomb interactions, which are long-range interactions (potential ~ 1/R).



● protons

● electrons

} thermal equilibrium $T_e = T_p = T$ -> Boltzmann distribution

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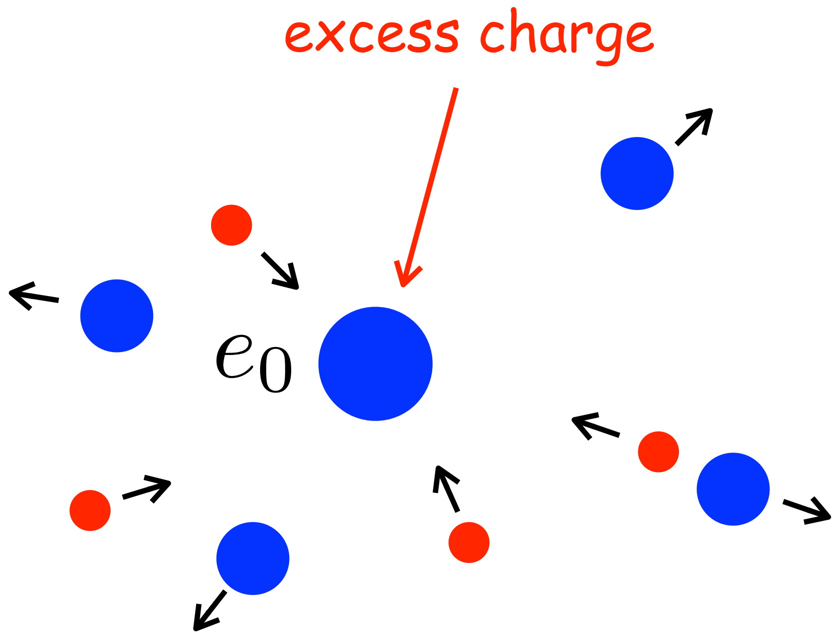

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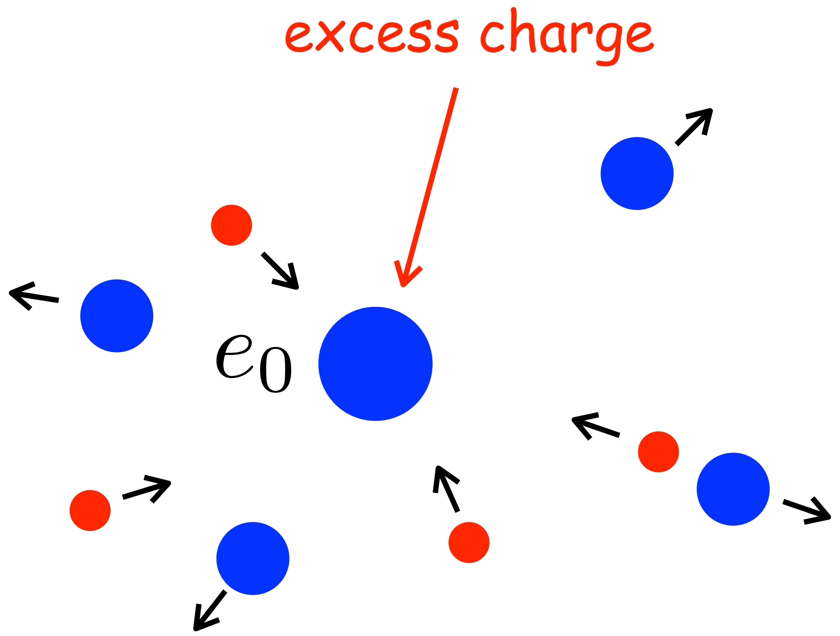
$$\nabla \cdot \vec{E} = -\nabla^2 \phi = 4\pi \varrho = 4\pi e(n_i - n_e) + 4\pi e_0 \delta(\vec{R})$$



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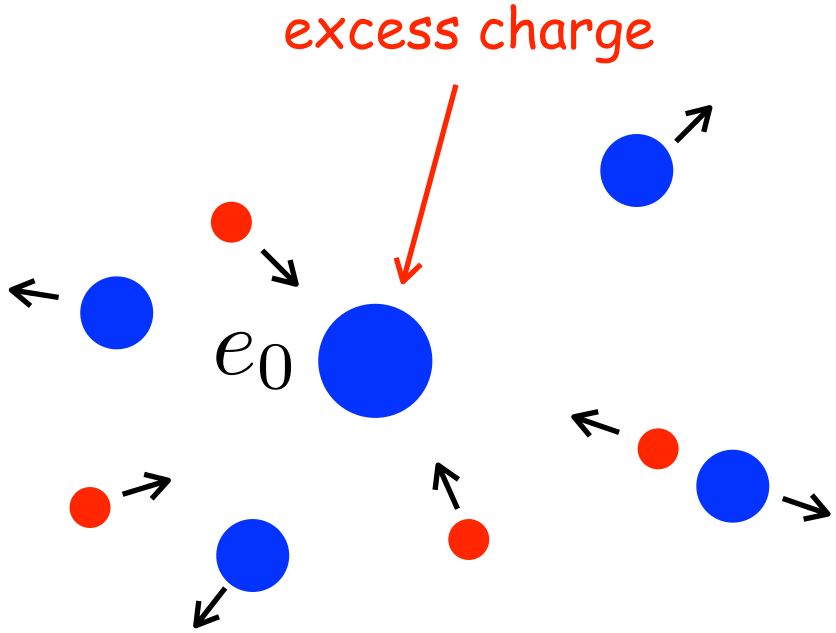

$$\begin{cases} n_e = n_0 \exp\left[-\frac{(-e\phi)}{kT}\right] \\ n_i = n_0 \exp\left[-\frac{(e\phi)}{kT}\right] \end{cases}$$

● protons

● electrons

$\}$ thermal equilibrium $T_e = T_p = T$ -> Boltzmann distribution

Quasi-neutrality

$$\nabla^2\phi = 4\pi n_0 e\left[\exp\left(\frac{e\phi}{kT}\right) - \exp\left(\frac{-e\phi}{kT}\right)\right] - 4\pi e_0\delta(\vec{R})$$

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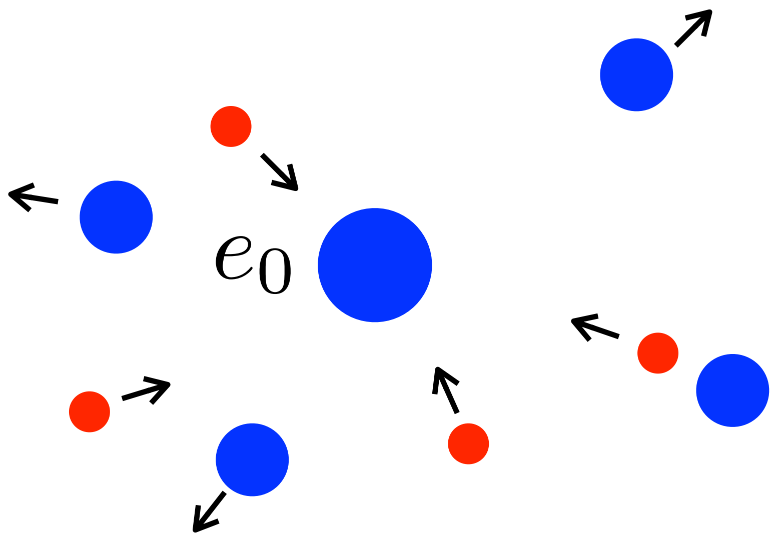
$$\nabla^2\phi = 8\pi n_0 e\frac{e\phi}{kT} - 4\pi e_0\delta(\vec{R})$$

$$\begin{aligned}\frac{1}{R^2}\frac{\mathrm{d}}{\mathrm{d}R}\left(R^2\frac{\mathrm{d}\phi}{\mathrm{d}R}\right) &= 8\pi n_0 e\frac{e\phi}{kT} - 4\pi e_0\delta(\vec{R}) \\ &= \frac{\phi}{\lambda^2} - 4\pi e_0\delta(\vec{R})\end{aligned}$$

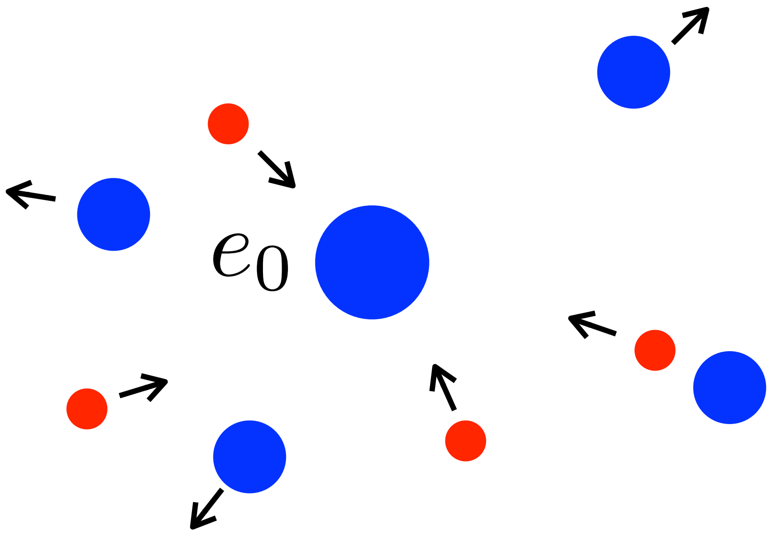
$$\boxed{\lambda = \left(\frac{kT}{8\pi n_0 e^2}\right)^{1/2}}$$

Quasi-neutrality: Debye length



$$\phi = \frac{e_0}{R}\, e^{-\frac{R}{\lambda}}$$

Quasi-neutrality: Debye length



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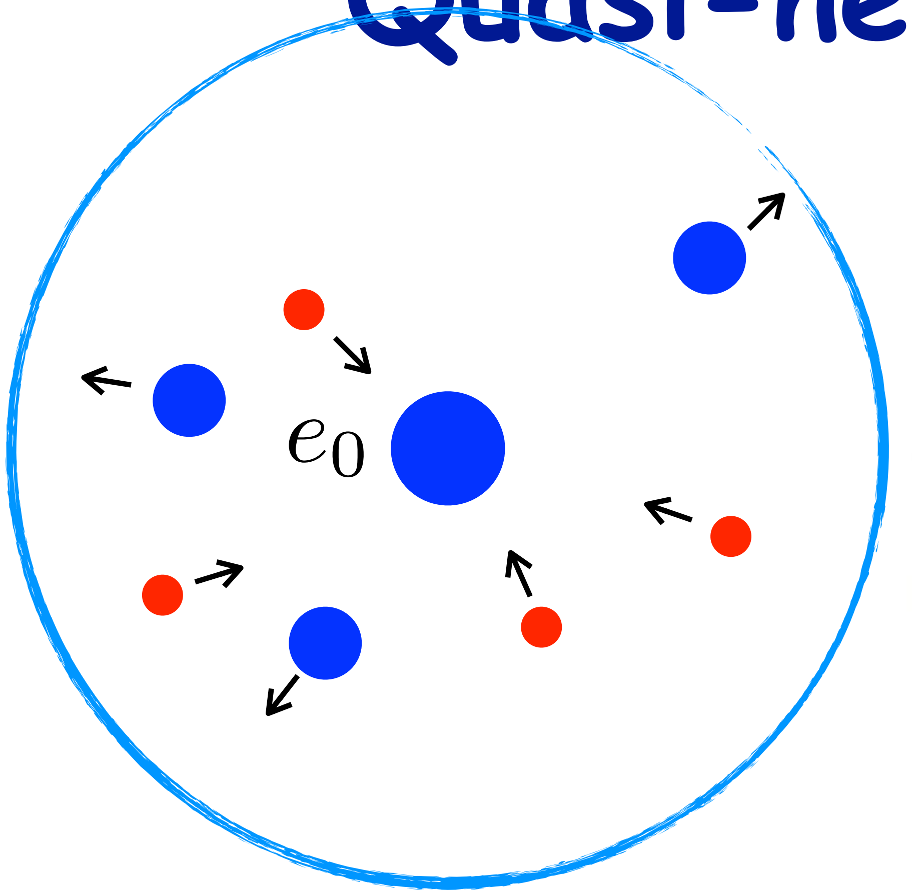
Coulomb potential

Quasi-neutrality: Debye length



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screened!

Coulomb potential

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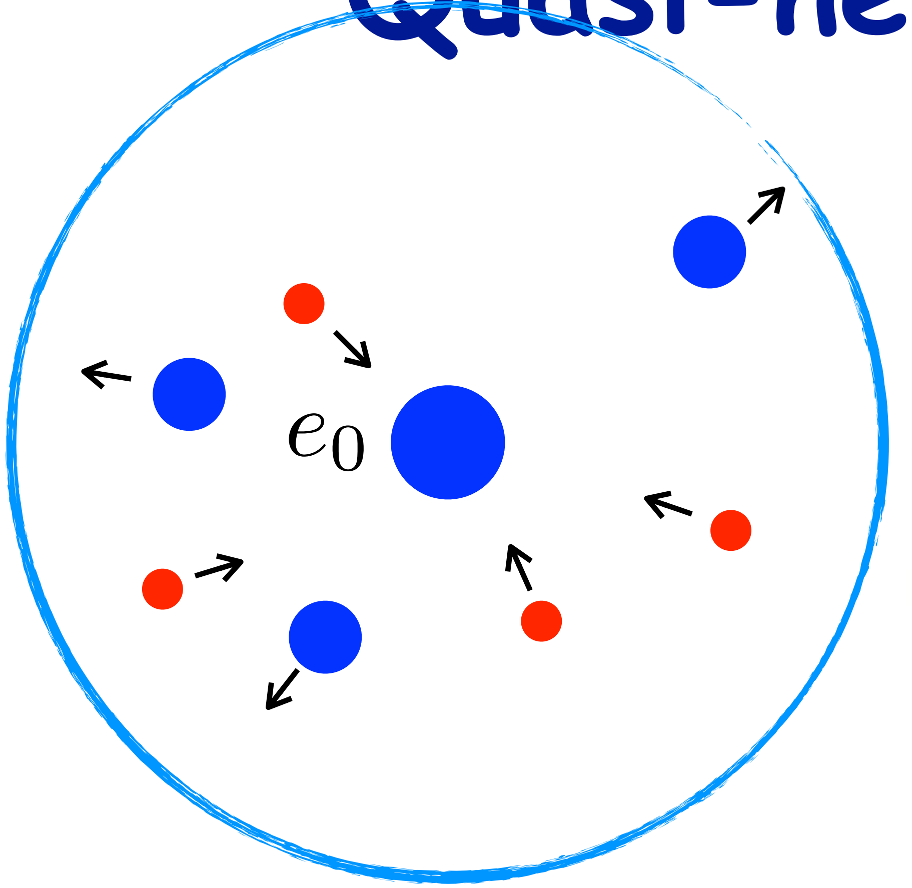

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screened!

Coulomb potential

Excess charges are screened on a scale called Debye length

$$\lambda = \left(\frac{kT}{8\pi n_0 e^2}\right)^{1/2} \sim 5 \times 10^2 \left(\frac{T}{10^4 \text{ K}}\right)^{1/2} \left(\frac{n_0}{\text{cm}^{-3}}\right)^{-1/2} \text{cm}$$

↑ extremely small!

it does NOT depend on the charge excess!

How long it takes?

dimensionally —>

$$\tau \sim \frac{\lambda}{v_{th}}$$

plasma frequency

$$\omega_p \sim 1/\tau$$

How long it takes?

dimensionally —>

$$\tau \sim \frac{\lambda}{v_{th}} = \sqrt{\frac{m_e}{16\pi n_0 e^2}} \approx 10^{-5} \left(\frac{n_0}{\mathrm{cm}^{-3}}\right)^{-1/2} \mathrm{s}$$

$$\frac{1}{2} m_e v_{th}^2 \sim kT \rightarrow v_{th} \sim \sqrt{\frac{2kT}{m_e}}$$

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time to react to the presence of an electric charge

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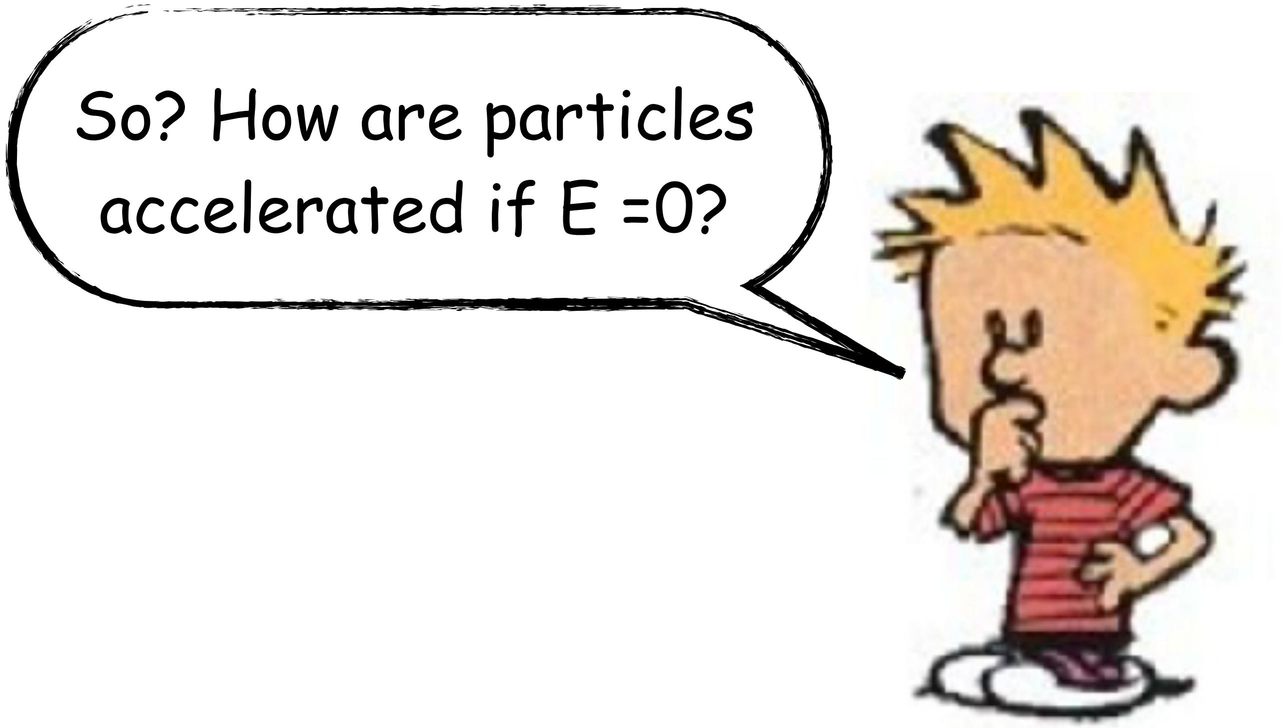

Way-out: time varying B

We DO need electric fields to accelerate particles!

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Maxwell equations

$$\nabla \vec{E} = 4\pi \varrho$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \vec{B} = 0$$

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

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$$\nabla \vec{E} = 4\pi \varrho \; = 0$$ —> plasma quasi-neutrality

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Faraday law

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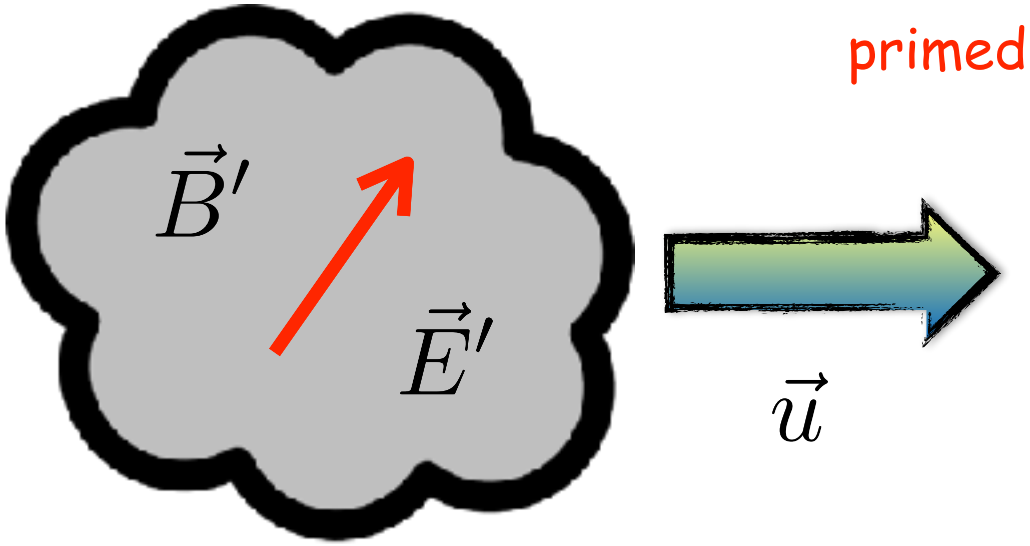
A time varying magnetic field acts as a source of electric field!

Equivalent way to see that: change rest frame

Consider a magnetised cloud of plasma moving at a (non relativistic) velocity u



primed quantities —> cloud frame

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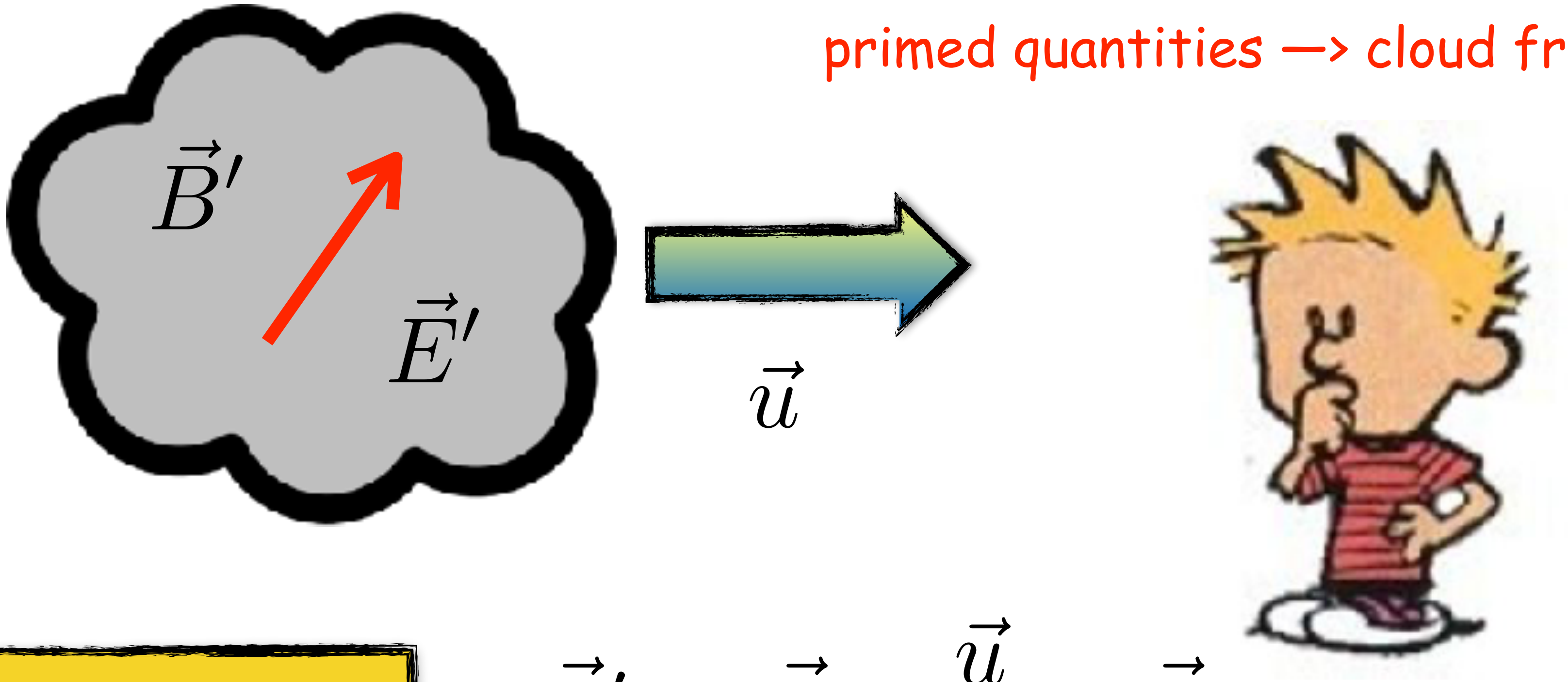

Lorentz transformation

$$\vec{E}' = \vec{E} + \frac{\vec{u}}{c} \times \vec{B}$$

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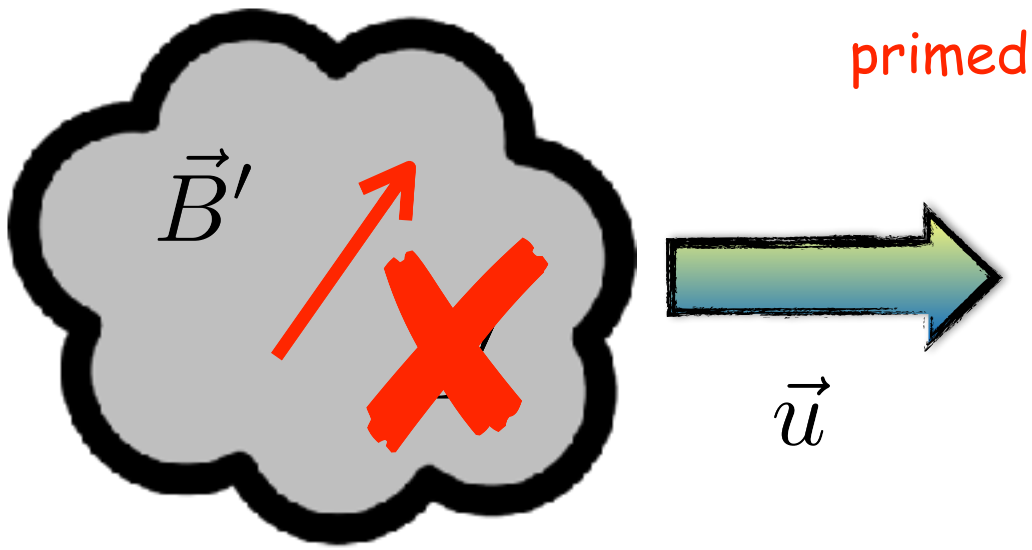

Lorentz transformation

$$\vec{E}' = \vec{E} + \frac{\vec{u}}{c} \times \vec{B}$$

$$0 = \vec{E} + \frac{\vec{u}}{c} \times \vec{B} \longrightarrow \vec{E} = -\frac{\vec{u}}{c} \times \vec{B}$$

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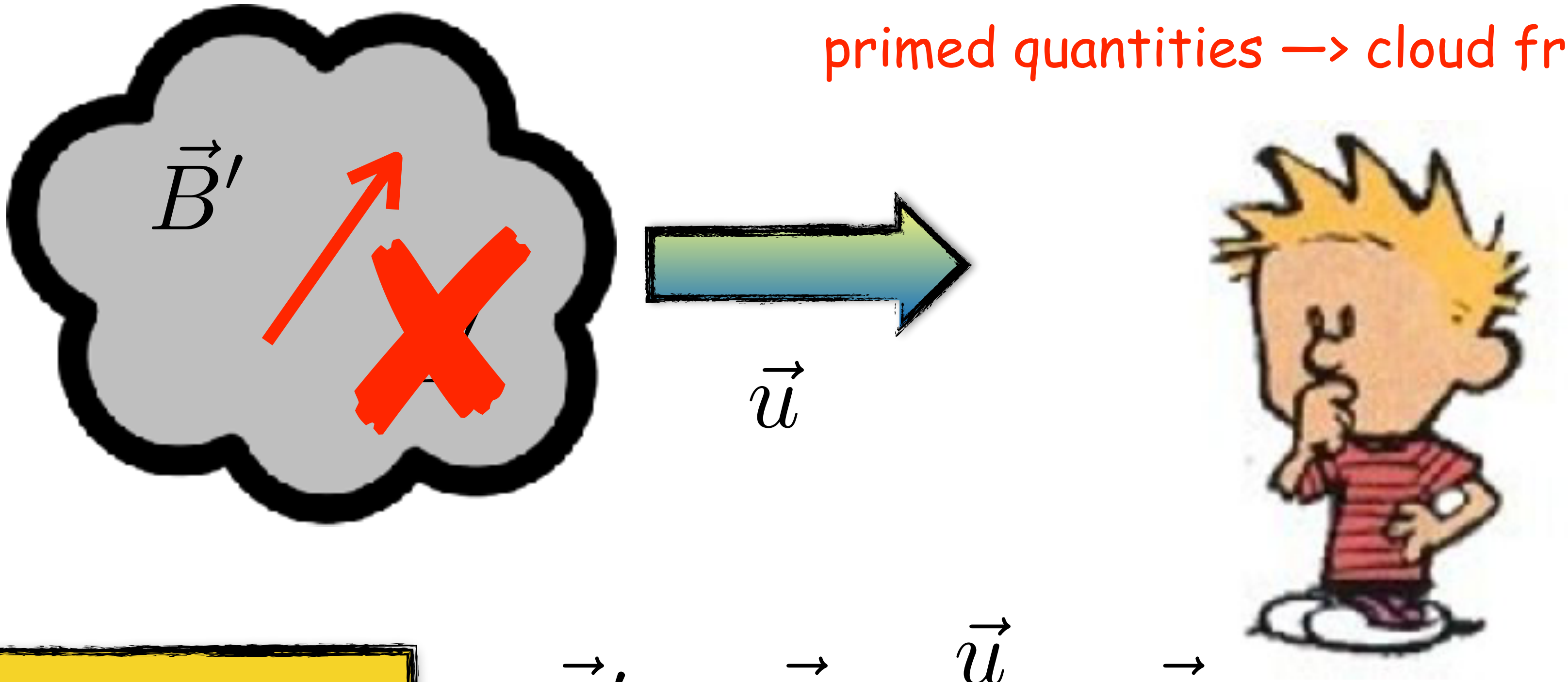

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an observer in the lab frame sees an electric field!

Order of magnitude estimates of the induced electric field

time-varying B-field

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

Order of magnitude estimates of the induced electric field

time-varying B-field

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

characteristic length

$$\nabla\times \quad \rightarrow \quad \frac{1}{L}$$

$$\frac{\partial}{\partial t} \quad \rightarrow \quad \frac{1}{T}$$

characteristic time

Order of magnitude estimates of the induced electric field

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characteristic time

characteristic velocity

$$E \approx \frac{L}{T}\frac{B}{c} \approx \frac{U}{c}B$$

Order of magnitude estimates of the induced electric field

time-varying B-field

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characteristic length

$$\nabla \times \quad \rightarrow \quad \frac{1}{L}$$

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characteristic time

$$\Rightarrow \quad E \approx \frac{L}{T}\frac{B}{c} \approx \frac{U}{c}B$$

characteristic velocity

Lorentz transformation

$$\vec{E} = -\frac{\vec{u}}{c} \times \vec{B} \quad \Rightarrow \quad E \approx \frac{U}{c}B$$

The Hillas criterion

Let's go back to the results obtained for the simplest accelerator

$$\mathcal{E}_{max} = q\ E\ L$$

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$$\Rightarrow$$

electric charge velocity

$$\mathscr{E}_{max} = \frac{q}{c}\ U\ B\ L$$

size

B-field

The Hillas criterion

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electric charge

velocity

B-field

size

acceleration rate —>

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t} = c\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}x} = cqE = q\ U\ B$$

The Hillas criterion

Let's go back to the results obtained for the simplest accelerator

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electric charge velocity

$$\mathscr{E}_{max} = \frac{q}{c}\;U\;B\;L$$

B-field size

acceleration rate —>

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t} = c\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}x} = cqE = q\;U\;B$$

Larmor radius —>

$$R_L = \frac{\mathscr{E}}{qB}$$

$$R_L^{max} = \left(\frac{U}{c}\right) L$$

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Let's go back to the results obtained for the simplest accelerator

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$$\Rightarrow$$

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B-field size

acceleration rate —>

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Larmor radius —>

$$R_L = \frac{\mathscr{E}}{qB} \Rightarrow R_L^{max} = \left(\frac{U}{c}\right) L$$

acceleration time —>

$$\tau_{acc}^{H} = \frac{\mathscr{E}_{max}}{\mathrm{d}\mathscr{E}/\mathrm{d}t} = \left(\frac{U}{c}\right)^{-1} \frac{R_L}{c}$$

The Hillas criterion

Let's go back to the results obtained for the simplest accelerator

$$\mathcal{E}_{max} = q\ E\ L$$

$$E \approx \frac{U}{c} B$$

electric charge

velocity

$$\mathcal{E}_{max} = q \frac{U}{c} B\ L$$

B-field

size

very general, we didn't assume anything about the nature of the accelerator!

acceleration rate

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = c\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}x} = cqE = q\ U\ B$$

radius —>

$$R_L = \frac{\mathcal{E}}{qB}$$

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acceleration time —>

$$\tau_{acc}^{H} = \frac{\mathcal{E}_{max}}{\mathrm{d}\mathcal{E}/\mathrm{d}t} = \left(\frac{U}{c}\right)^{-1} \frac{R_L}{c}$$

The Hillas criterion in numbers

$$\boxed{\mathscr{E}_{max} = \frac{q}{c}\, U\, B\, L \sim 3\ Z \left(\frac{U}{1000\ \mathrm{km/s}}\right)\left(\frac{B}{\mu\mathrm{G}}\right)\left(\frac{L}{\mathrm{pc}}\right) \mathrm{TeV}}$$

The Hillas criterion in numbers

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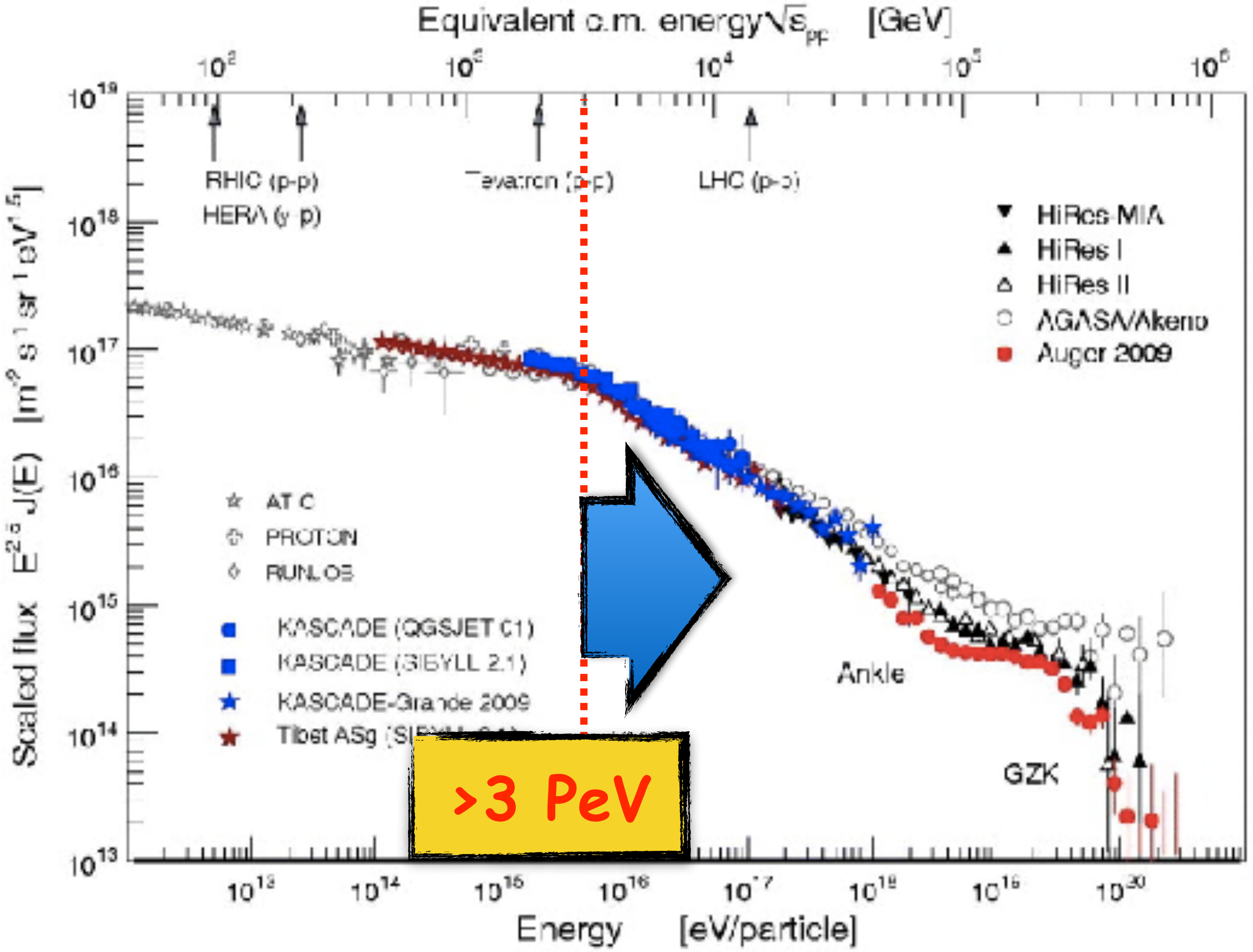

>3 PeV

The Hillas criterion in numbers

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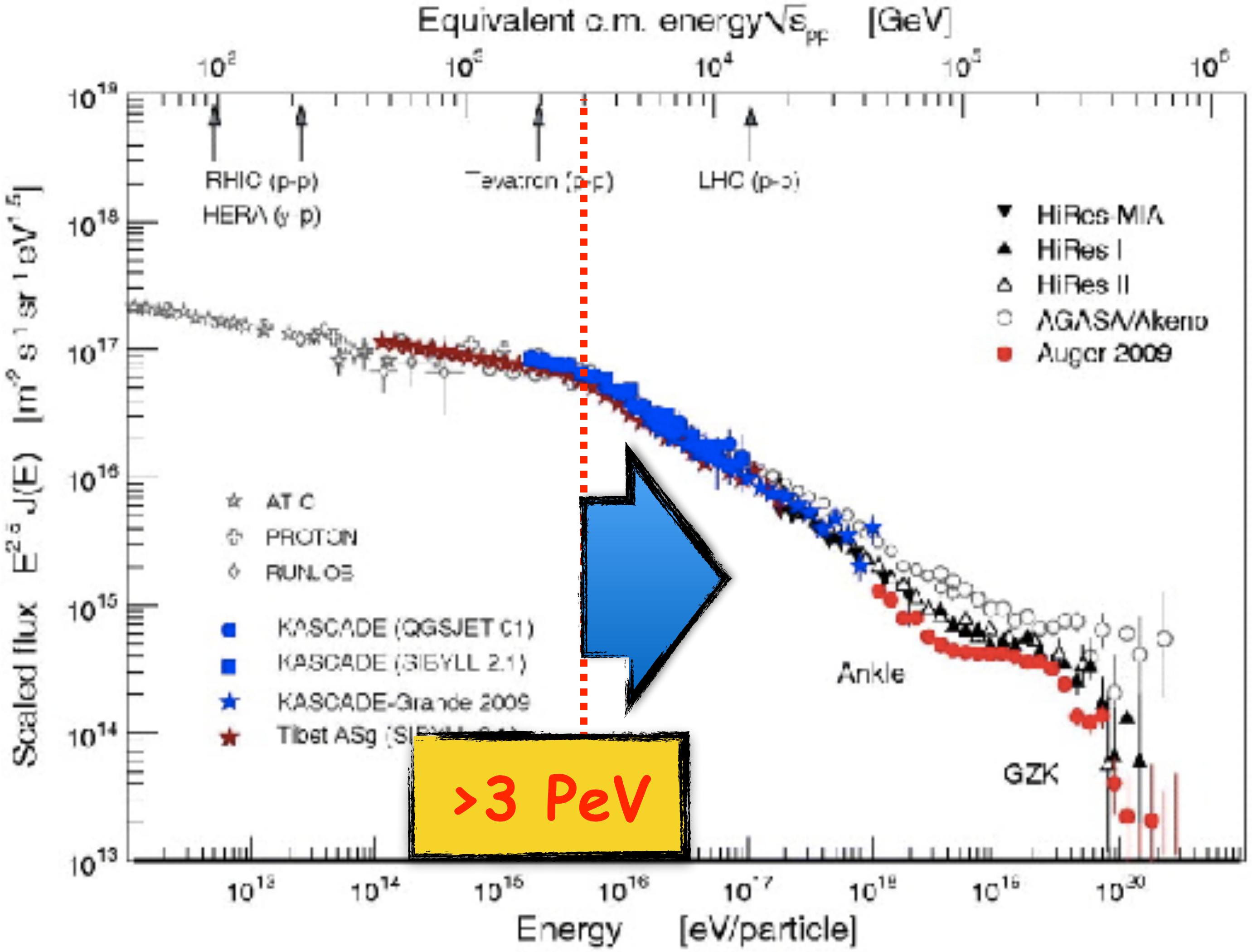


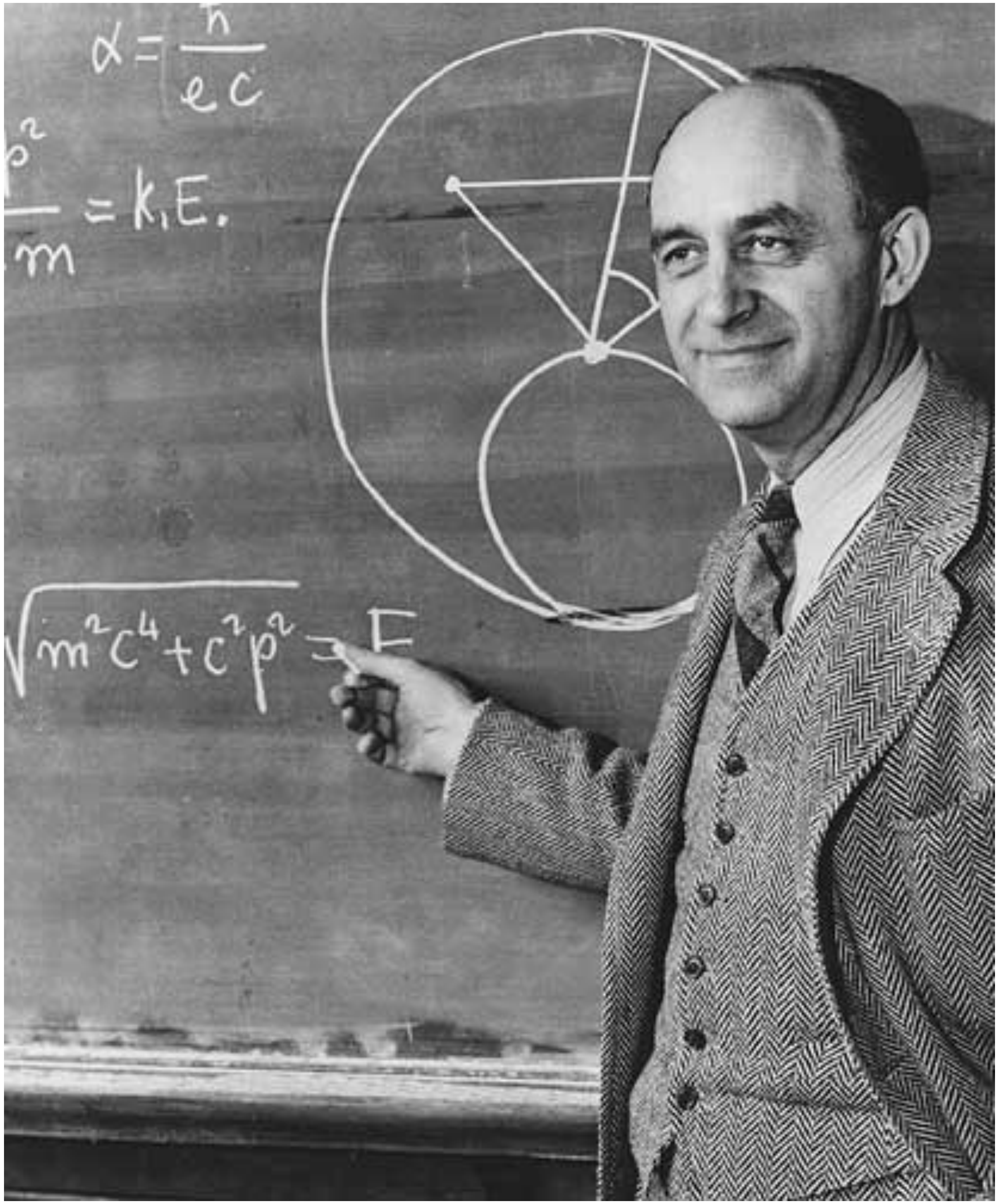
$$\tau_{acc} \sim \frac{\mathscr{E}}{\mathrm{d}\mathscr{E}/\mathrm{d}t} \sim \left(\frac{U}{c}\right)^{-1} \frac{R_L}{c} = 3 \left(\frac{L}{\text{pc}}\right)\ \text{yr}$$

Advise…

If you had to remember only one thing about cosmic ray acceleration, that would probably be the Hillas criterion. However, while this criterion imposes necessary conditions to accelerate particles, it tells us NOTHING about HOW particles are accelerated…

Note: from now on we indicate particle energies with E rather $\mathcal{E}$ as there is no longer ambiguity with electric field

Fermi idea



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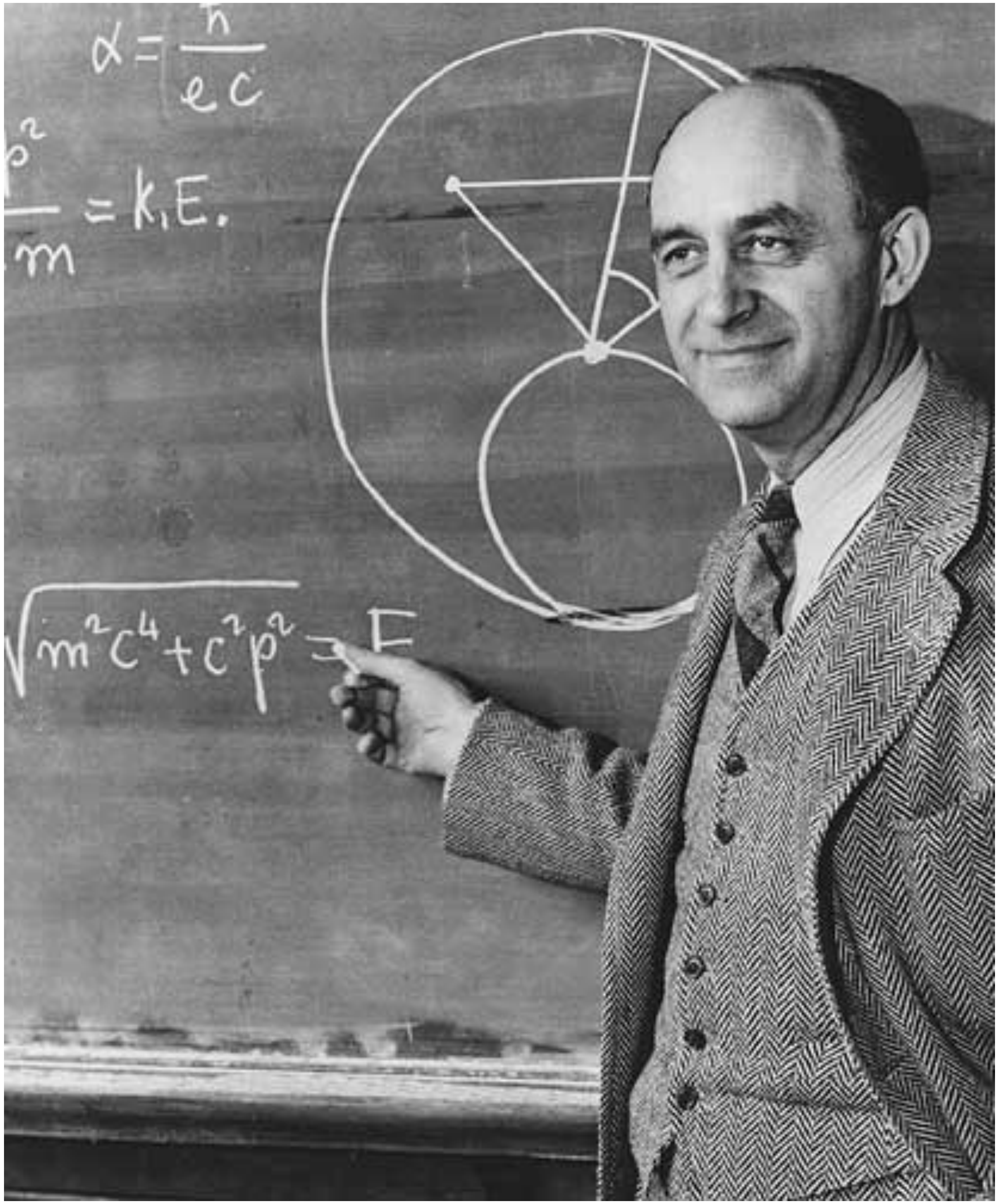
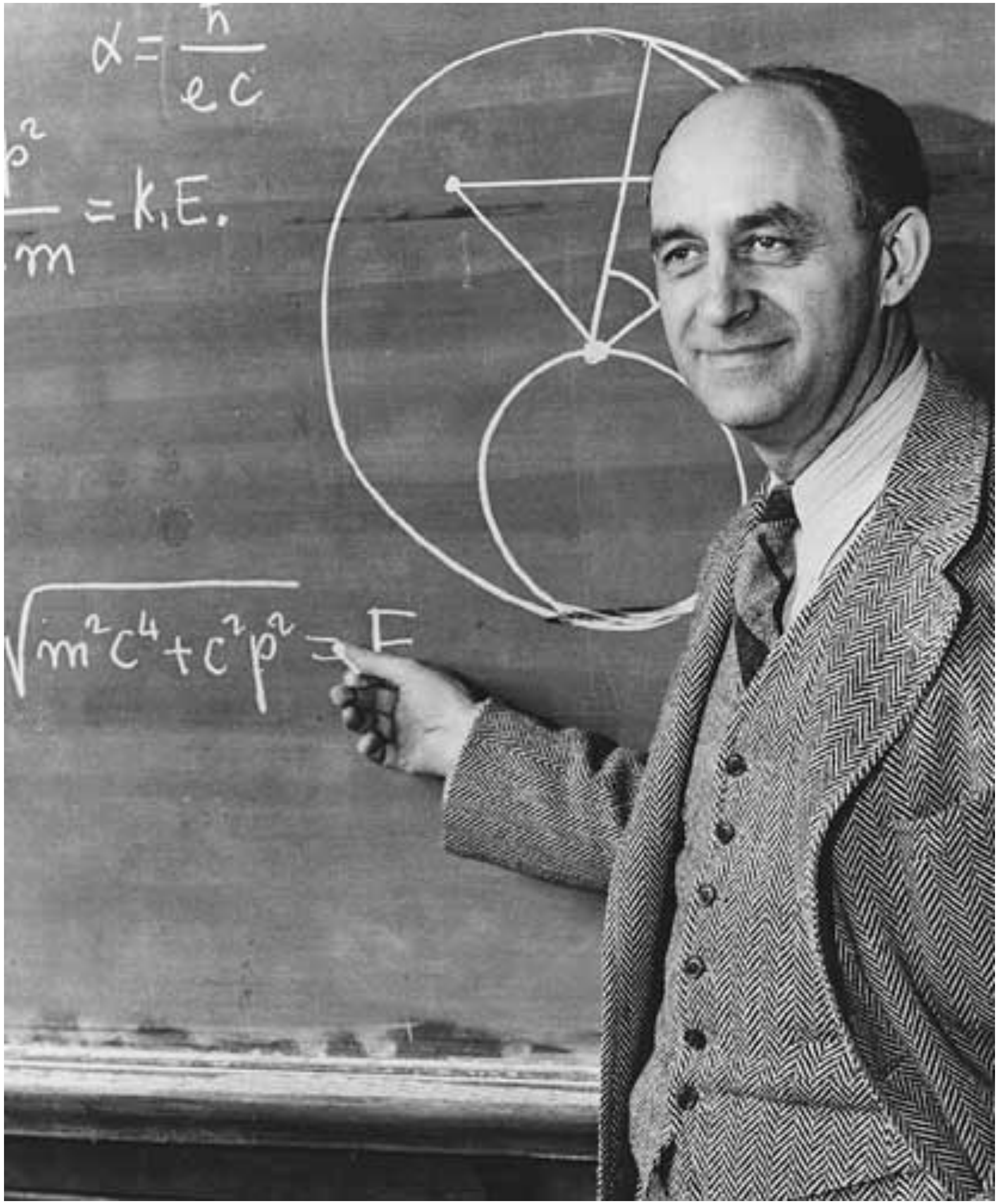


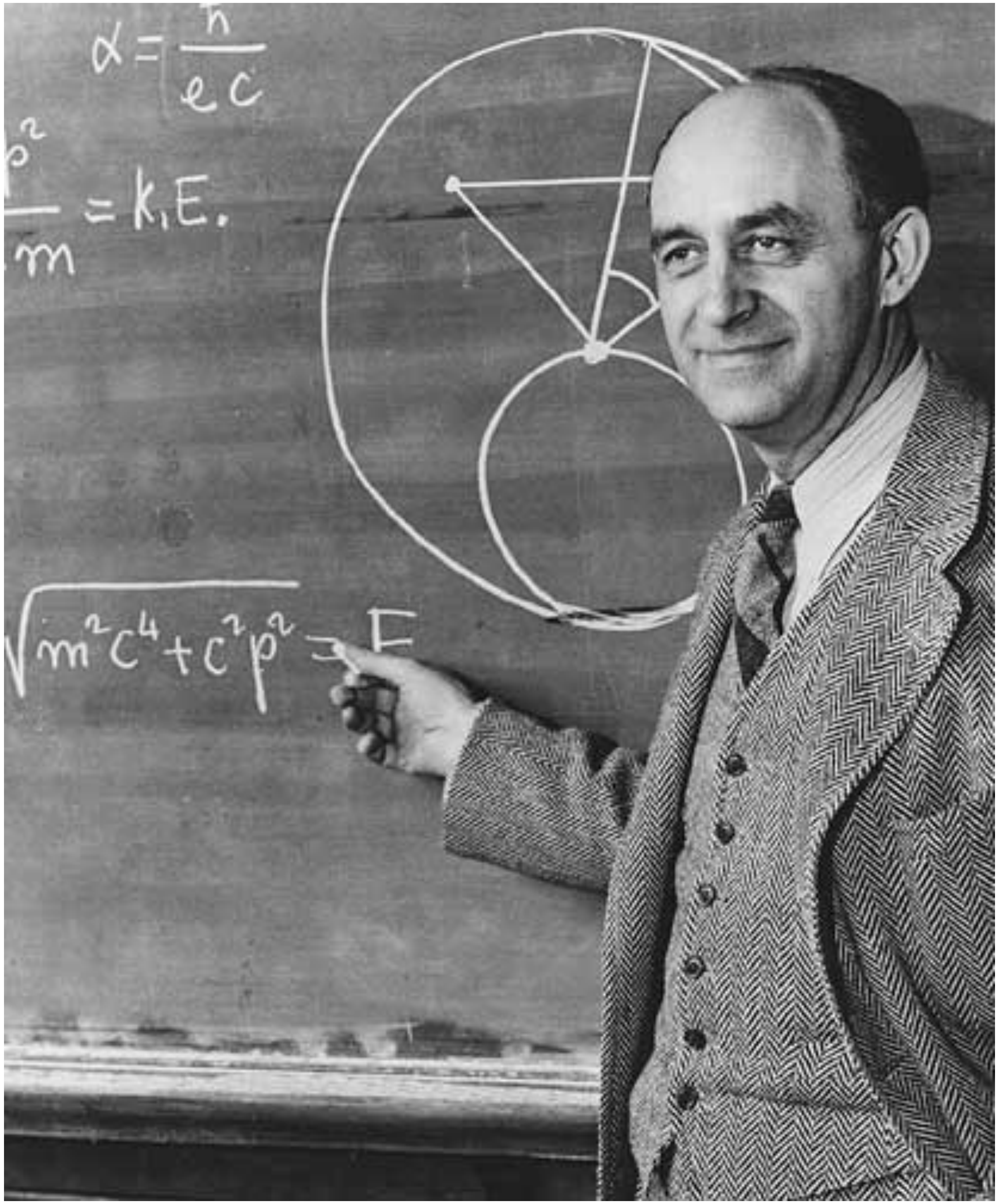
- The interstellar medium is:

Fermi idea



- The interstellar medium is:
  - inhomogeneous

Fermi idea



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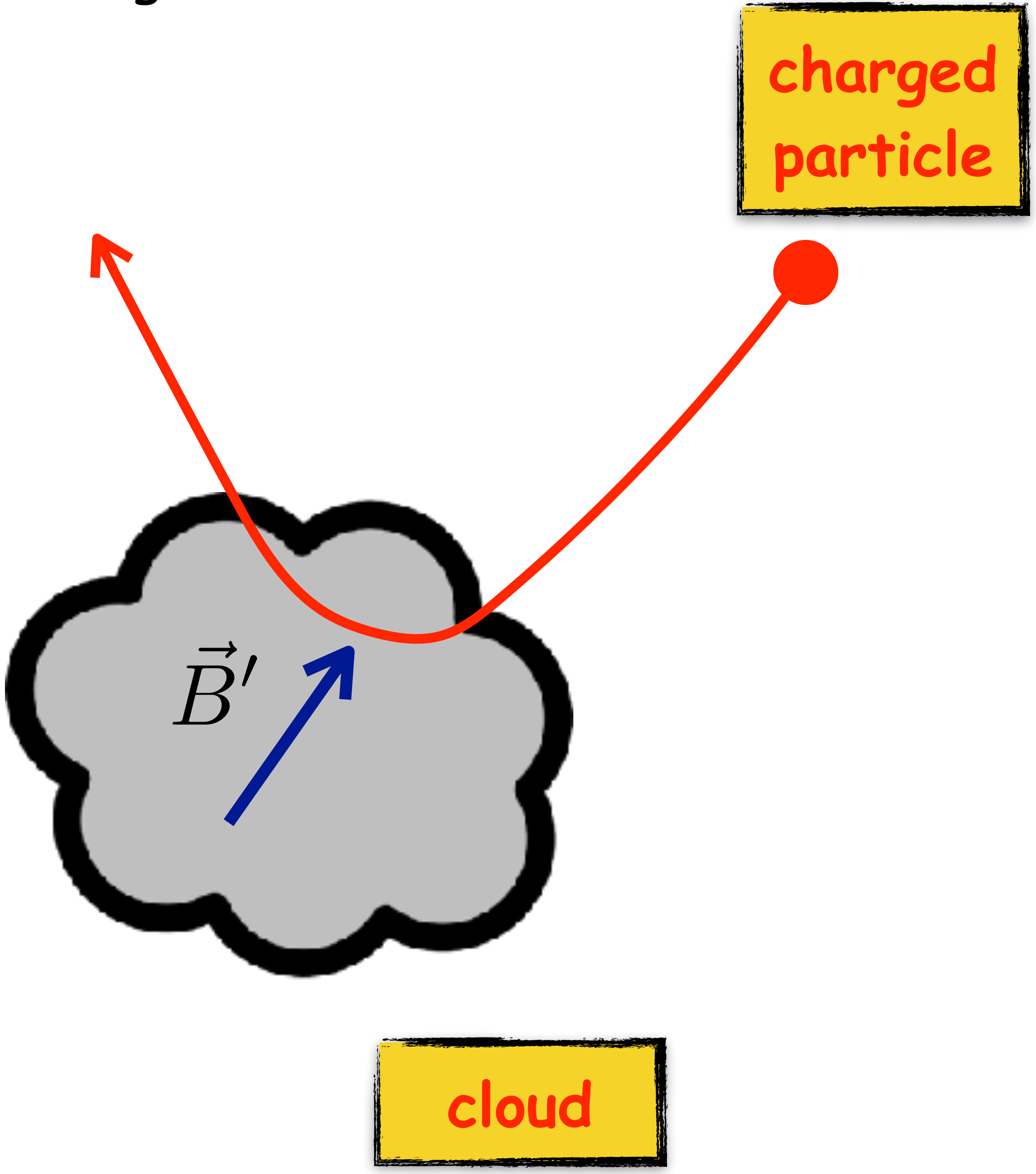
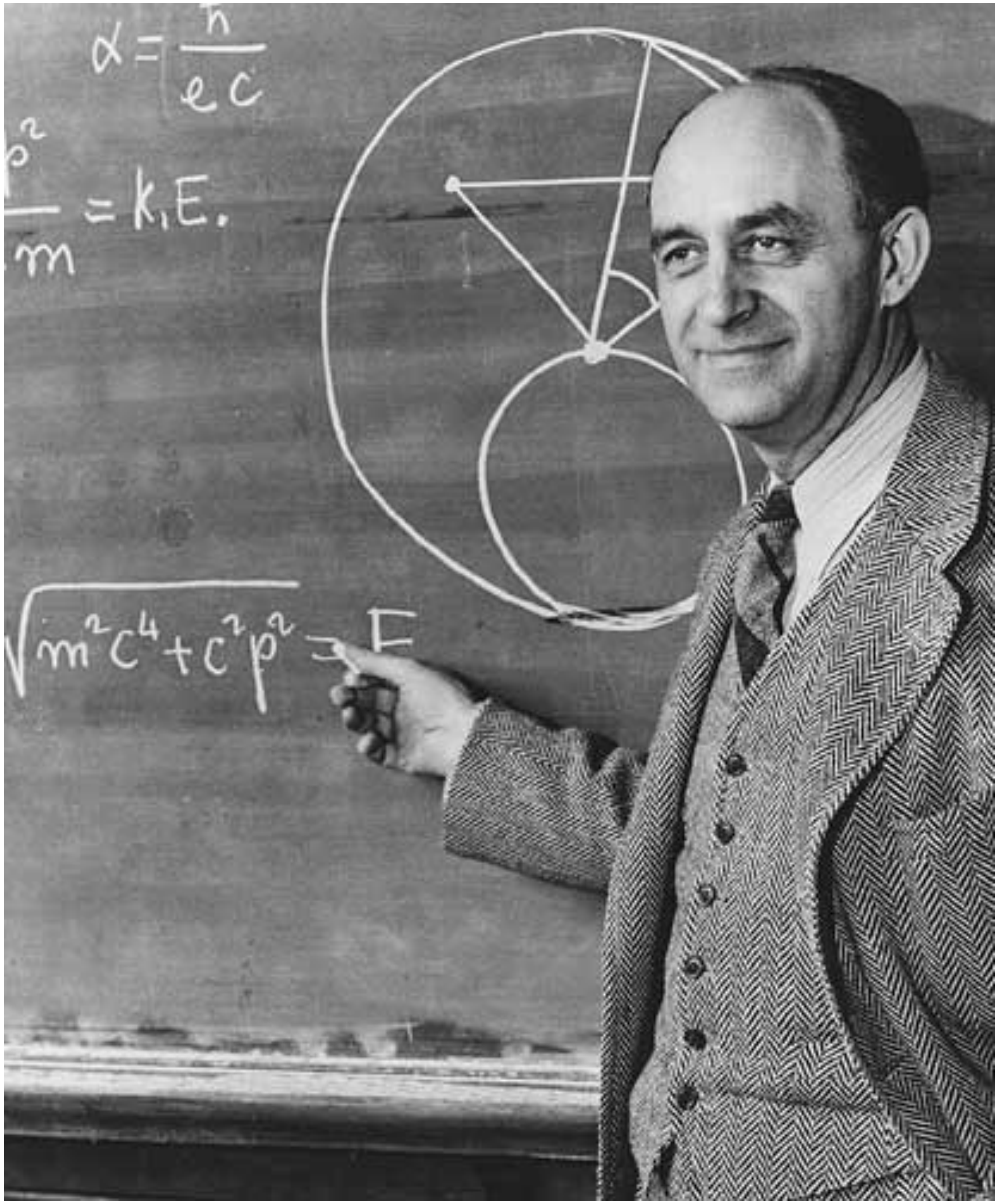


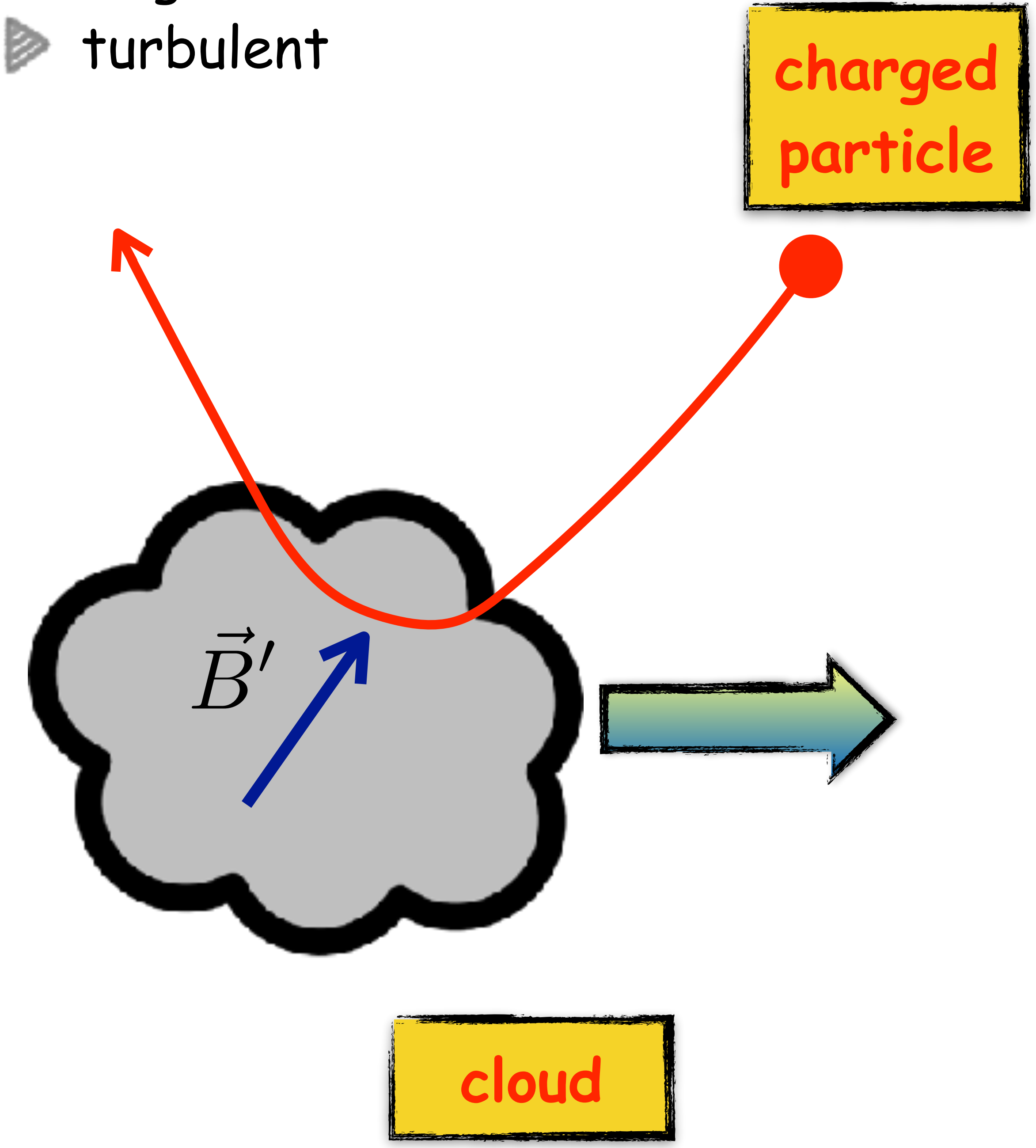
charged particle

cloud

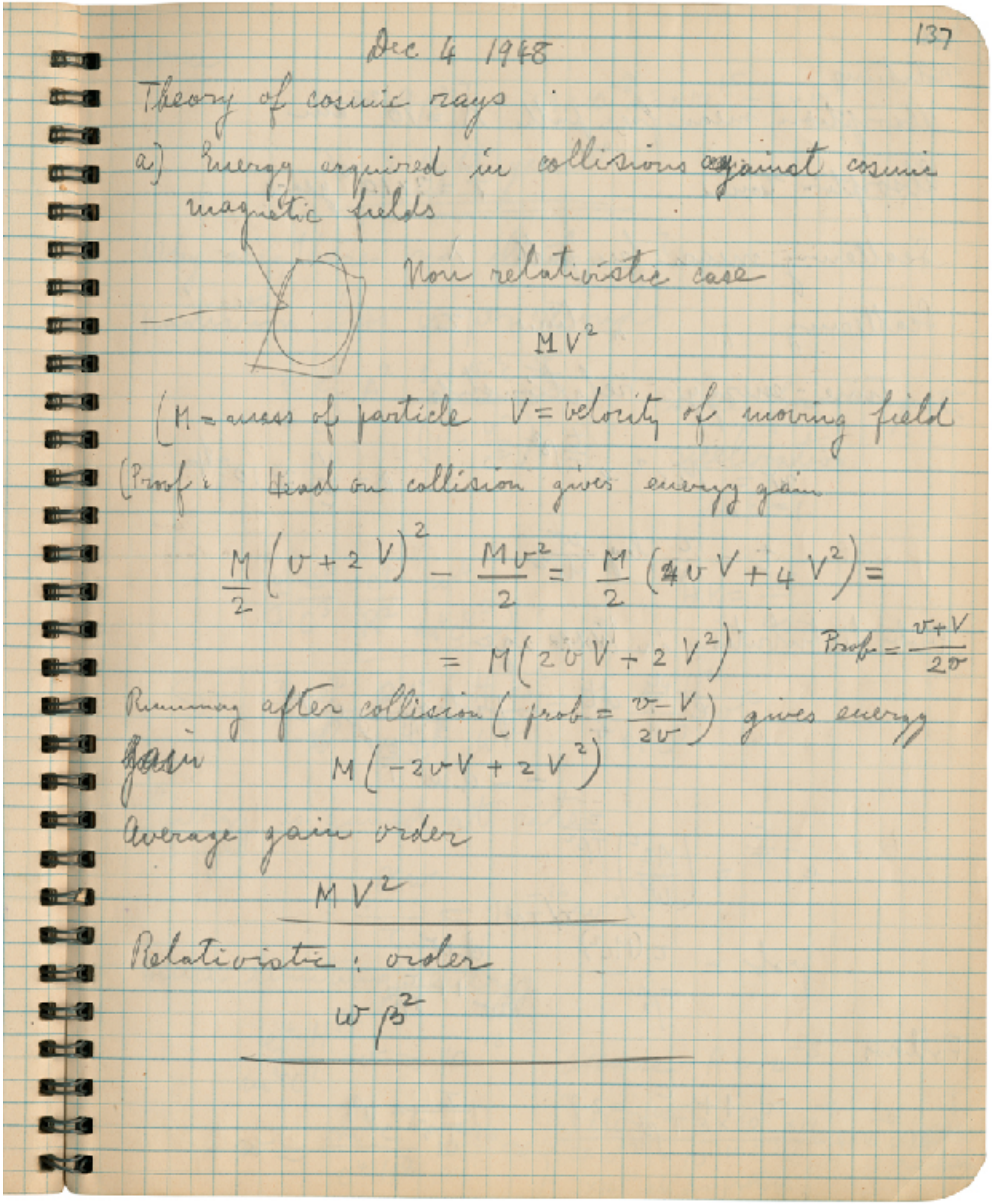
Fermi idea



- The interstellar medium is:
  - inhomogeneous
  - magnetised
  - turbulent



Fermi's notebook
December 1948



137

Dec 4 1948

Theory of cosmic rays.

a) Energy acquired in collisions against cosmic magnetic fields.

Non relativistic case

$$MV^2$$

(M = mass of particle V = velocity of moving field

(Proof: Head on collision gives energy gain

$$\frac{M}{2}(v+2V)^2 - \frac{Mv^2}{2} = \frac{M}{2}(4vV+4V^2) =$$

$$= M(2vV+2V^2)$$ Prob $= \frac{v+V}{2v}$

Running after collision (prob $= \frac{v-V}{2v}$) gives energy gain

$$M(-2vV+2V^2)$$

Average gain order

$$MV^2$$

Relativistic: order

$$w\beta^2$$

From Fermi's notebook

Fermi started by considering a non-relativistic problem involving elastic collisions between two very different (in mass) solid bodies

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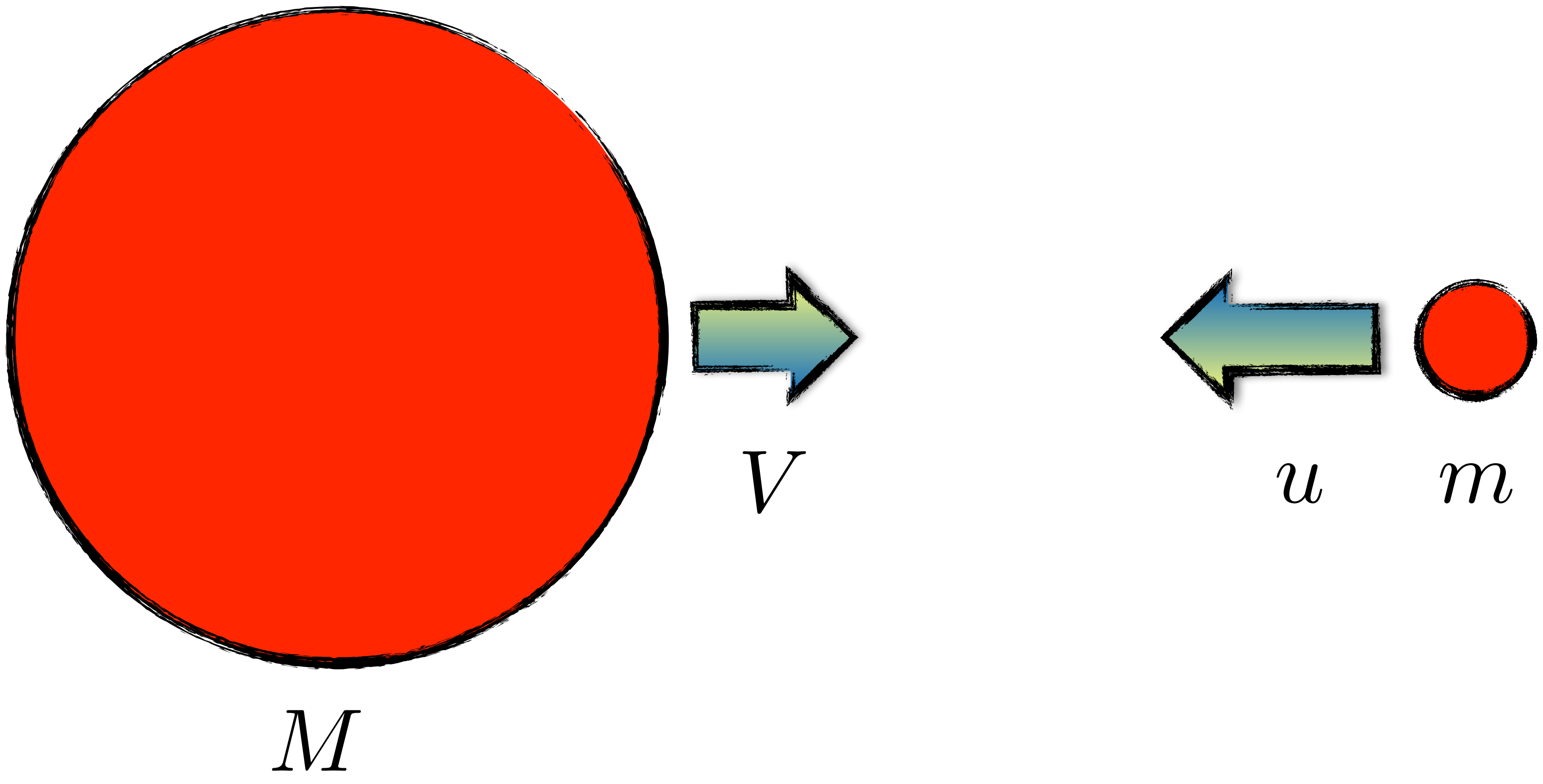
head-on collision

$$M \gg m$$



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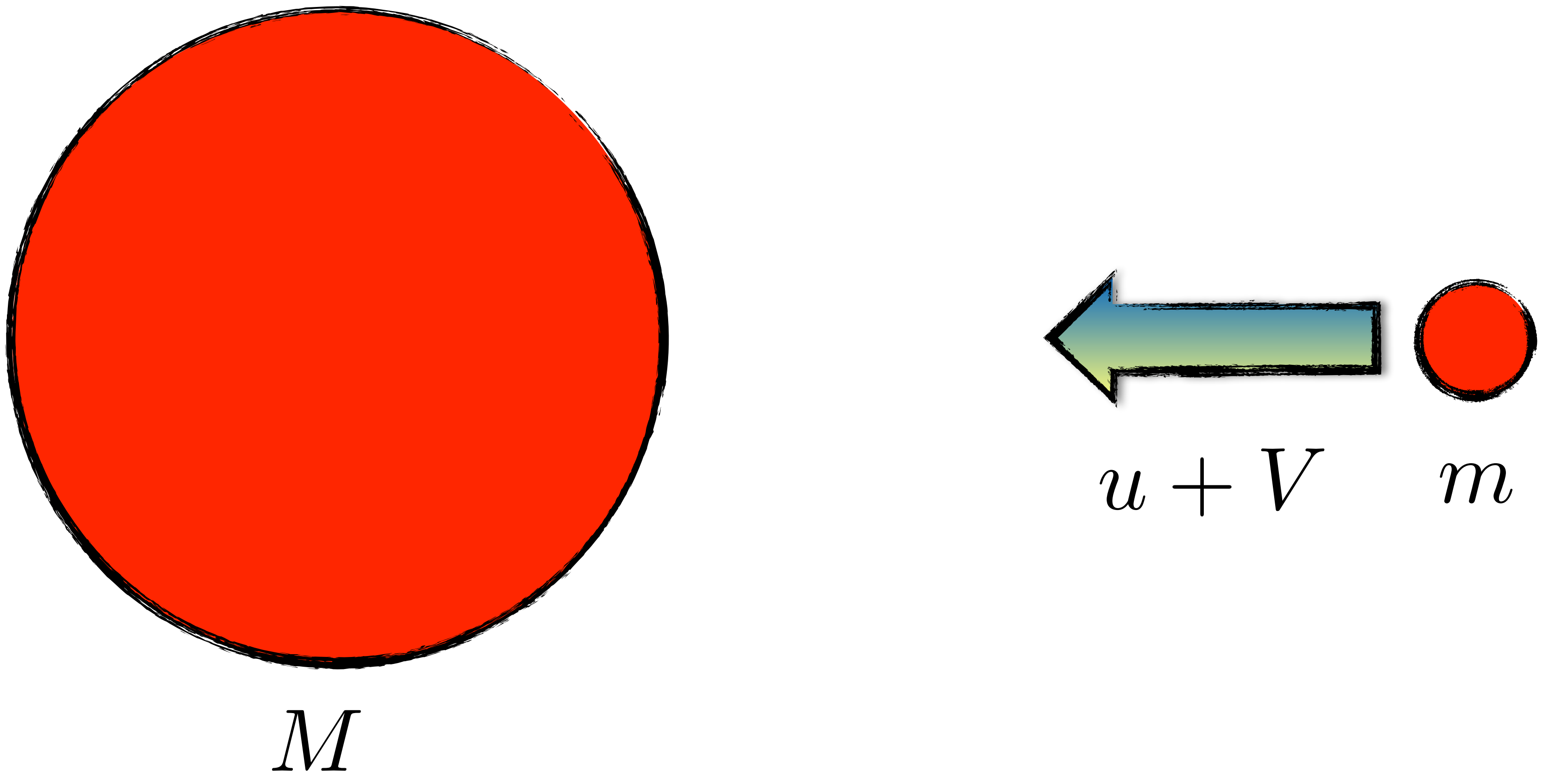
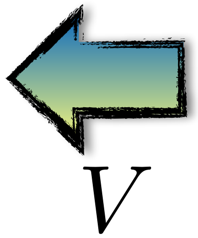


add  to move to the rest frame where M is at rest

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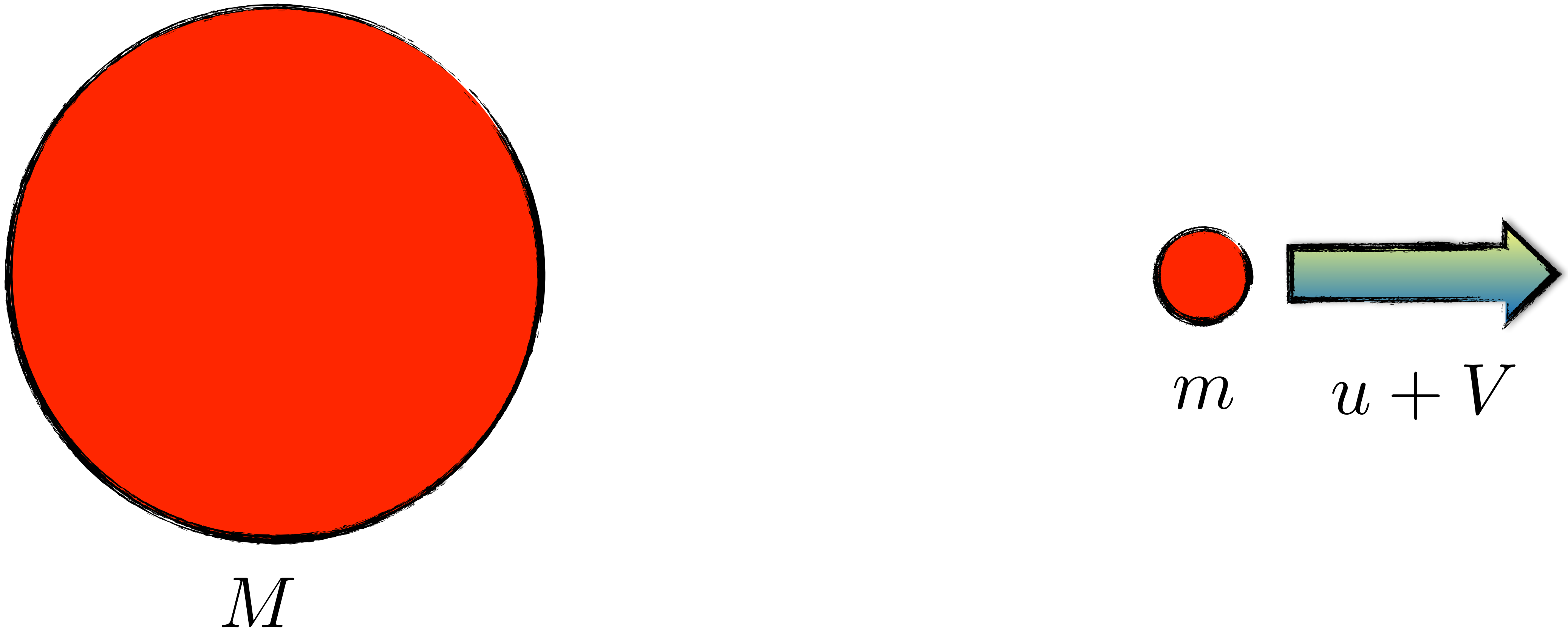

elastic collision: same velocity but in the opposite direction

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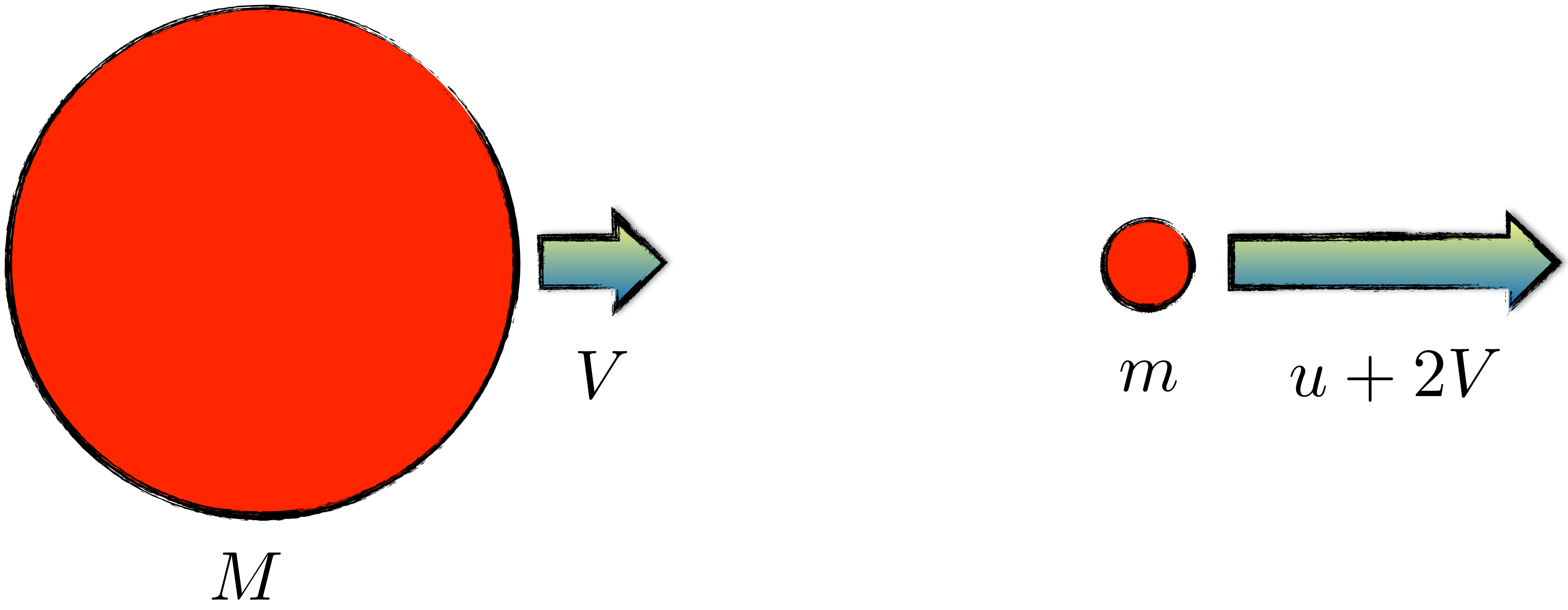

add $\xrightarrow[V]{}$ to move back to the lab rest frame

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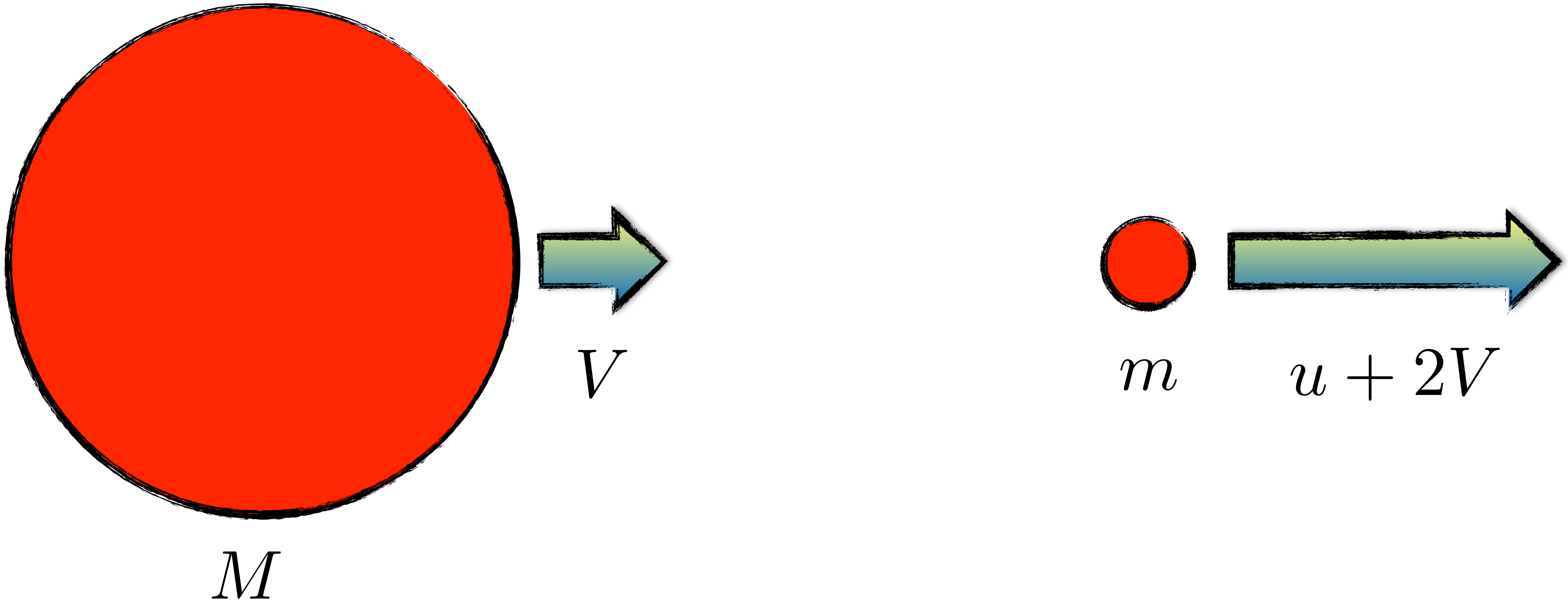

Energy gain -> $$\frac{1}{2}m(u+2V)^2 - \frac{1}{2}mu^2 = 2m(V^2 + uV)$$

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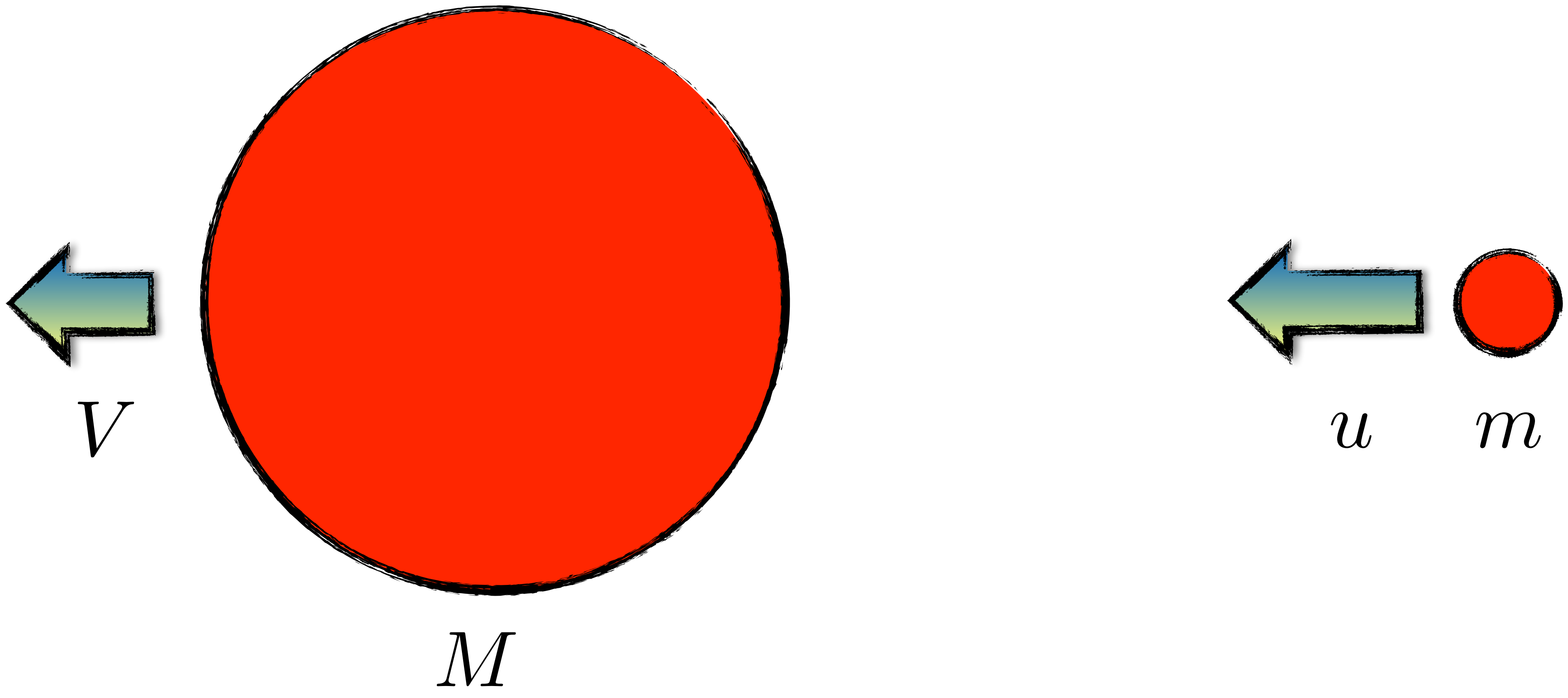

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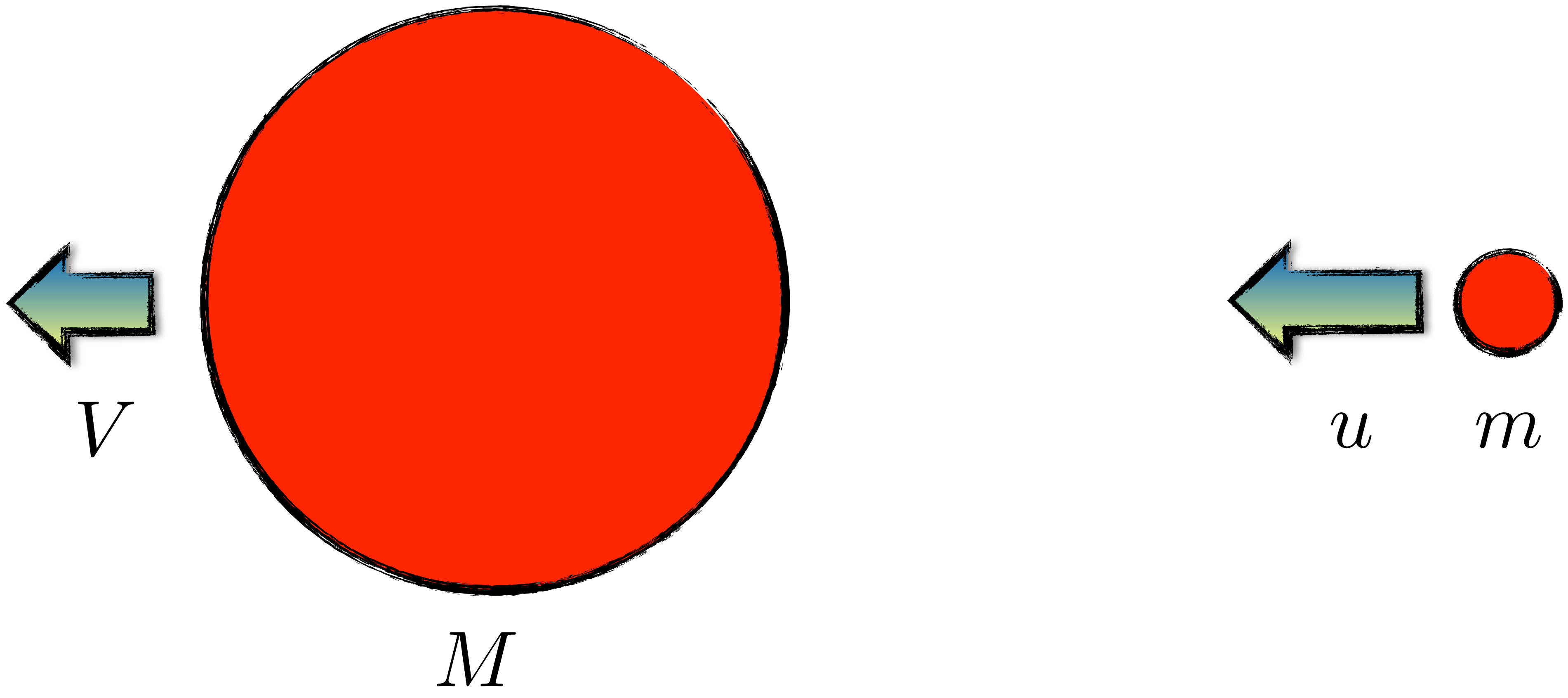

Energy gain -> $$\frac{1}{2}m(u-2V)^2 - \frac{1}{2}mu^2 = 2m(V^2 - uV)$$

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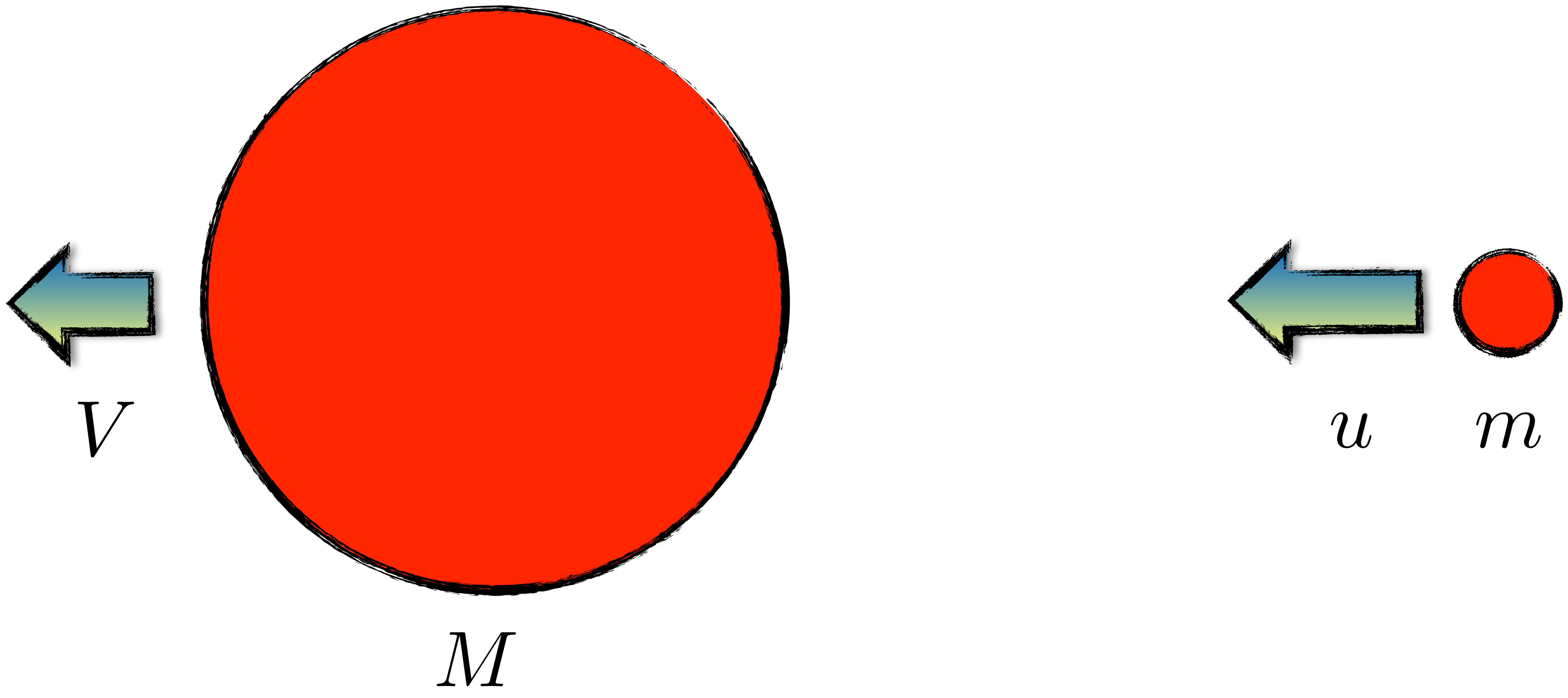

Energy **loss** $\frac{1}{2}m(u-2V)^2 - \frac{1}{2}mu^2 = 2m(V^2 - uV) \quad < 0$

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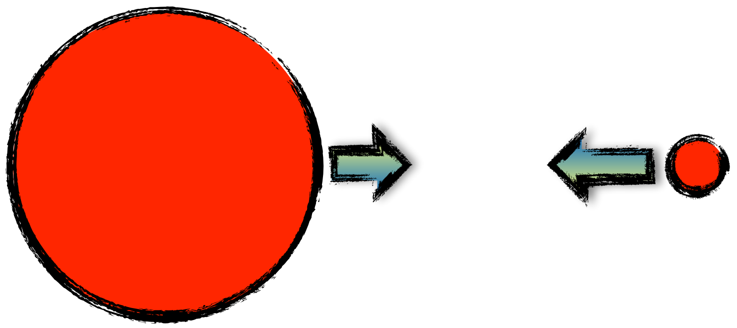
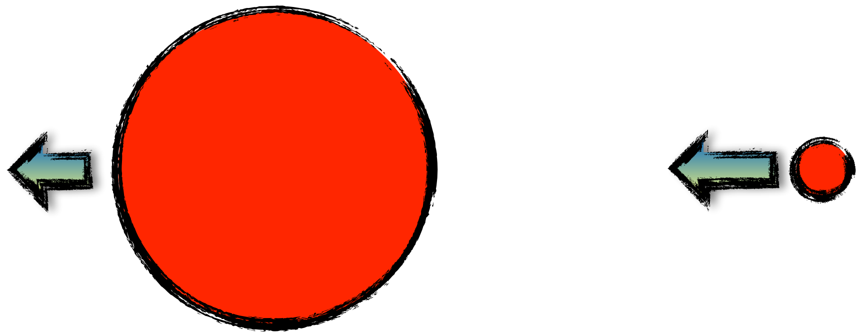
head-on collision -> gain



running after collision -> loss



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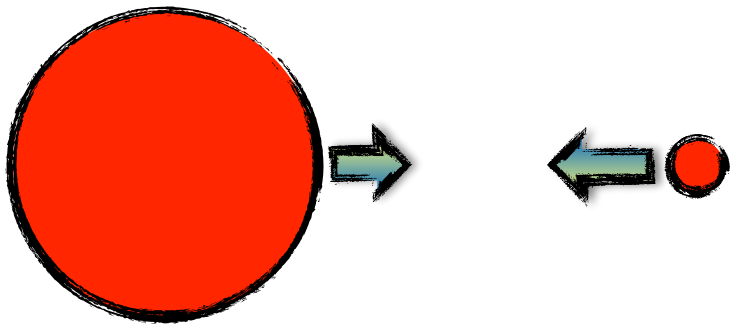
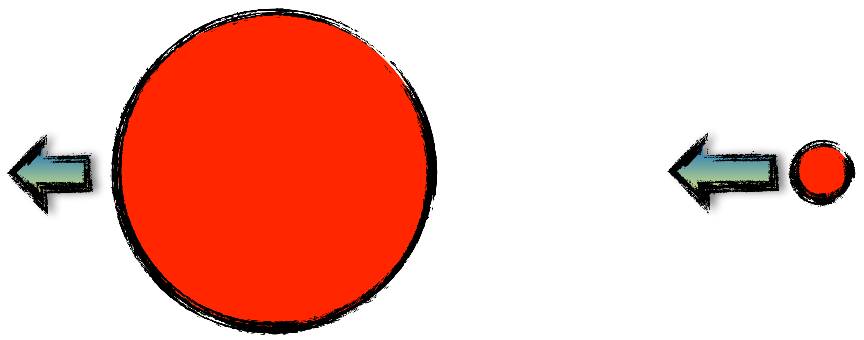


$$\frac{u+V}{2u}$$

probability

running after collision -> loss



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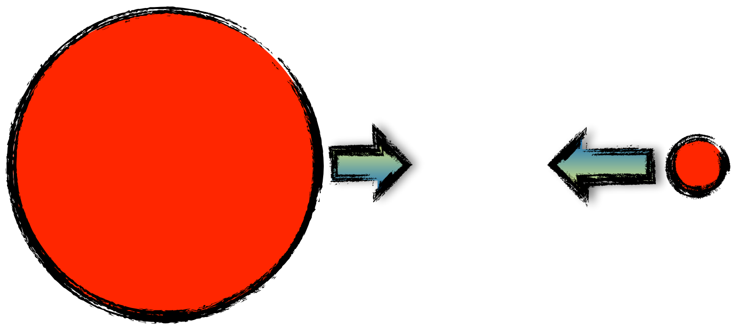

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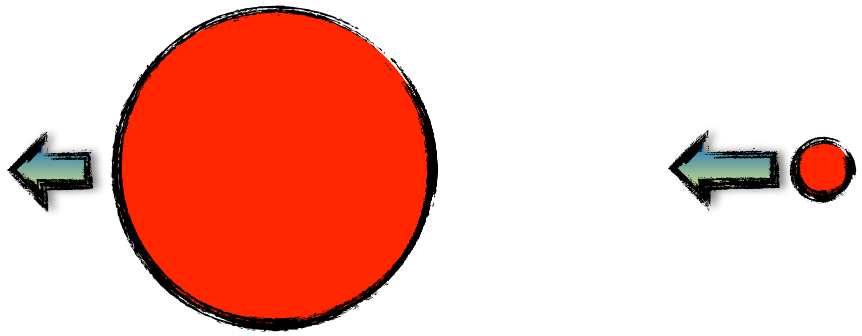

$$\frac{u-V}{2u}$$

$$\Delta E = 2m(V^2 + uV)\frac{u+V}{2u} + 2m(V^2 - uV)\frac{u-V}{2u} = 4mV^2$$

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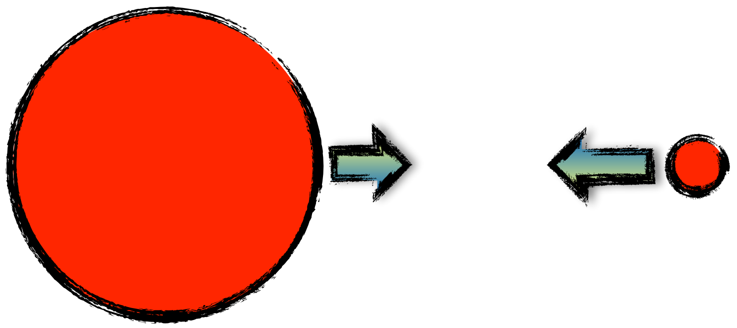

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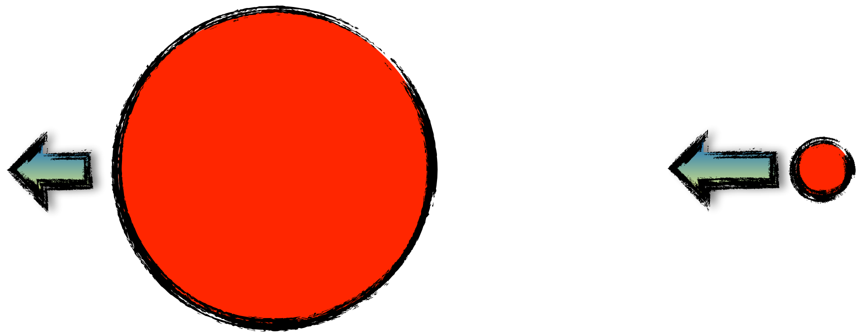

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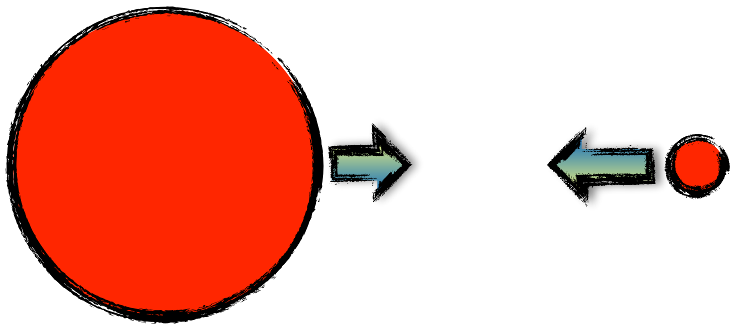

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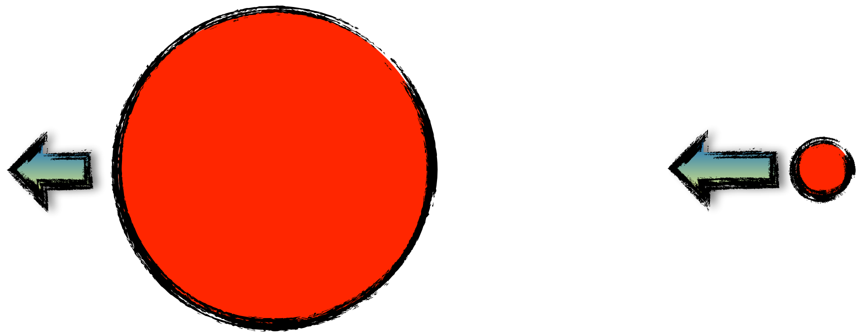

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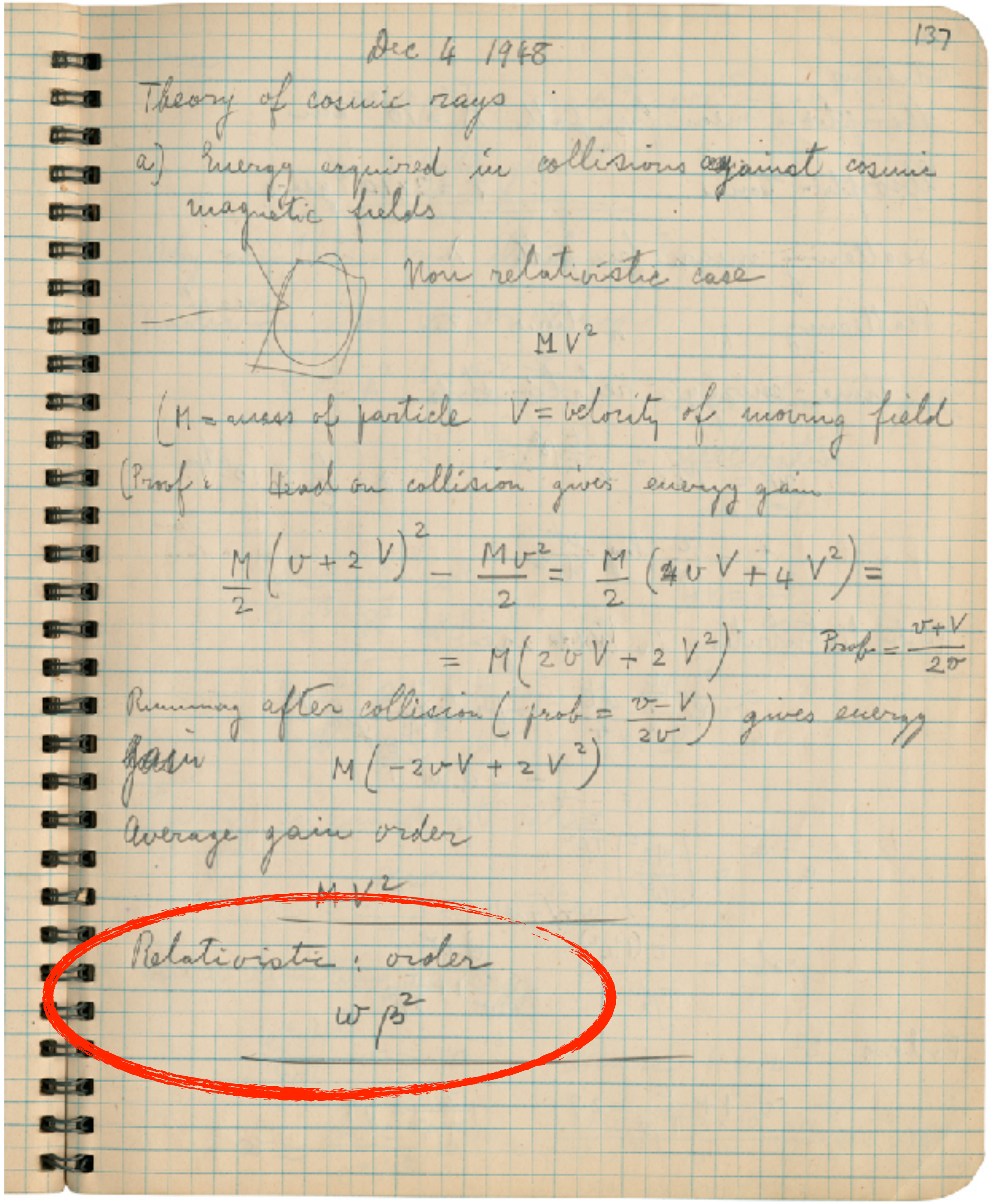
second order

From Fermi's notebook

Fermi ends his notes saying that for the relativistic case (u —> c) one should expect:



137

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Average gain order

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Relativistic: order

$$\omega\beta^2$$

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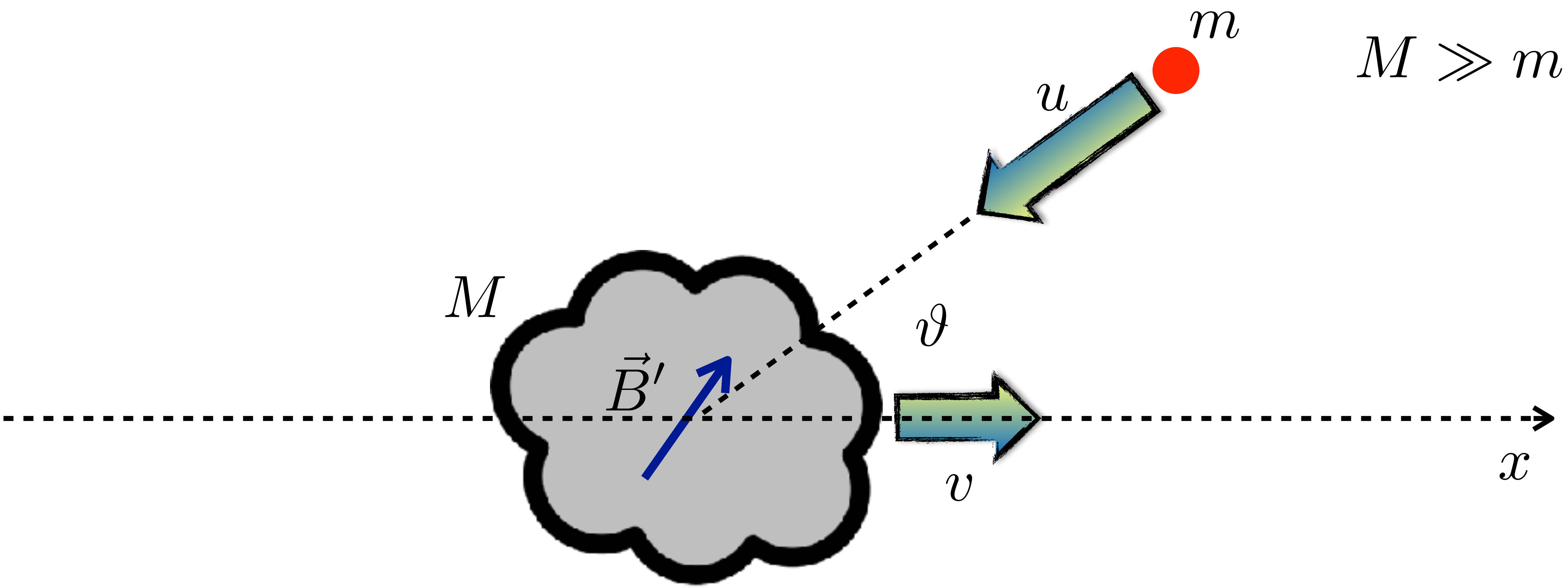
Fermi II

From Fermi's paper

Fermi (1949, 1954)



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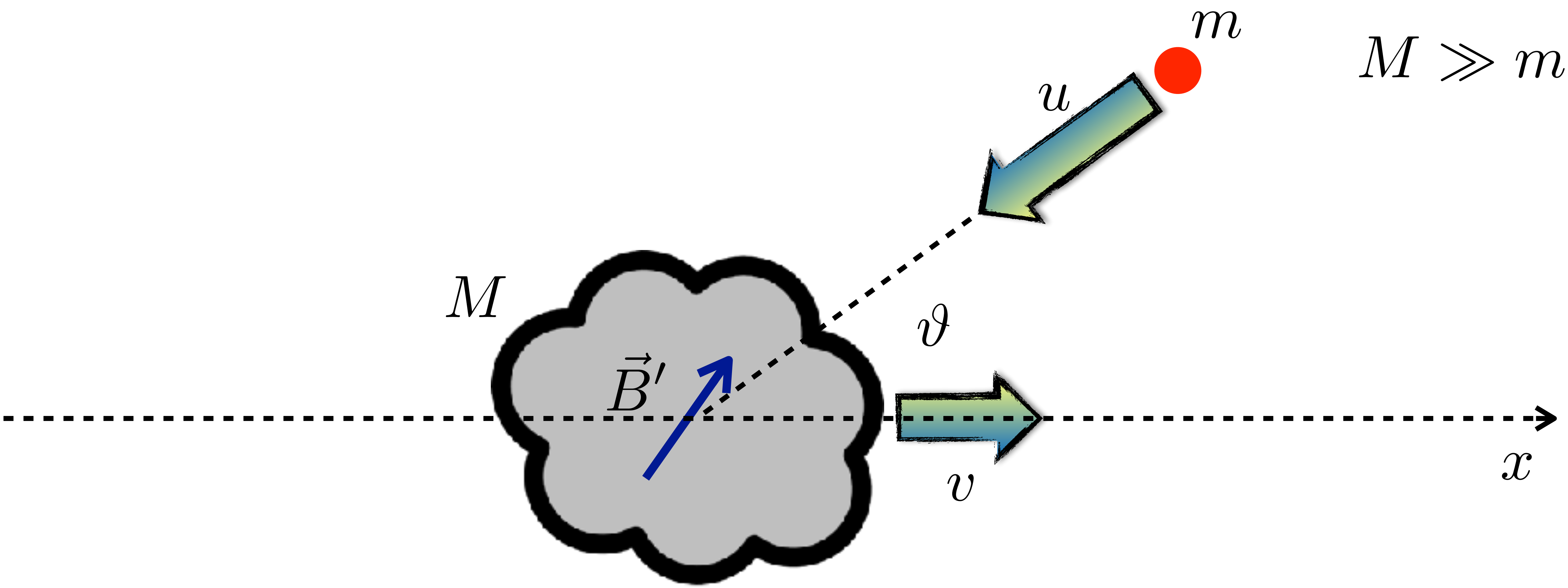

energy of the particle in the cloud frame —> $E' = \gamma_v \left(E + vp\cos\vartheta \right)$

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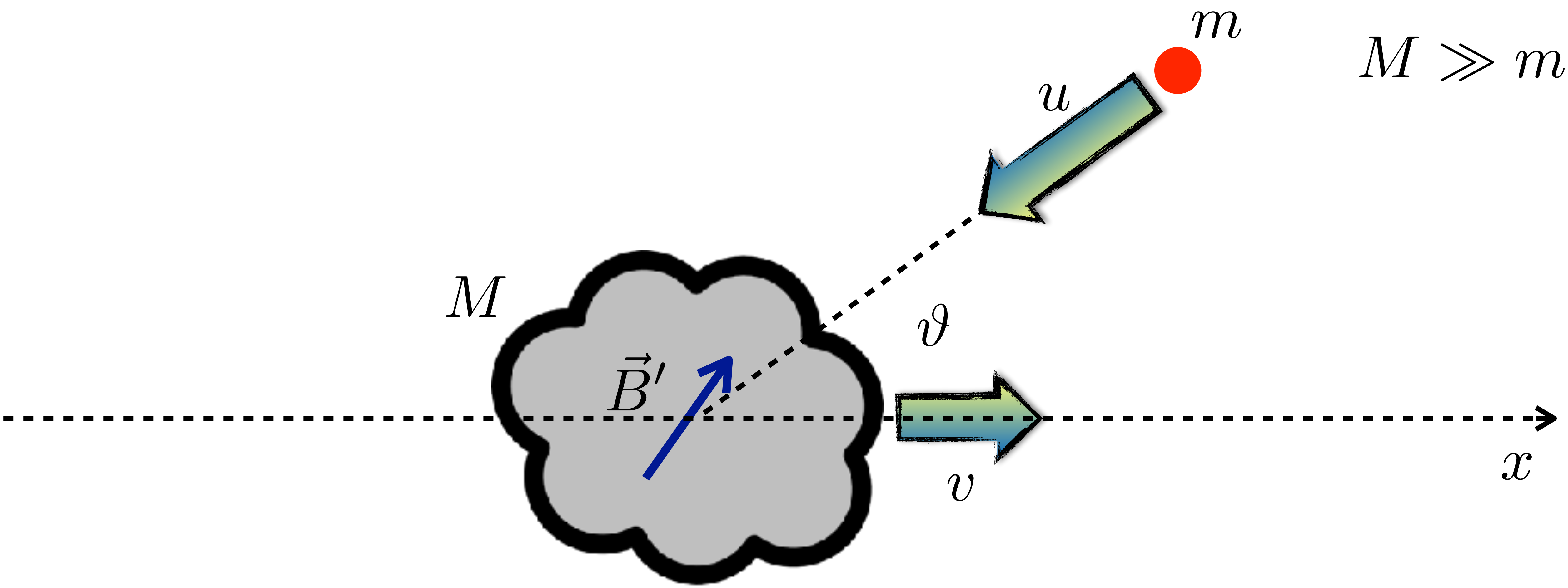

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From Fermi's paper

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elastic scattering
in the cloud frame

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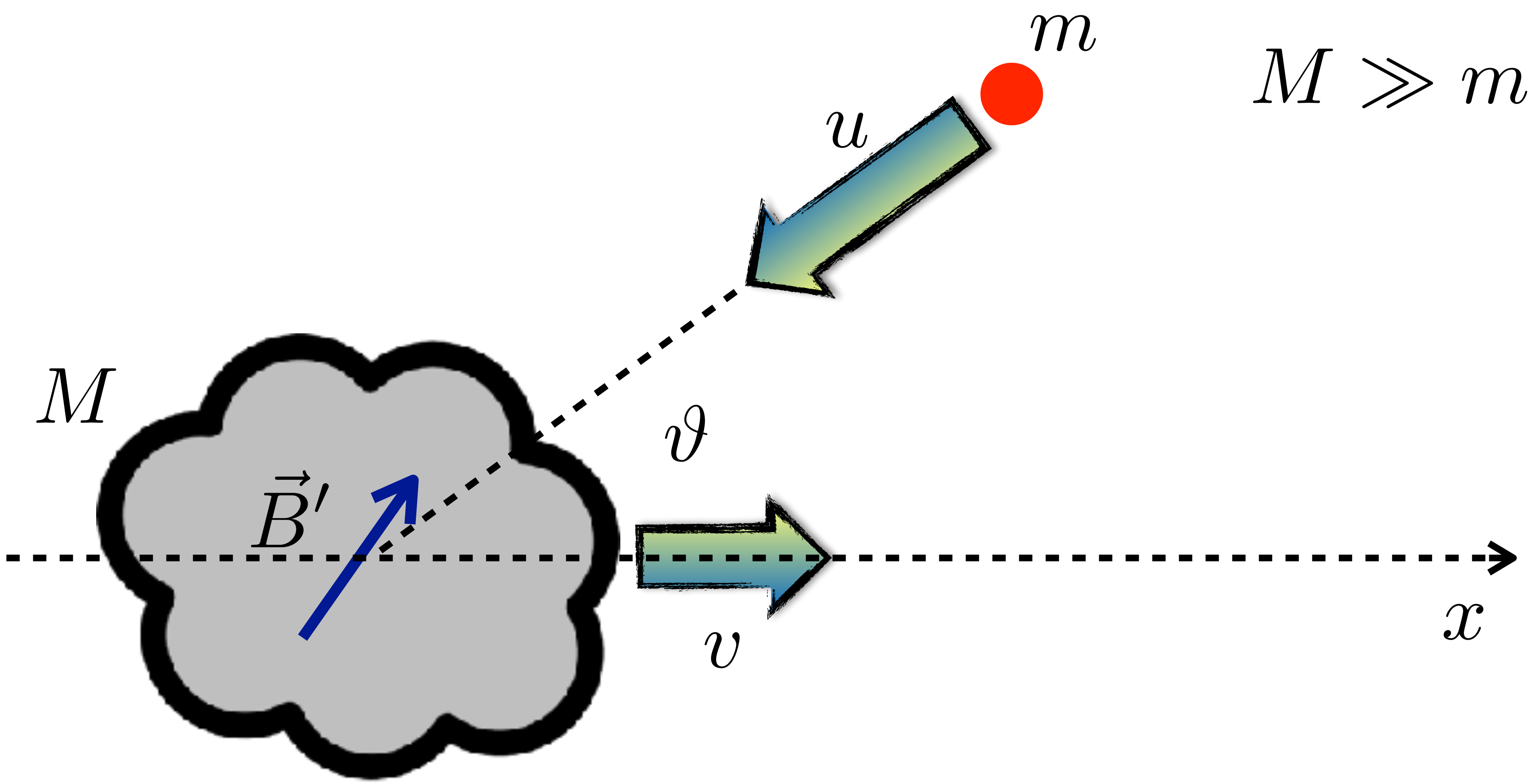

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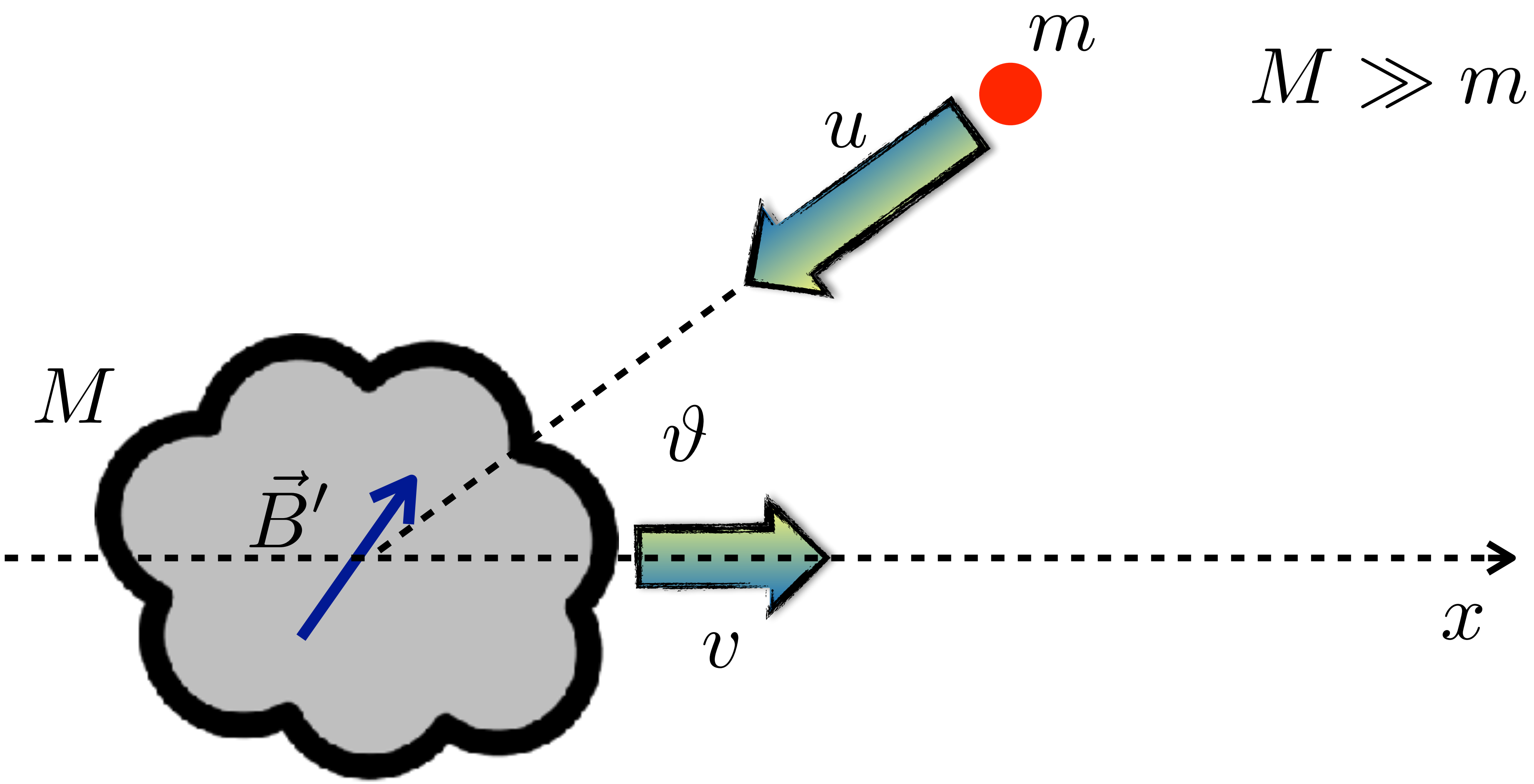

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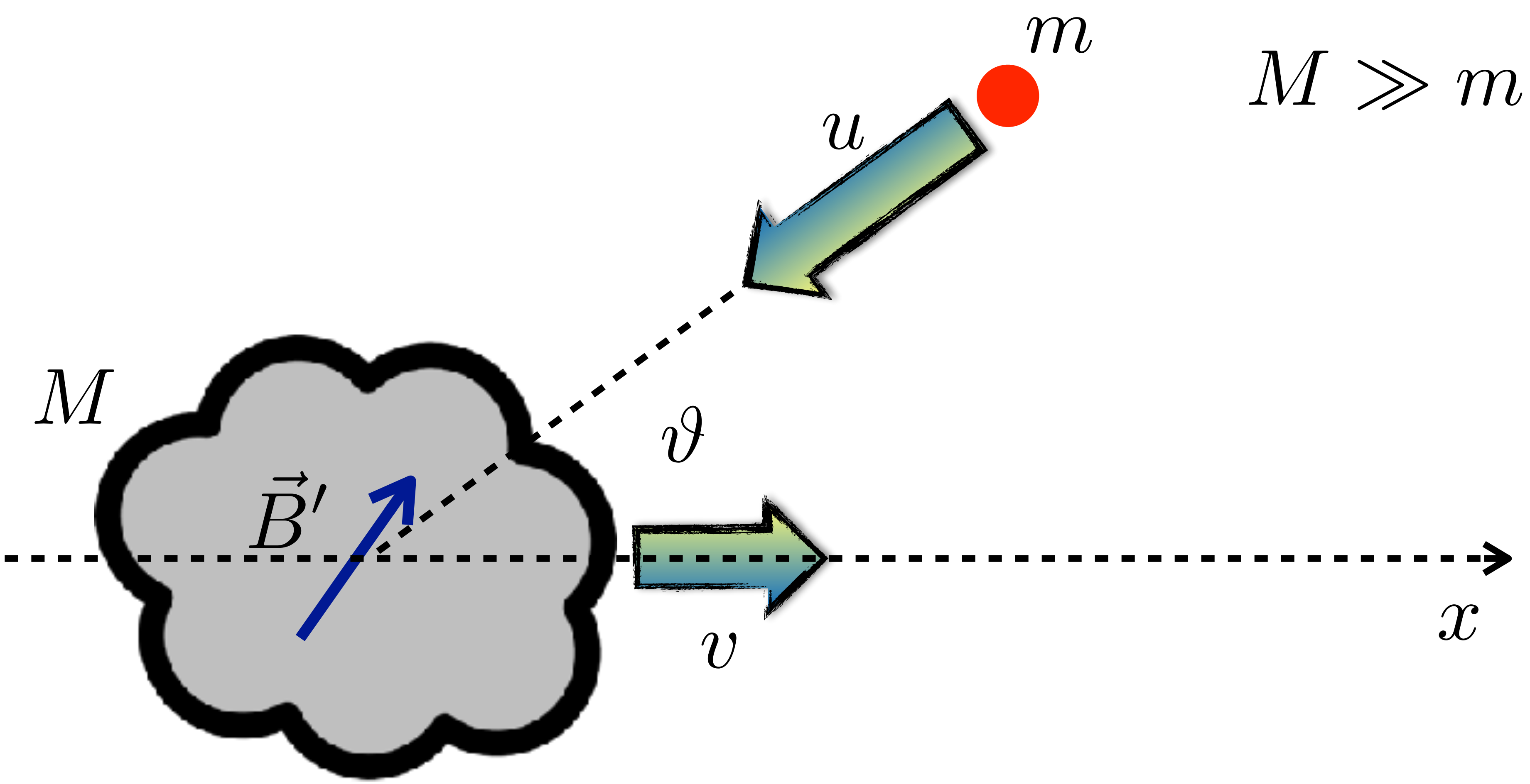

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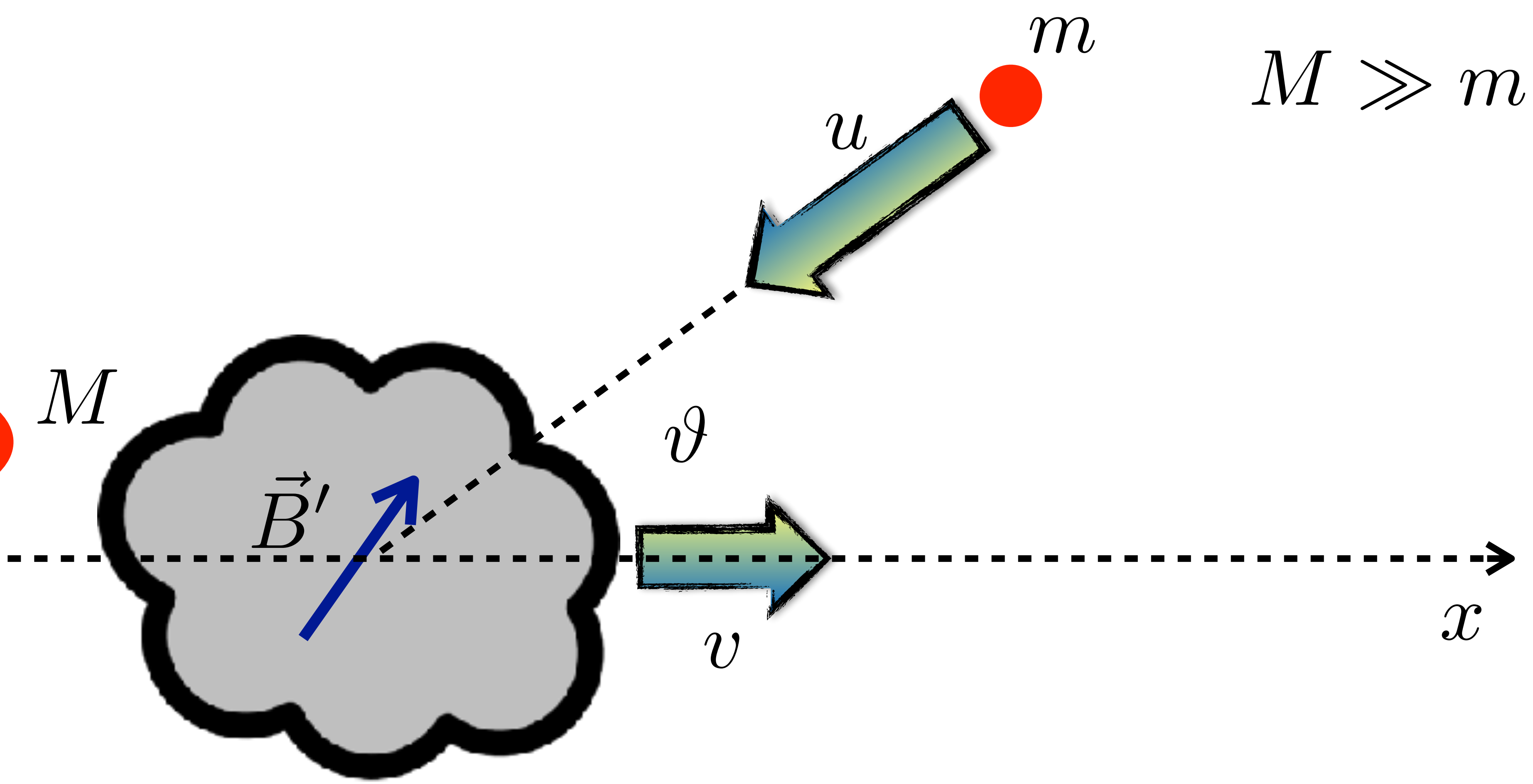

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Probablylity to find θ in [θ,θ+dθ] —>

$$\sin\vartheta \mathrm{d}\vartheta = -\mathrm{d}\cos\vartheta$$

From Fermi's paper

Fermi (1949, 1954)

$$\cos\vartheta = x \qquad\qquad u \longrightarrow c$$

$$\boxed{\left\langle \frac{\Delta E}{E} \right\rangle} = \left(\frac{2v}{c}\right) \frac{\int_{-1}^{1} \mathrm{d}x \; x \left[1 + \frac{v}{c} x\right]}{\int_{-1}^{1} \mathrm{d}x \left[1 + \frac{v}{c} x\right]} + 2\left(\frac{v}{c}\right)^2 =$$

$$= \frac{2}{3}\left(\frac{v}{c}\right)^2 + 2\left(\frac{v}{c}\right)^2 = \frac{8}{3}\left(\frac{v}{c}\right)^2$$

Energy gain, second order, as expected in the initial (notebook) estimate!

Slightly different approach

Fermi considered clouds as magnetic mirrors, in other derivations clouds are considered as scattering centres

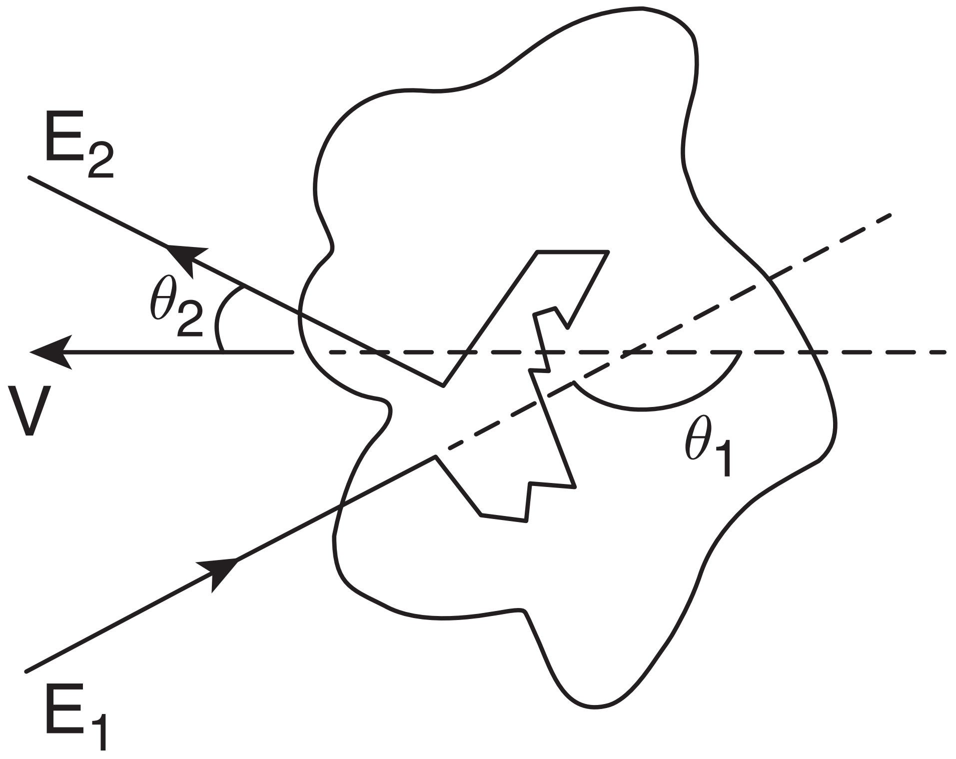

Figure from Gaisser's book

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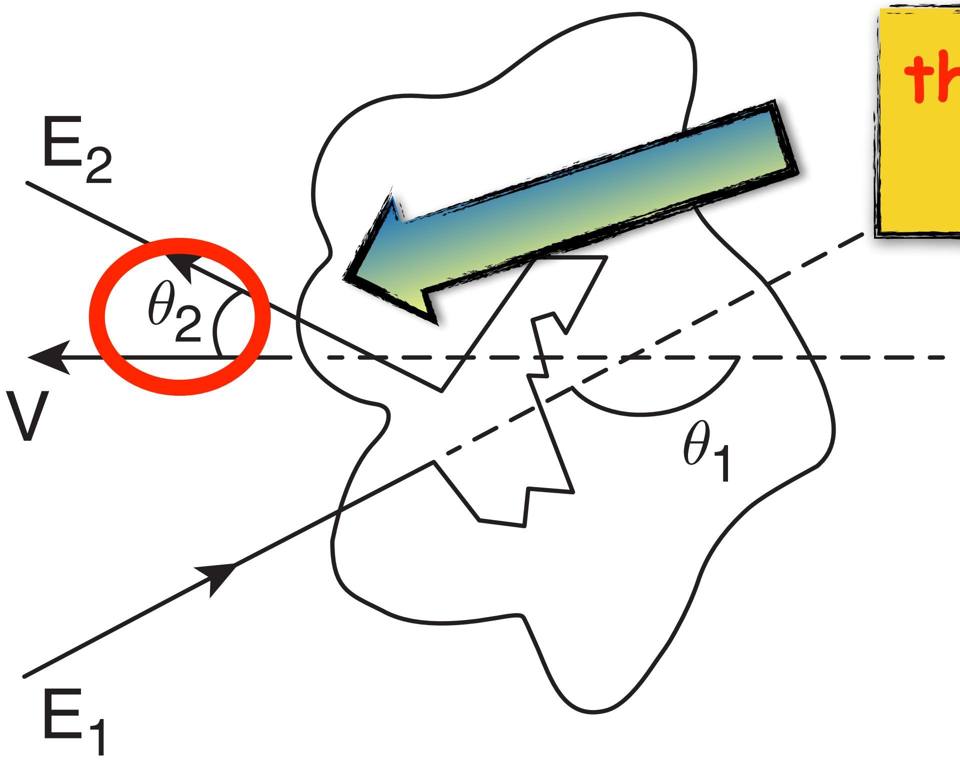

this introduces a dependency on the exit angle

$$\frac{\Delta E}{E} = \beta\left[\cos(\vartheta'_{out}) - \cos(\vartheta_{in})\right] + \beta^2\left[1 - \cos(\vartheta_{in})\cos(\vartheta'_{out})\right]$$

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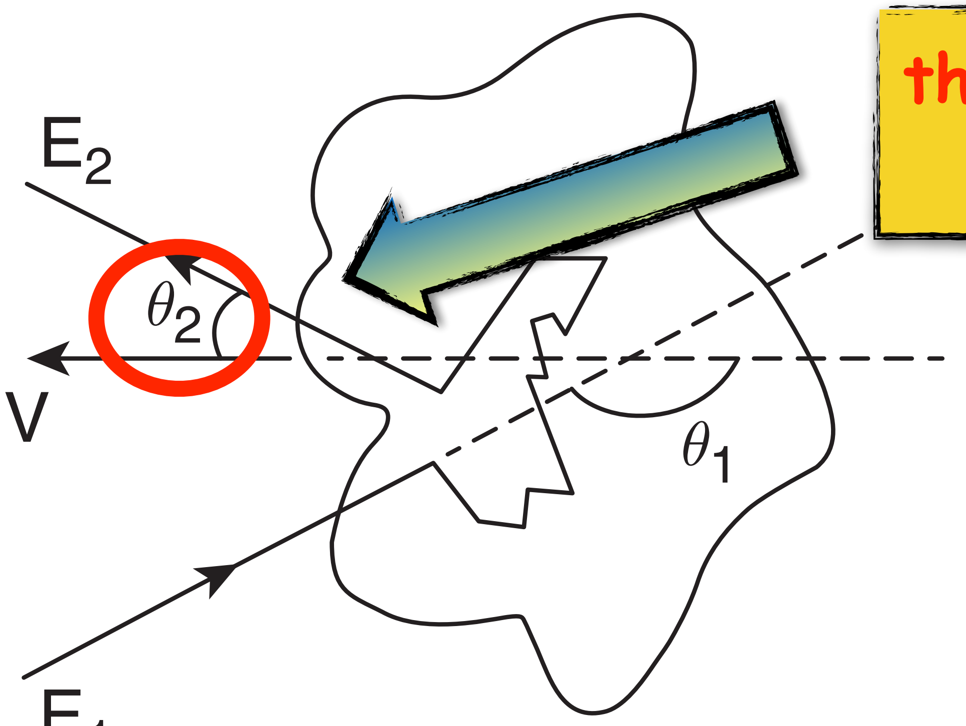

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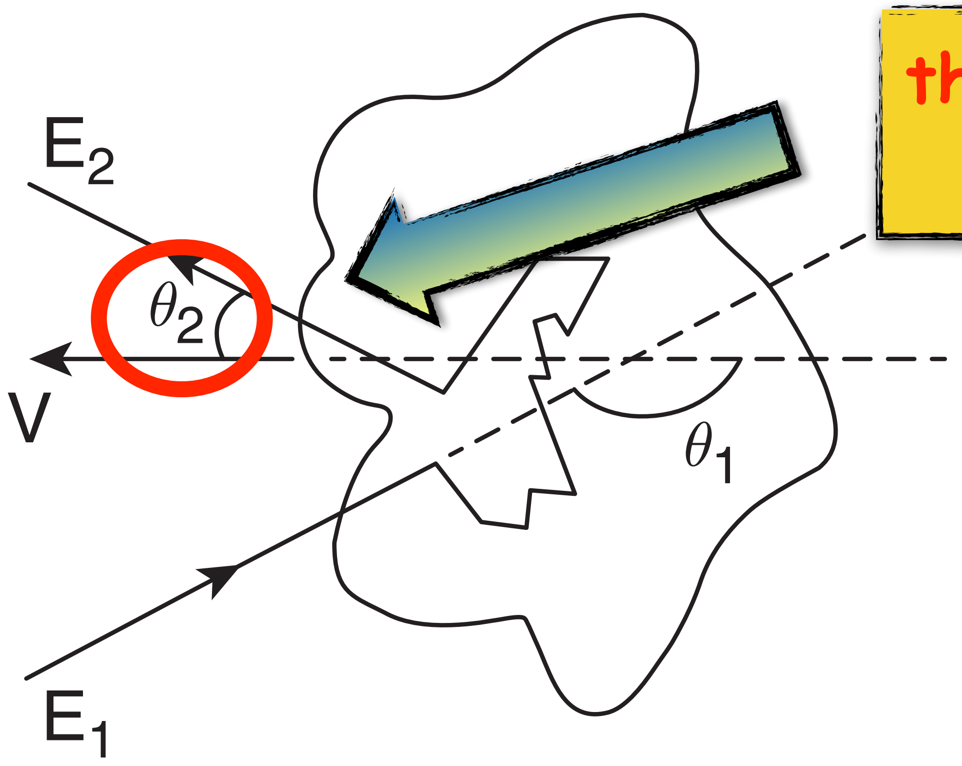

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$$\left\langle \frac{\Delta E}{E} \right\rangle \approx \frac{4}{3}\,\beta^2$$

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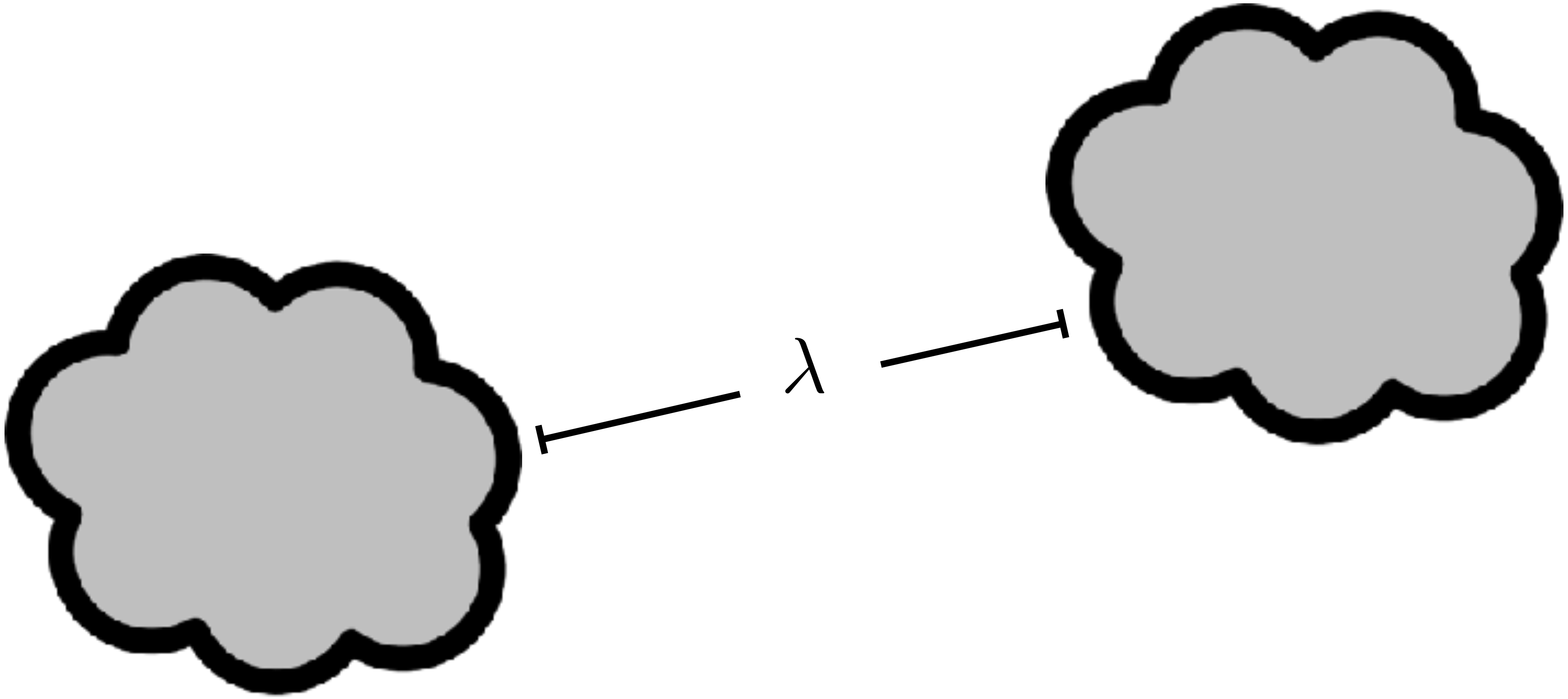
$$\frac{\Delta E}{E} = \beta\left[\cos(\vartheta'_{out}) - \cos(\vartheta_{in})\right] + \beta^2\left[1 - \cos(\vartheta_{in})\cos(\vartheta'_{out})\right]$$

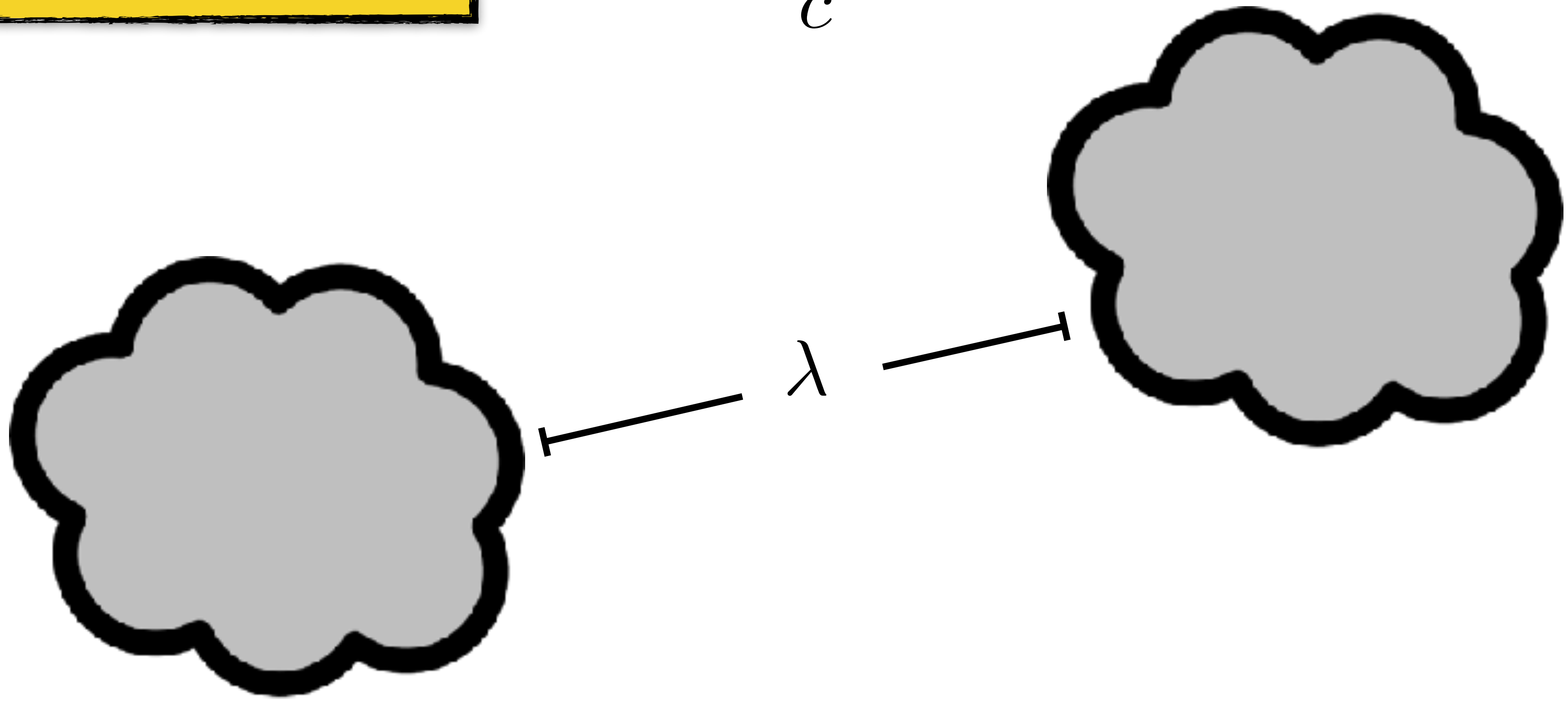
Acceleration rate



Acceleration rate

time between collisions

$$\tau_c \approx \frac{\lambda}{c}$$



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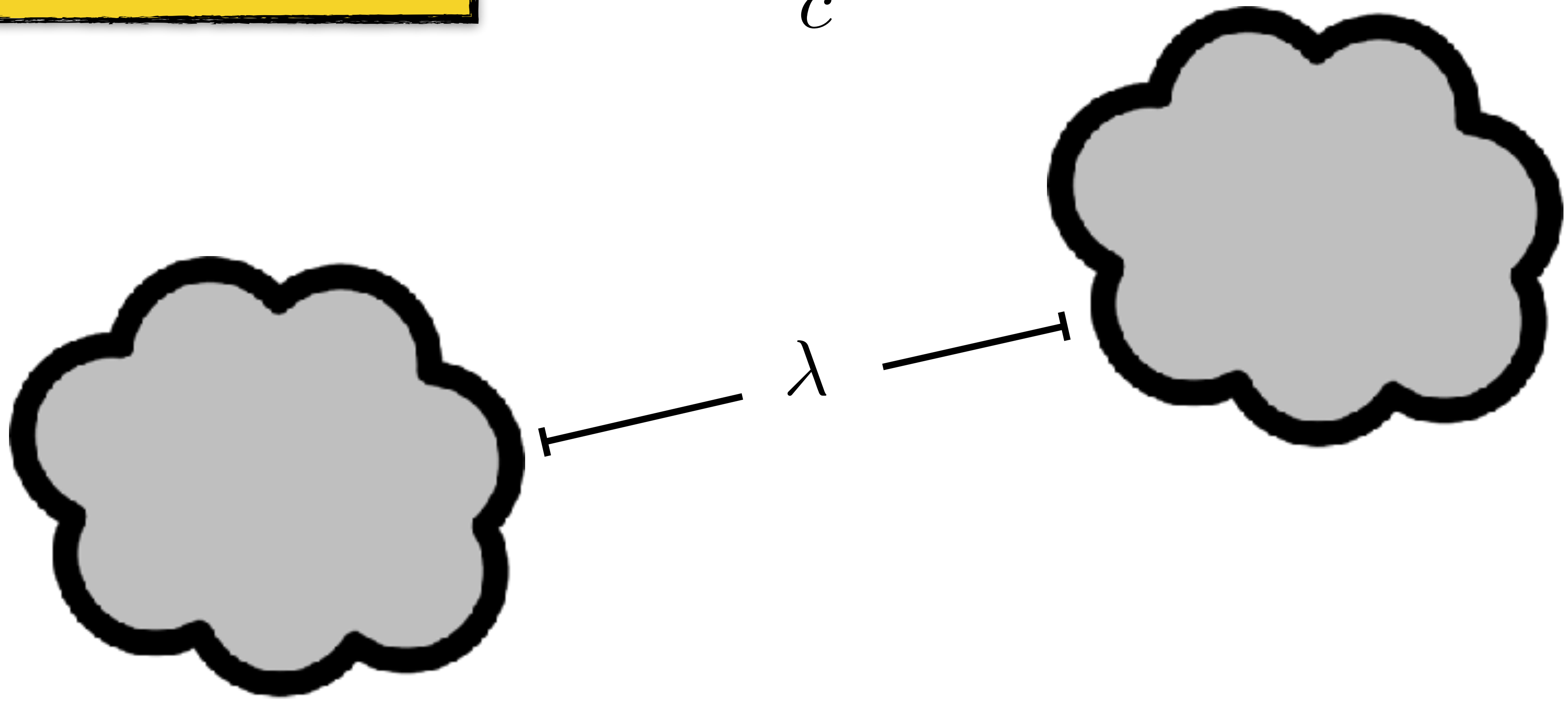

acceleration rate —>

$$\frac{\mathrm{d}E}{\mathrm{d}t} \sim \frac{\Delta E}{\tau_c} = \alpha \left(\frac{v}{c}\right)^2 \frac{E}{\tau_c} = \alpha \frac{v^2}{\lambda c} E$$

Good or bad? Compare with Hillas!

Hillas acceleration rate —>

Larmor radius ↘

$$\frac{\mathrm{d}E}{\mathrm{d}t} = qvB = \frac{v}{c}\left(\frac{R_L}{c}\right)^{-1} E$$

Good or bad? Compare with Hillas!

Larmor radius

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$$\frac{\mathrm{d}E}{\mathrm{d}t} = qvB = \frac{v}{c}\left(\frac{R_L}{c}\right)^{-1} E$$

Hillas acceleration time —>

$$\tau_{acc}^{H} = \left(\frac{v}{c}\right)^{-1} \frac{R_L}{c}$$

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Hillas acceleration time —>

$$\tau_{acc}^{H} = \left(\frac{v}{c}\right)^{-1} \frac{R_L}{c}$$

for any acceleration mechanism —>

$$\tau_{acc} = \eta \tau_{acc}^{H}$$

where of course: $\eta > 1$

Fermi II acceleration time

$$\tau_{acc} = \frac{\lambda c}{\alpha v^2} = \alpha^{-1}\frac{\lambda}{R_L}\left(\frac{v}{c}\right)^{-2}\frac{R_L}{c} = \alpha^{-1}\frac{\lambda}{R_L}\left(\frac{v}{c}\right)^{-1}\tau_{acc}^{H}$$

energy
independent

$$\eta = \alpha^{-1}\frac{\lambda}{R_L}\left(\frac{v}{c}\right)^{-1}$$

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order unity

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>> 1

we are very far from the optimal (idealised) Hillas rate

How to build a simple transport equation

Take a generic acceleration rate: $\frac{\mathrm{d}E}{\mathrm{d}t} = b(E)$

How to build a simple transport equation

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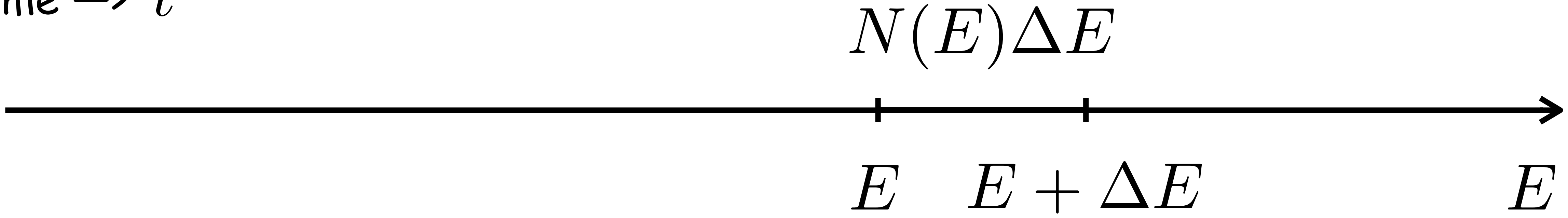
time –> t

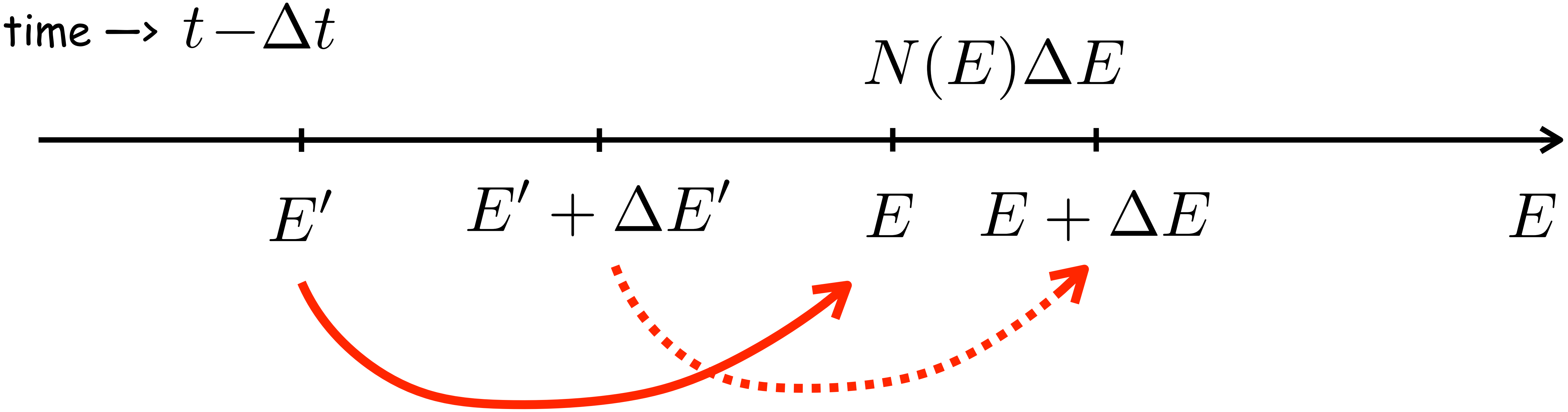


$N(E)\Delta E$

E $E + \Delta E$ E

How to build a simple transport equation

Take a generic acceleration rate: $\dfrac{\mathrm{d}E}{\mathrm{d}t} = b(E)$



time —> $t - \Delta t$

$N(E)\Delta E$

E' $E' + \Delta E'$ E $E + \Delta E$ E

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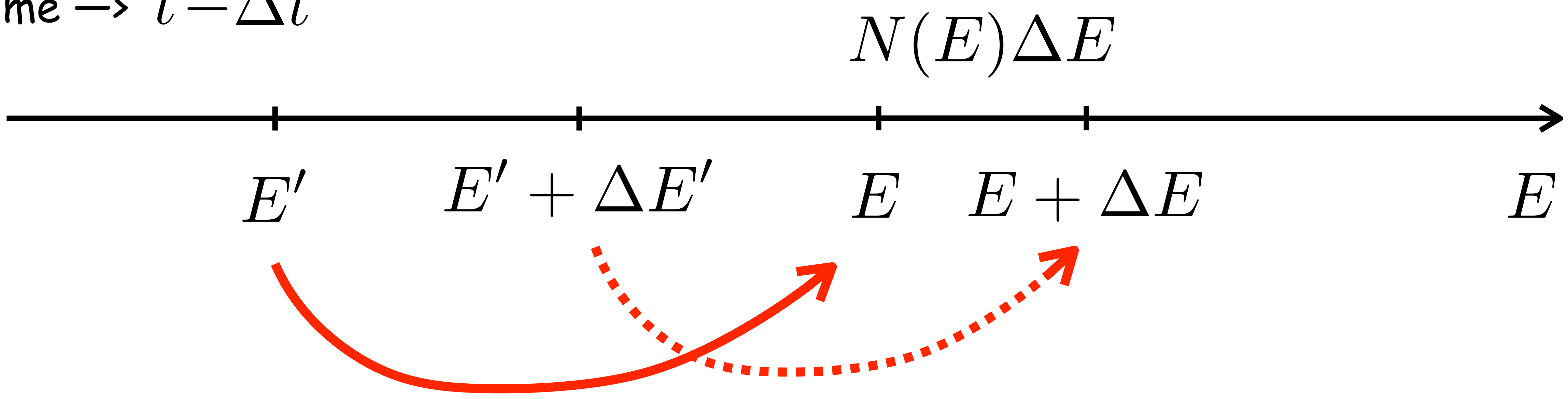
time —> $t - \Delta t$



$$E = E' + b(E)\Delta t$$

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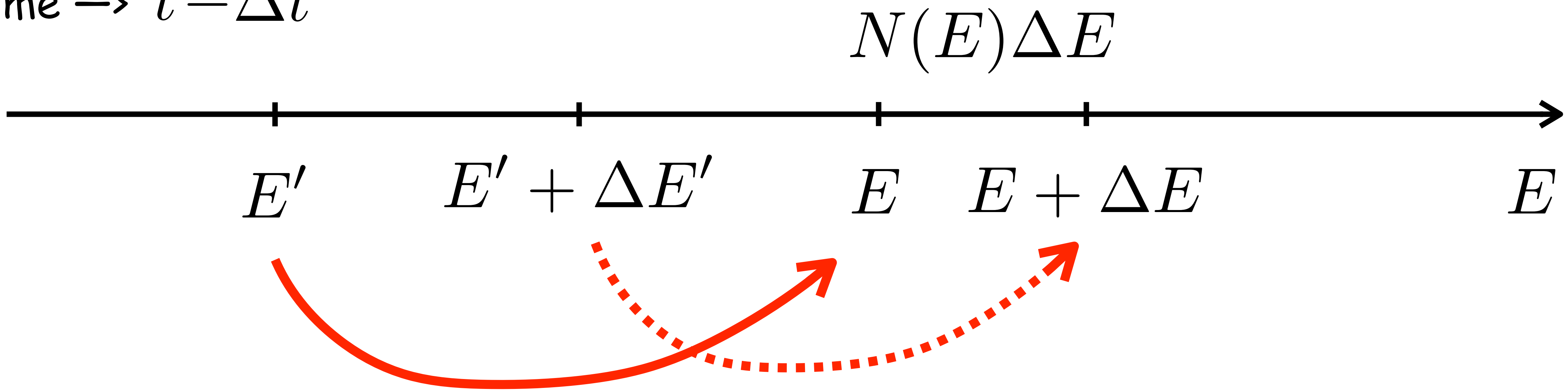

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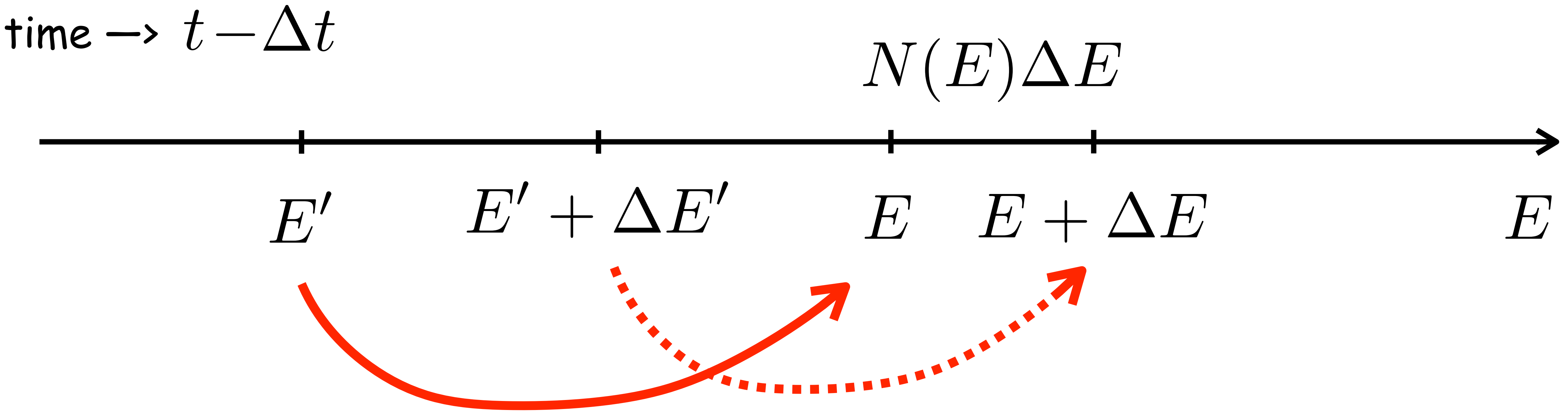

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How to build a simple transport equation

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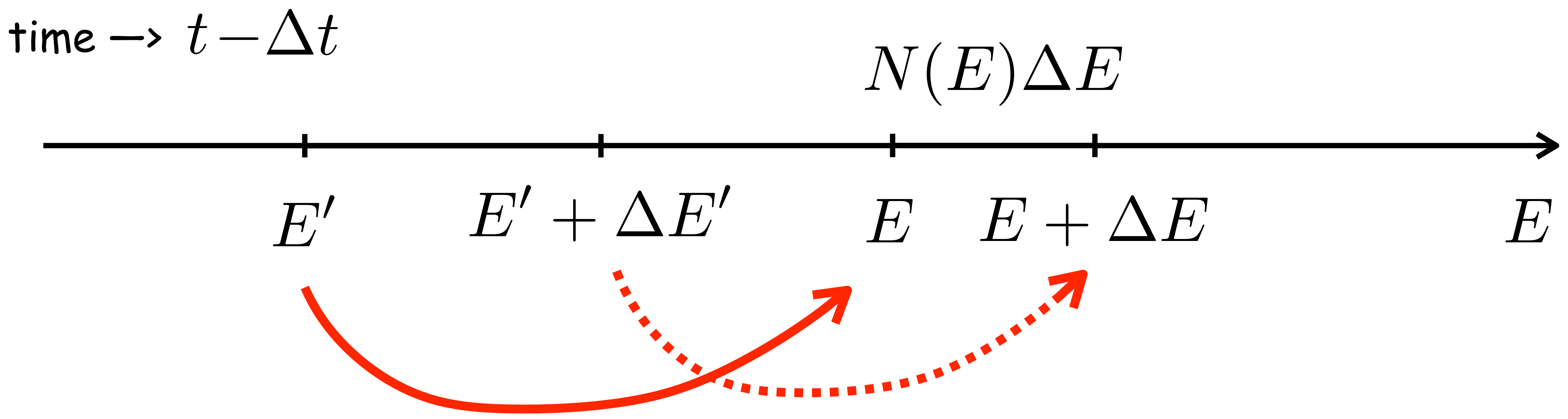

$$E = E' + b(E)\Delta t$$

$$E + \Delta E = E' +$$

subtract ↓

$$)\Delta t = E' + \Delta E' + b(E)\Delta t + \frac{\mathrm{d}b(E)}{\mathrm{d}E}\Delta E \Delta t$$

$$\Delta E = \Delta E' + \frac{\mathrm{d}b(E)}{\mathrm{d}E}\Delta E \Delta t$$

How to build a simple transport equation

$$N(E, t + \Delta t)\Delta E = N(E', t)\Delta E'$$

$$E = E' + b(E)\Delta t$$

$$\Delta E = \Delta E' + \frac{\mathrm{d}b(E)}{\mathrm{d}E}\Delta E \Delta t$$

How to build a simple transport equation

$$N(E, t + \Delta t)\Delta E = N(E', t)\Delta E' = N(E - b(E)\Delta t, t)\left[\Delta E - \frac{\mathrm{d}b(E)}{\mathrm{d}E}\Delta E \Delta t\right]$$

$$E = E' + b(E)\Delta t$$

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$$\downarrow$$

$$N(E,t) - \frac{\mathrm{d}N(E,t)}{\mathrm{d}E}b(E)\Delta t$$

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$$N(E,t) - \frac{\mathrm{d}N(E,t)}{\mathrm{d}E}b(E)\Delta t$$

$$N(E,t+\Delta t) = N(E,t) - \left[N(E,t)\frac{\mathrm{d}b(E)}{\mathrm{d}E} + \frac{\mathrm{d}N(E,t)}{\mathrm{d}E}b(E)\right]\Delta t$$

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$$\frac{\mathrm{d}N(E)}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}E}\left[b(E)N(E)\right]$$

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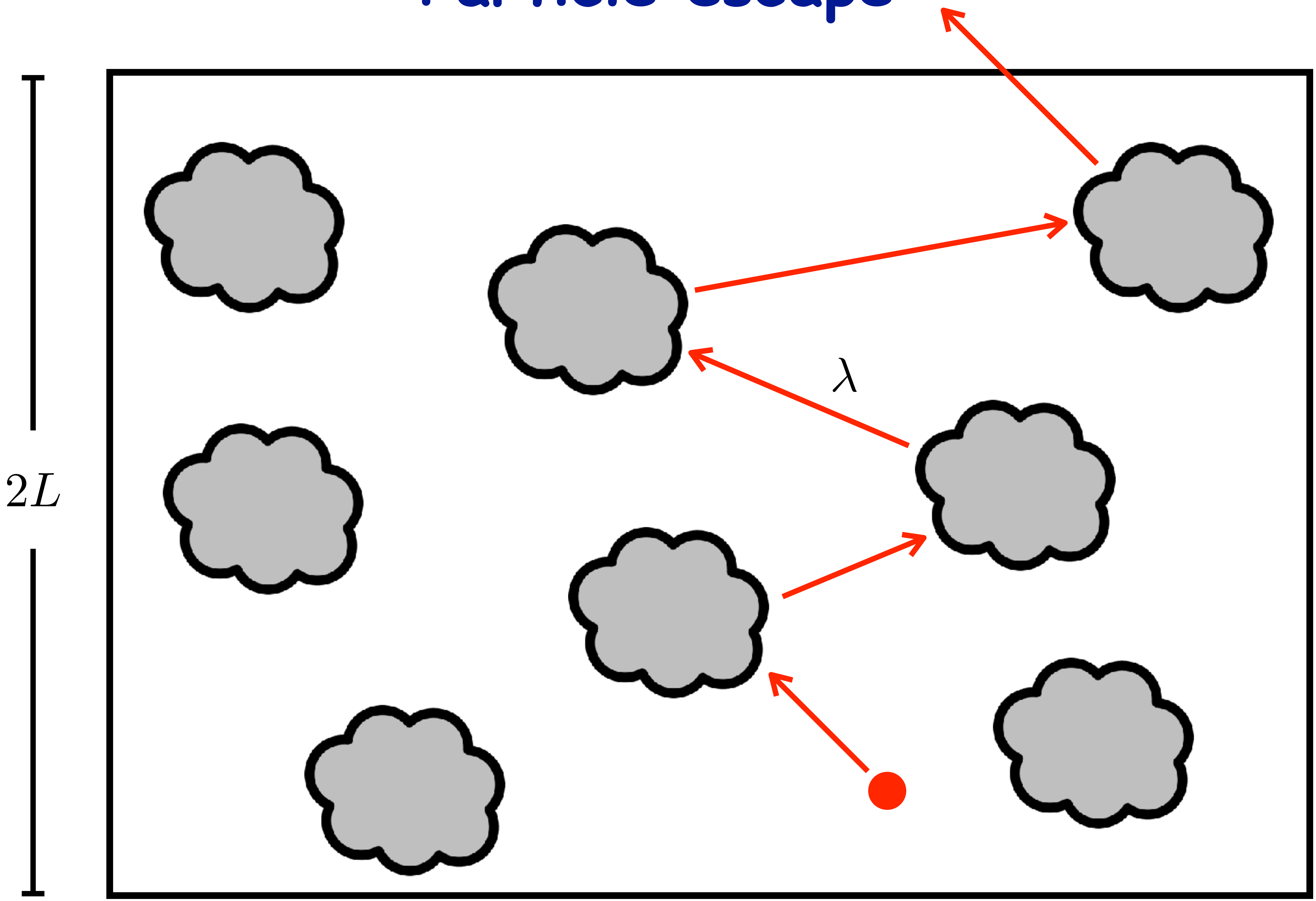
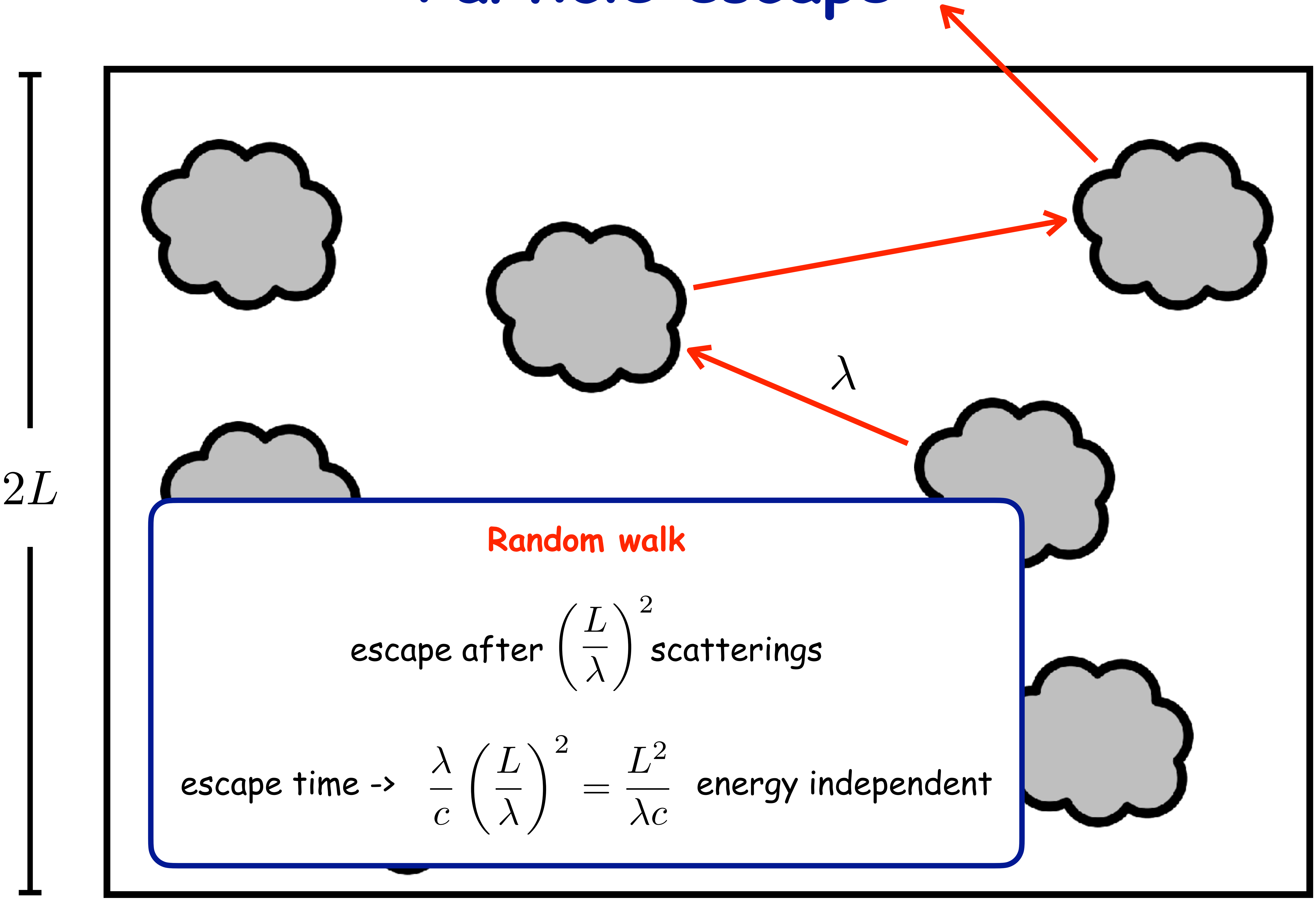
Particle escape



Particle escape



Random walk

escape after $\left(\frac{L}{\lambda}\right)^2$ scatterings

escape time -> $\frac{\lambda}{c}\left(\frac{L}{\lambda}\right)^2 = \frac{L^2}{\lambda c}$ energy independent

Spectrum of accelerated particles

$$\frac{\mathrm{d}N(E)}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}E}\left[b(E)N(E)\right] - \frac{N(E)}{\tau_{esc}}$$

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$$\boxed{N(E) \propto E^{-\left(1+\frac{\tau_{acc}}{\tau_{esc}}\right)}}$$

power law!

Fine tuning problem?

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$$\boxed{N(E) \propto E^{-\left(1+\frac{\tau_{acc}}{\tau_{esc}}\right)}}$$

the ratio $\dfrac{\tau_{acc}}{\tau_{esc}}$ can be tuned to get ANY spectral slope

with this respect, the model is not very predictive

Problem #1: too slow

acceleration rate —>

$$\frac{\mathrm{d}E}{\mathrm{d}t} \sim \frac{\Delta E}{\tau_c} = \alpha \left(\frac{v}{c}\right)^2 \frac{E}{\tau_c} = \alpha \frac{v^2}{\lambda c} E$$

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cosmic rays remain confined within the Galaxy for ~10-20 million years, and then they escape in intergalactic space...

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τ_{acc}^{-1} ←

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brutal approach: in real life this is important up to ~GeV energies

Problem#2: injection

cosmic rays in the interstellar medium lose energy due to ionisation losses

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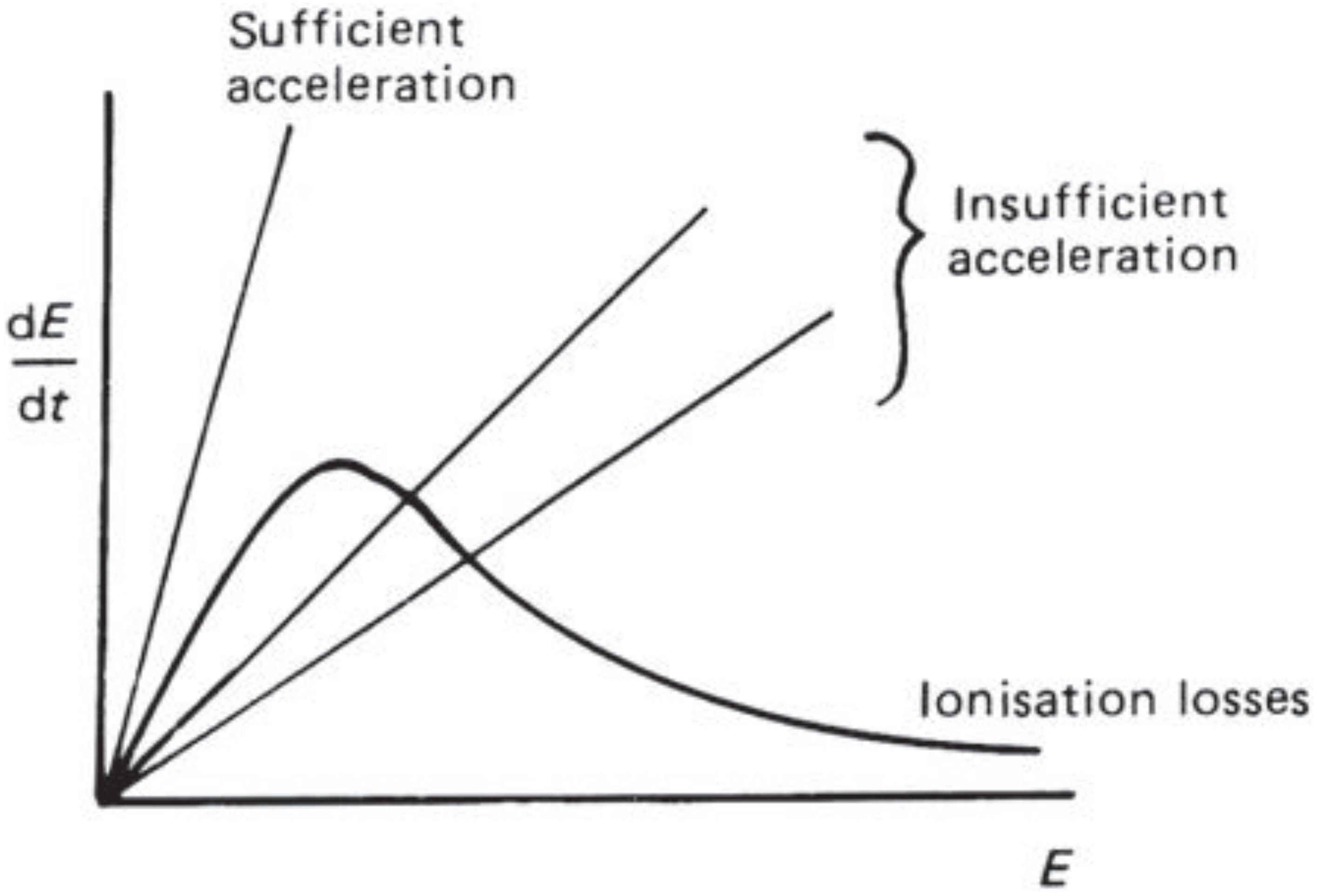
cosmic rays in the interstellar medium lose energy due to ionisation losses

Figure from Longair's book



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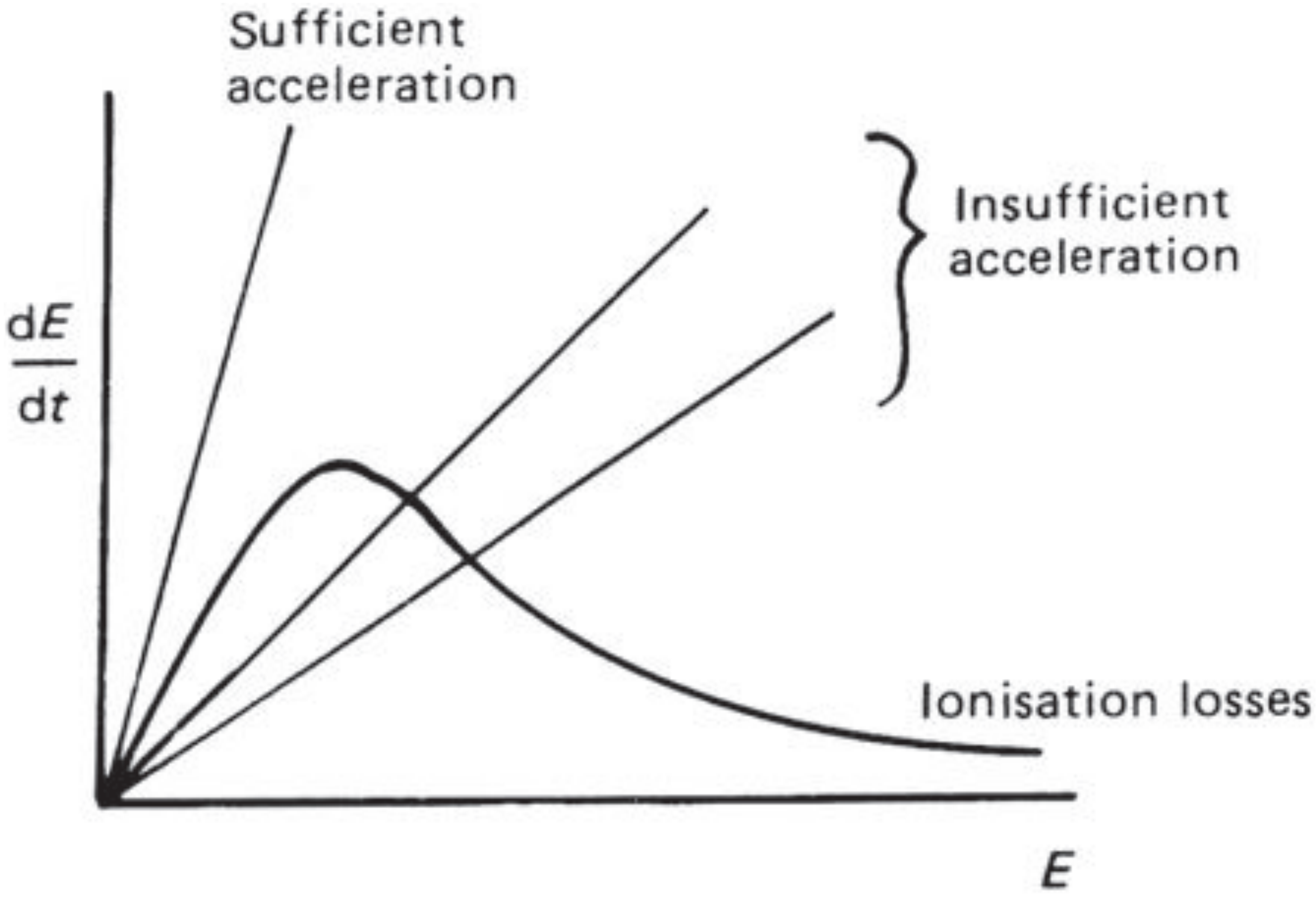
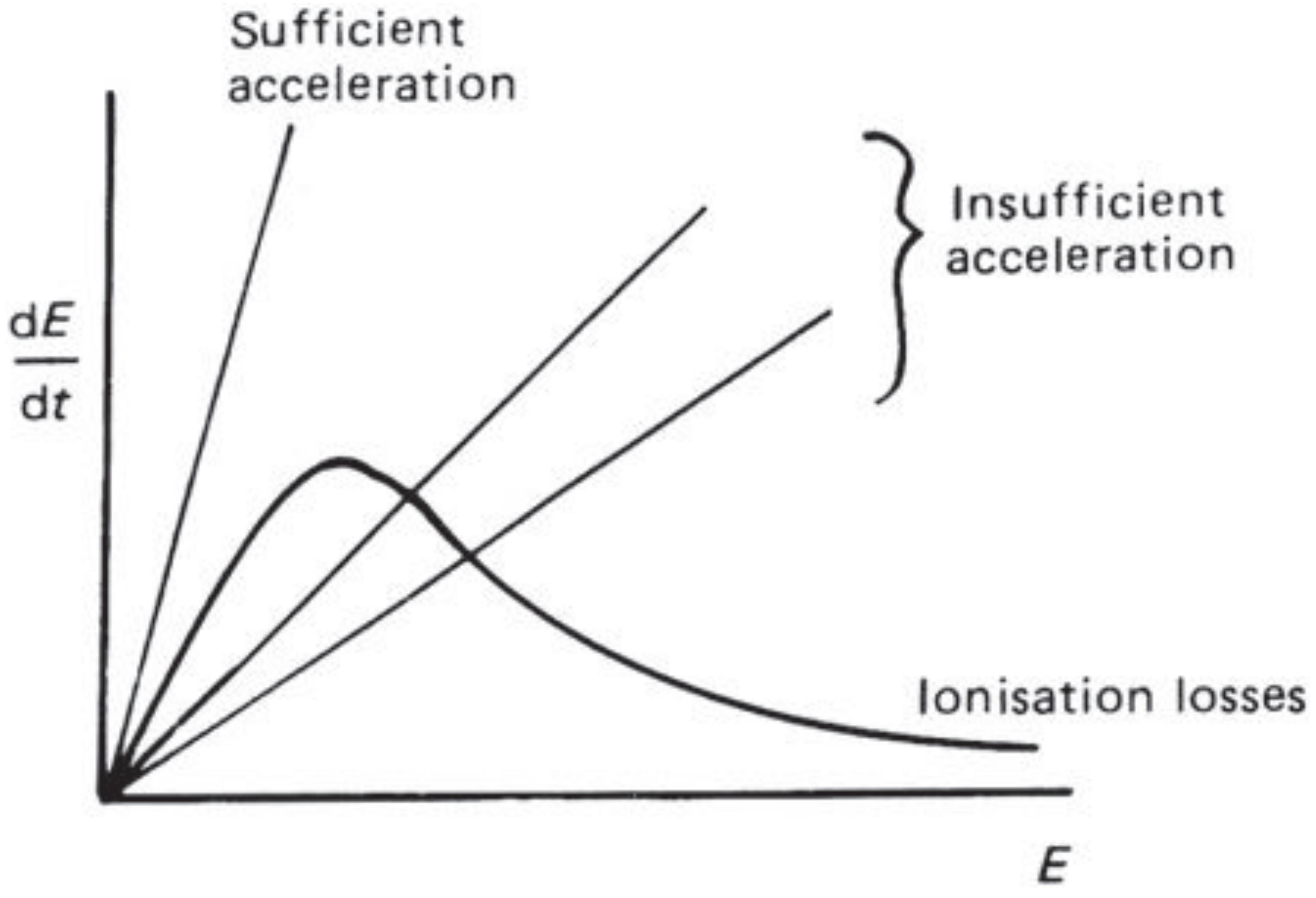


Figure from Longair's book

Two ways to overcome losses

Problem#2: injection

cosmic rays in the interstellar medium lose energy due to ionisation losses



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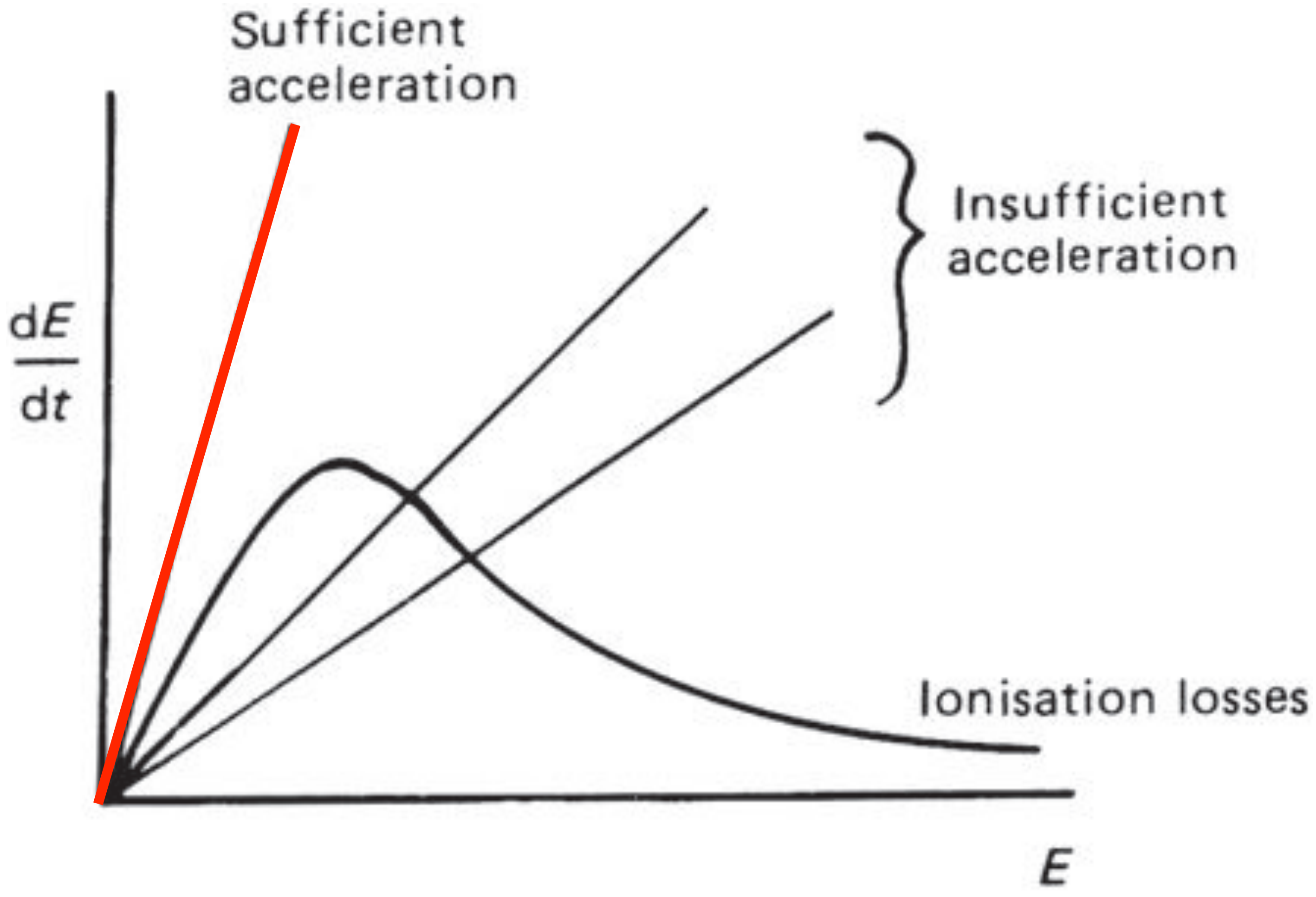

Figure from Longair's book

Two ways to overcome losses

- fast acceleration rate

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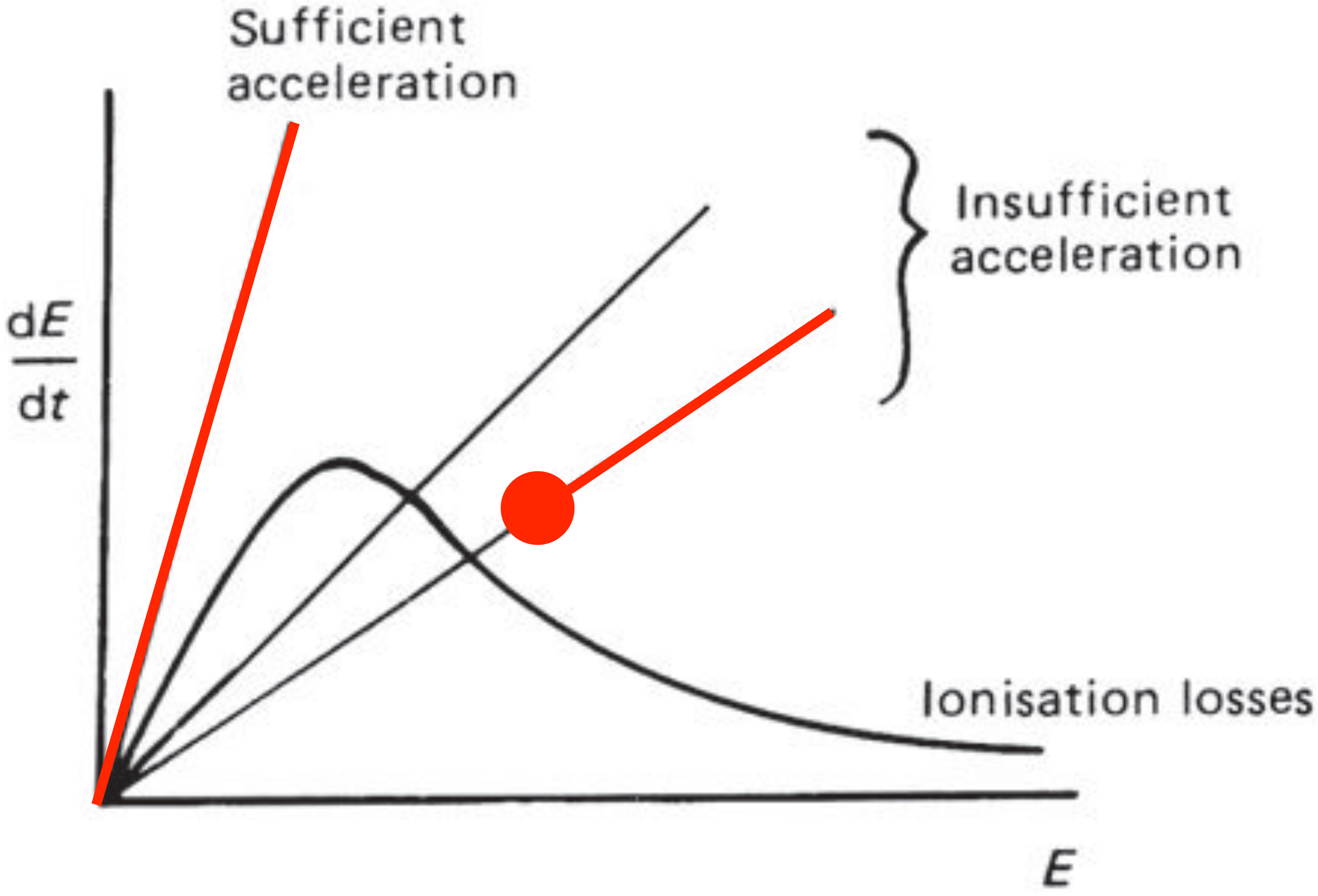

Figure from Longair's book

Two ways to overcome losses

- fast acceleration rate
- some process should inject particles at energies large enough to overcome losses

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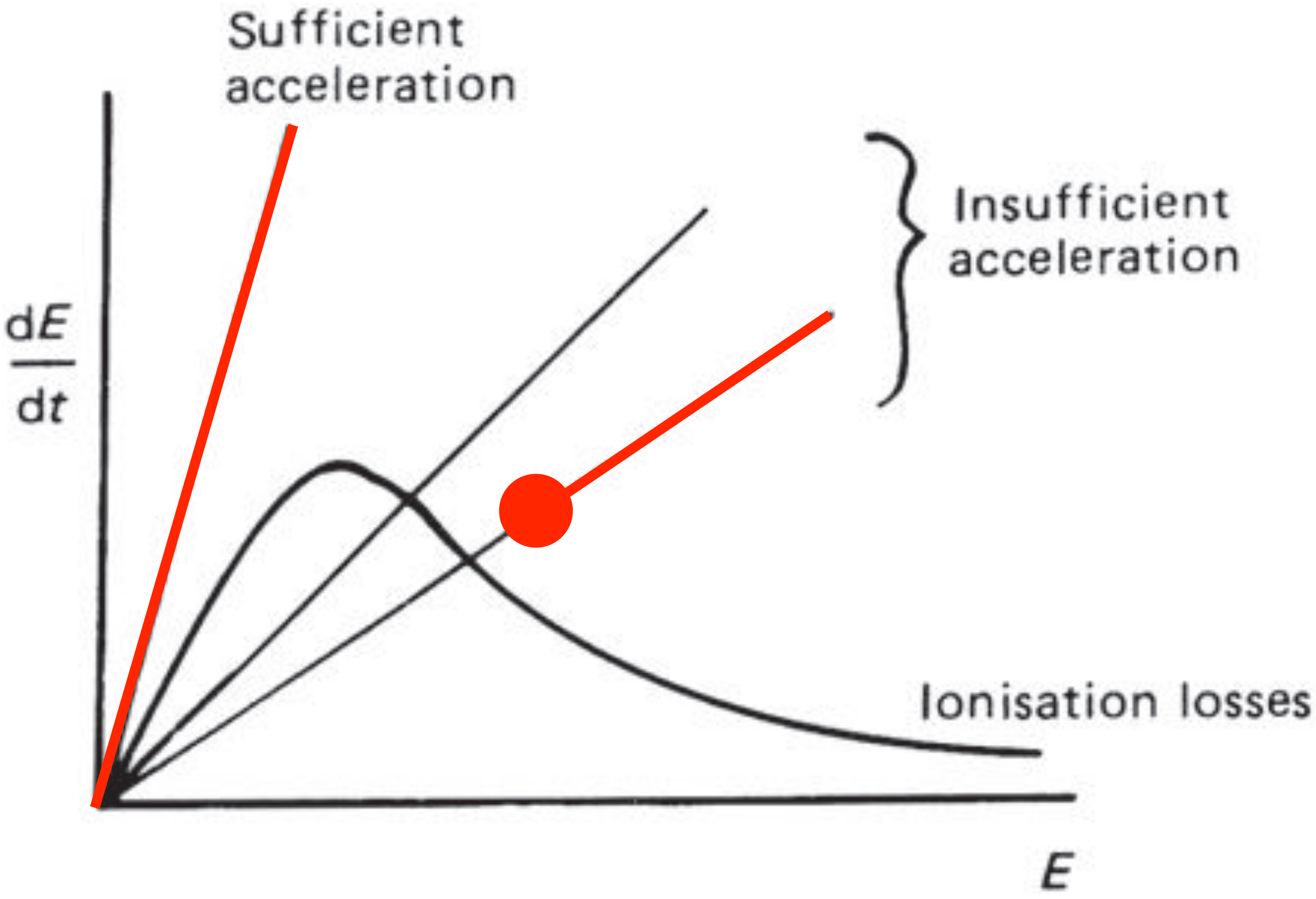
Figure from Longair's book



Two ways to overcome losses

- fast acceleration rate
- some process should inject particles at energies large enough to overcome losses

this is known as “injection problem” and belongs to all acceleration mechanisms

Root mean square change in energy

energy gain/loss in a interaction:

$$\frac{\Delta E}{E} = \frac{E'' - E}{E} = 2\frac{v}{c}\left[\frac{u}{c}\cos\vartheta + \frac{v}{c}\right]$$

Root mean square change in energy

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$$\frac{\Delta E}{E} = \frac{E'' - E}{E} = 2\frac{v}{c}\left[\frac{u}{c}\cos\vartheta + \frac{v}{c}\right]$$

$$(\Delta E)^2 = 4E^2\left(\frac{v}{c}\right)^2\left[\left(\frac{u}{c}\right)^2\cos^2\vartheta + \left(\frac{v}{c}\right)^2 + 2\frac{uv}{c^2}\cos\vartheta\right]$$

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$u \longrightarrow c$

keep only second order terms in v/c

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keep only second order terms in v/c

$$(\Delta E)^2 = 4E^2\left(\frac{v}{c}\right)^2\cos^2\vartheta$$

average over angles as already did to derive Fermi's result

$$\langle(\Delta E)^2\rangle = \frac{4}{3}E^2\left(\frac{v}{c}\right)^2$$

Systematic versus root mean square change in energy

systematic —>

$$\langle \Delta E \rangle = \frac{8}{3} E \left(\frac{v}{c} \right)^2$$

Systematic versus root mean square change in energy

systematic —>

$$\langle \Delta E \rangle = \frac{8}{3} E \left(\frac{v}{c} \right)^2$$

stochastic —>

$$\sqrt{\langle (\Delta E)^2 \rangle} = \frac{2}{\sqrt{3}} E \left(\frac{v}{c} \right)$$

Systematic versus root mean square change in energy

systematic —>

$$\langle \Delta E \rangle = \frac{8}{3} E \left(\frac{v}{c} \right)^2$$

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$$\sqrt{\langle (\Delta E)^2 \rangle} = \frac{2}{\sqrt{3}} E \left(\frac{v}{c} \right)$$

Systematic versus root mean square change in energy

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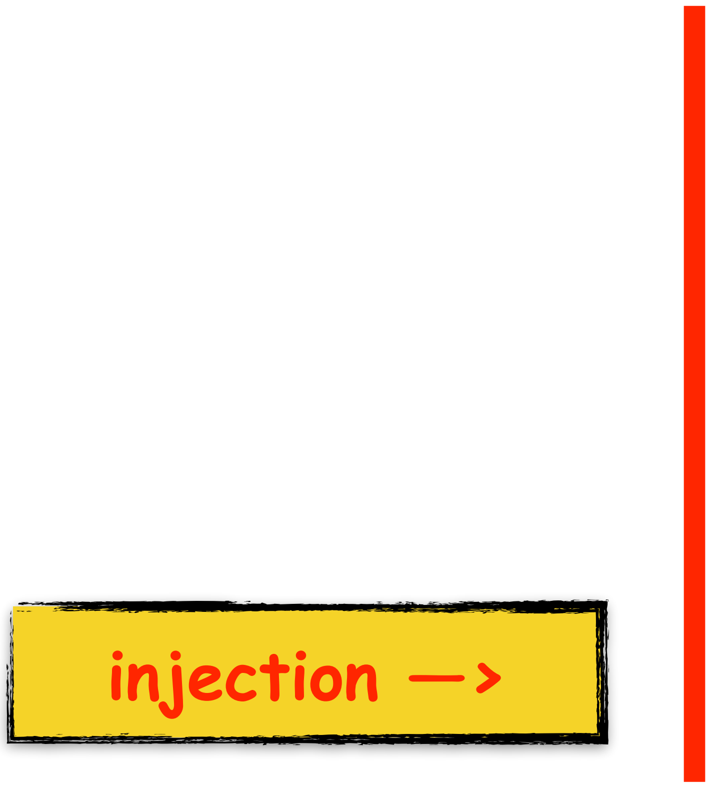
the transport equation we just derived takes into account only the systematic change in particle energy

Systematic versus root mean square change in energy

Figure from Mertsch (2011)

Systematic versus root mean square change in energy



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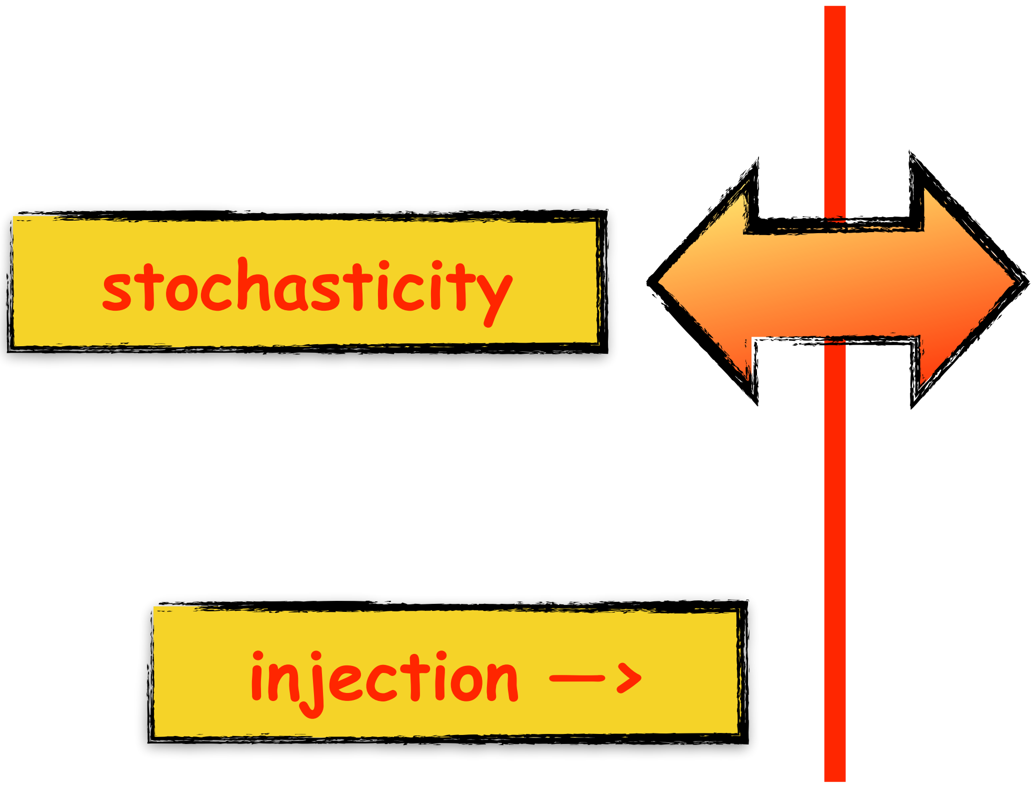

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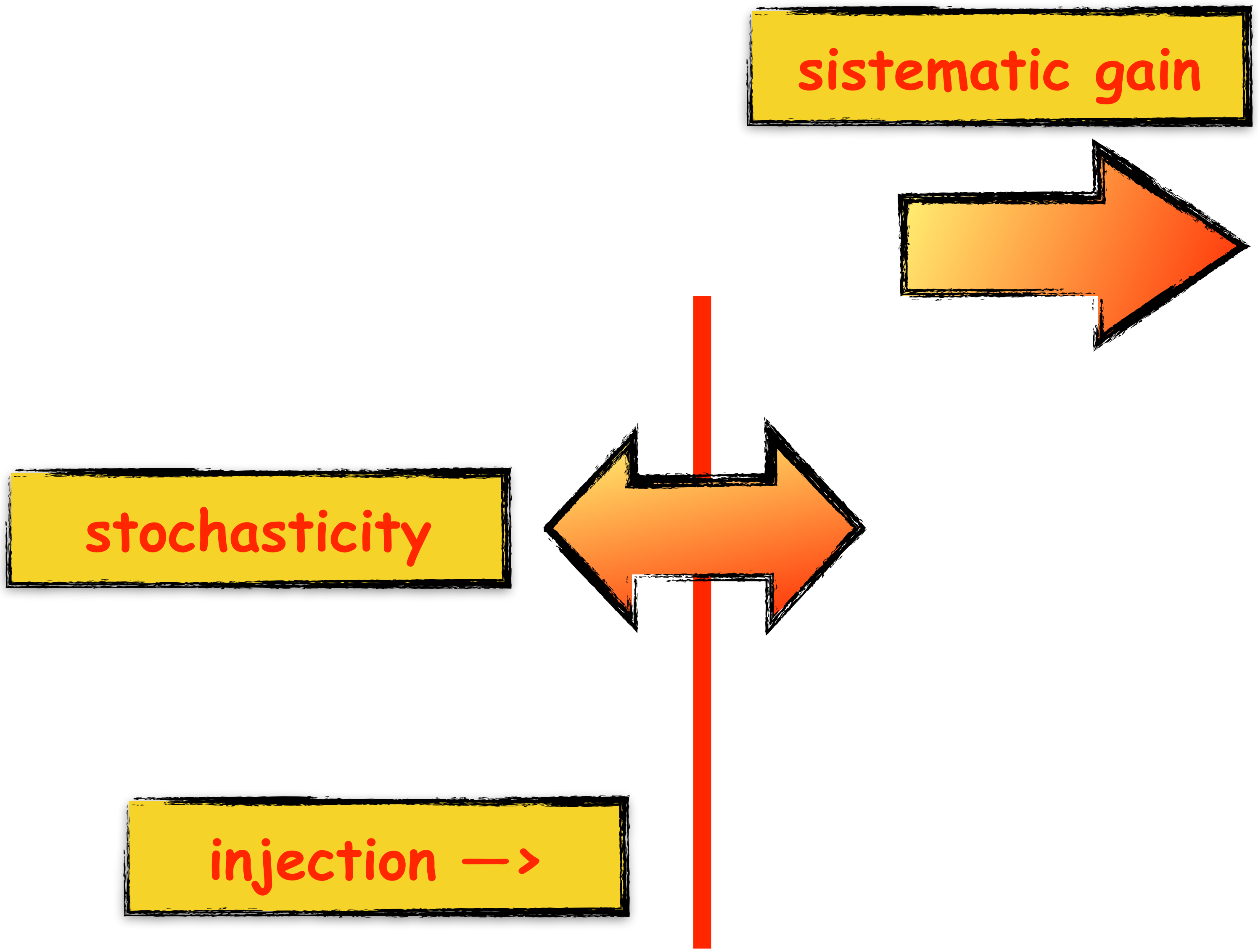

sistematic gain

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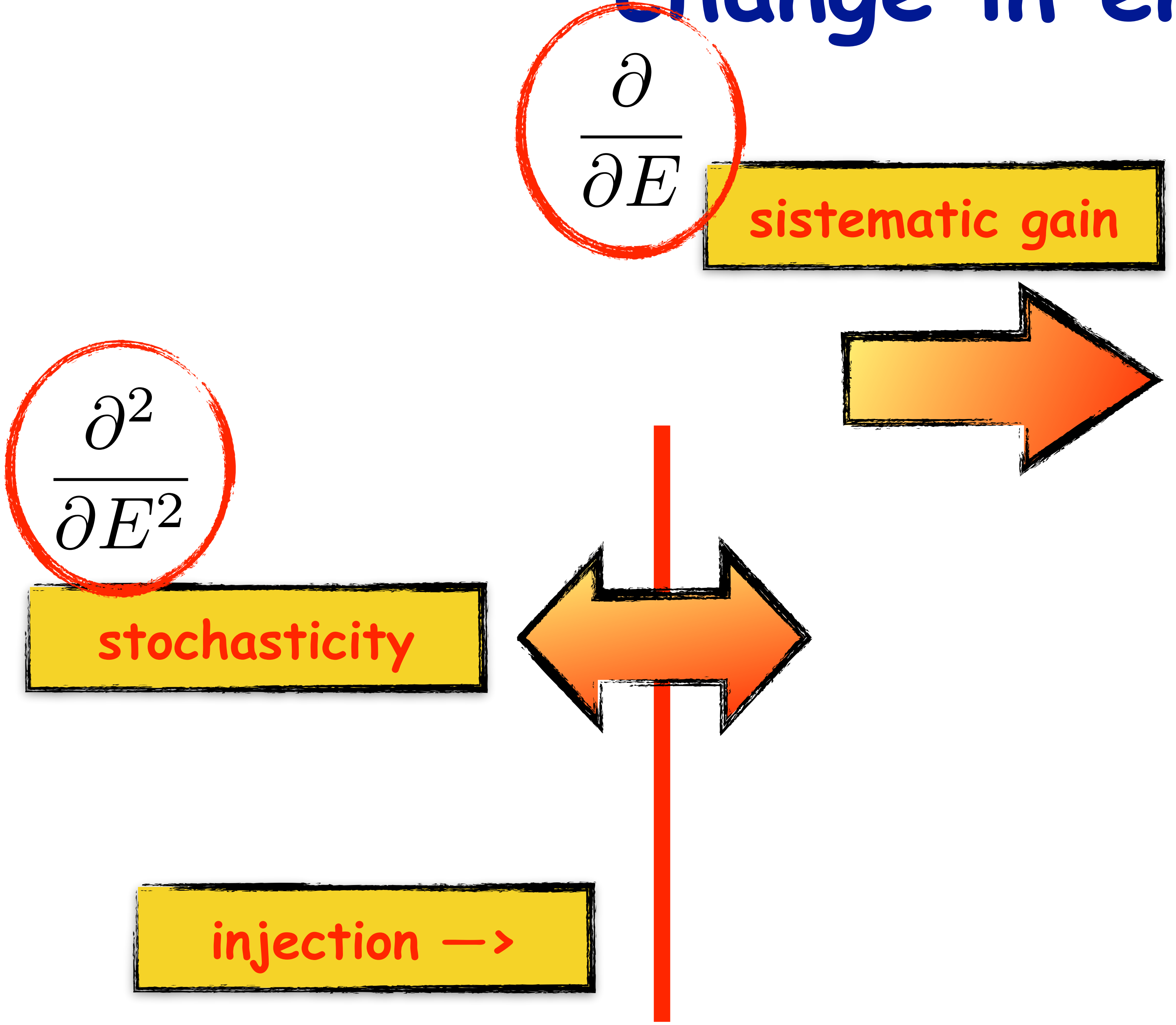

$$\frac{\partial}{\partial E}$$

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Diffusion coefficient in energy

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$$D_E = \frac{\langle (\Delta E)^2 \rangle}{\Delta t} = \frac{\langle (\Delta E)^2 \rangle}{\langle \Delta E \rangle} b(e) = \frac{b(E) E}{2}$$

A more complete transport equation

$$\frac{\mathrm{d}N(E)}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}E}\left[b(E)N(E)\right] - \frac{N(E)}{\tau_{esc}} + \frac{1}{2}\frac{\partial^2}{\partial E^2}\left[D_E(E)N(E)\right]$$

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can we still find a power law solution? $N(E) = N_0 E^{-s}$

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$$\boxed{s = \frac{3}{2}\left(1 + \frac{16}{9}\frac{\tau_{acc}}{\tau_{esc}}\right)^{1/2} - \frac{1}{2}}$$

Things to remember

Good things about the Fermi II mechanism

- Particles are accelerated!
- Systematic(gain) plus stochastic variation of particle energies
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- Particles are accelerated!
- Systematic(gain) plus stochastic variation of particle energies
- Power law spectra can be generated!

Bad things about the Fermi II mechanism

- It is too slow! (second order…)
- Injection problem (in fact, this is a problem of virtually any acceleration mechanism)
- Need to be fine tuned. The slope of the power law depends on physical parameters which are a priori unknown

Fermi I, or, Diffusive Shock Acceleration

What's next

Why is the second order Fermi mechanism so slow?

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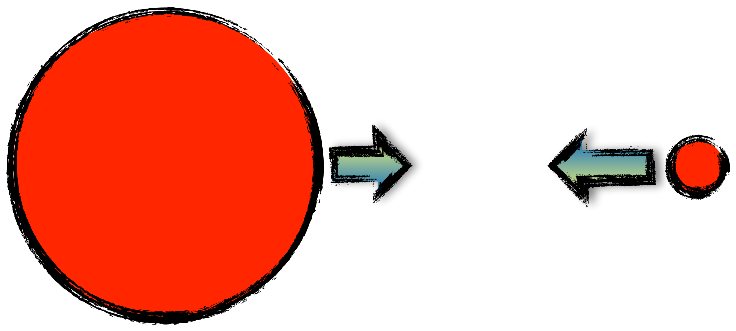
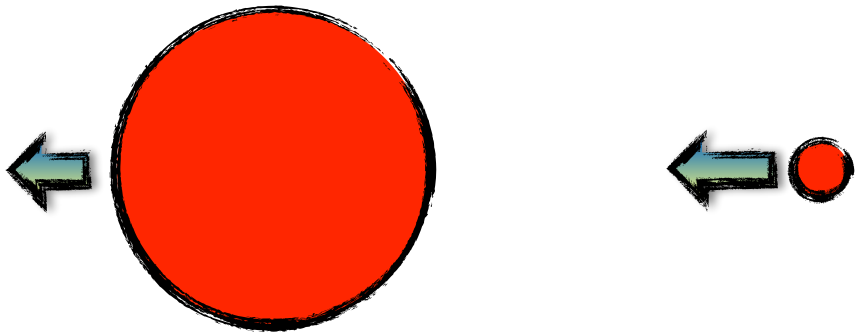
head-on collision -> gain



running after collision -> loss



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Can we do something about it?

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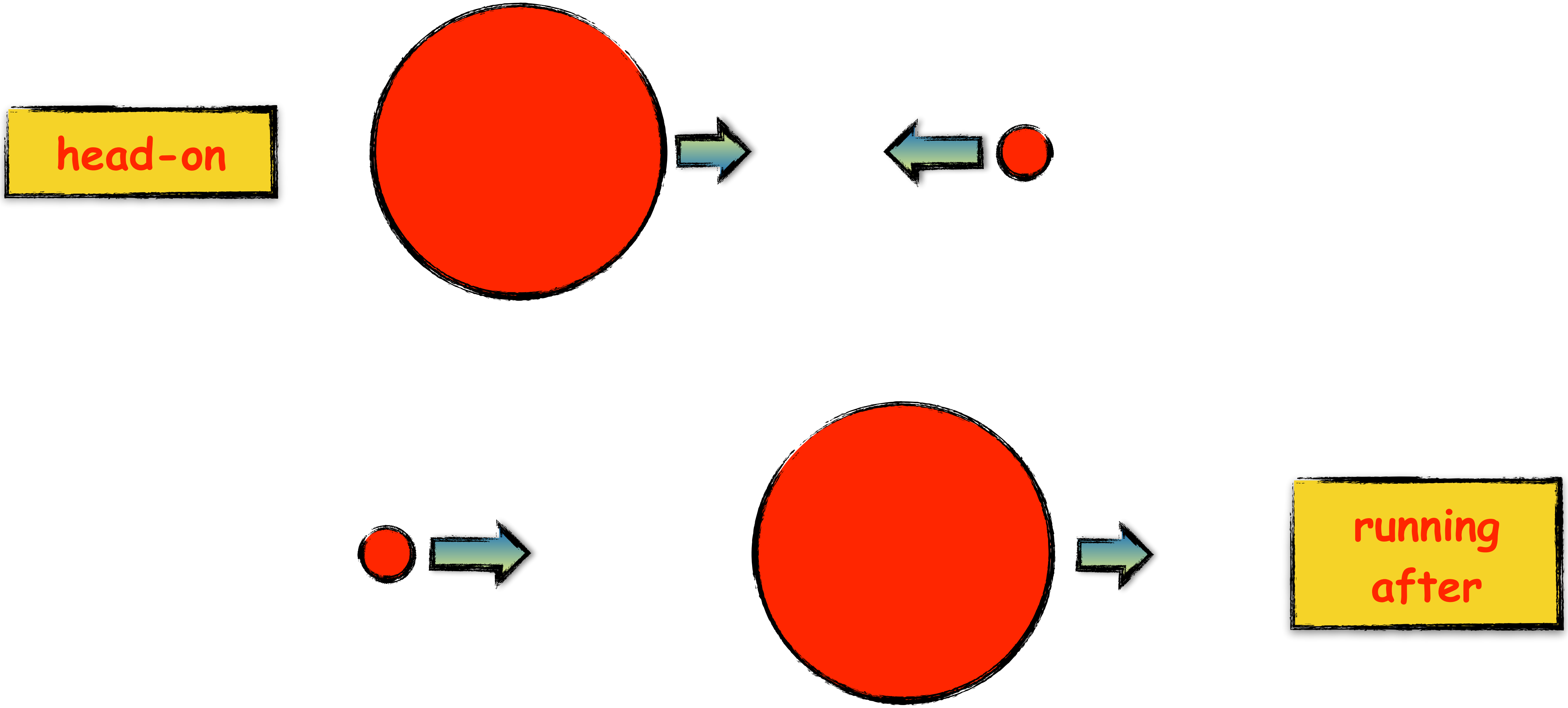

head-on

running after

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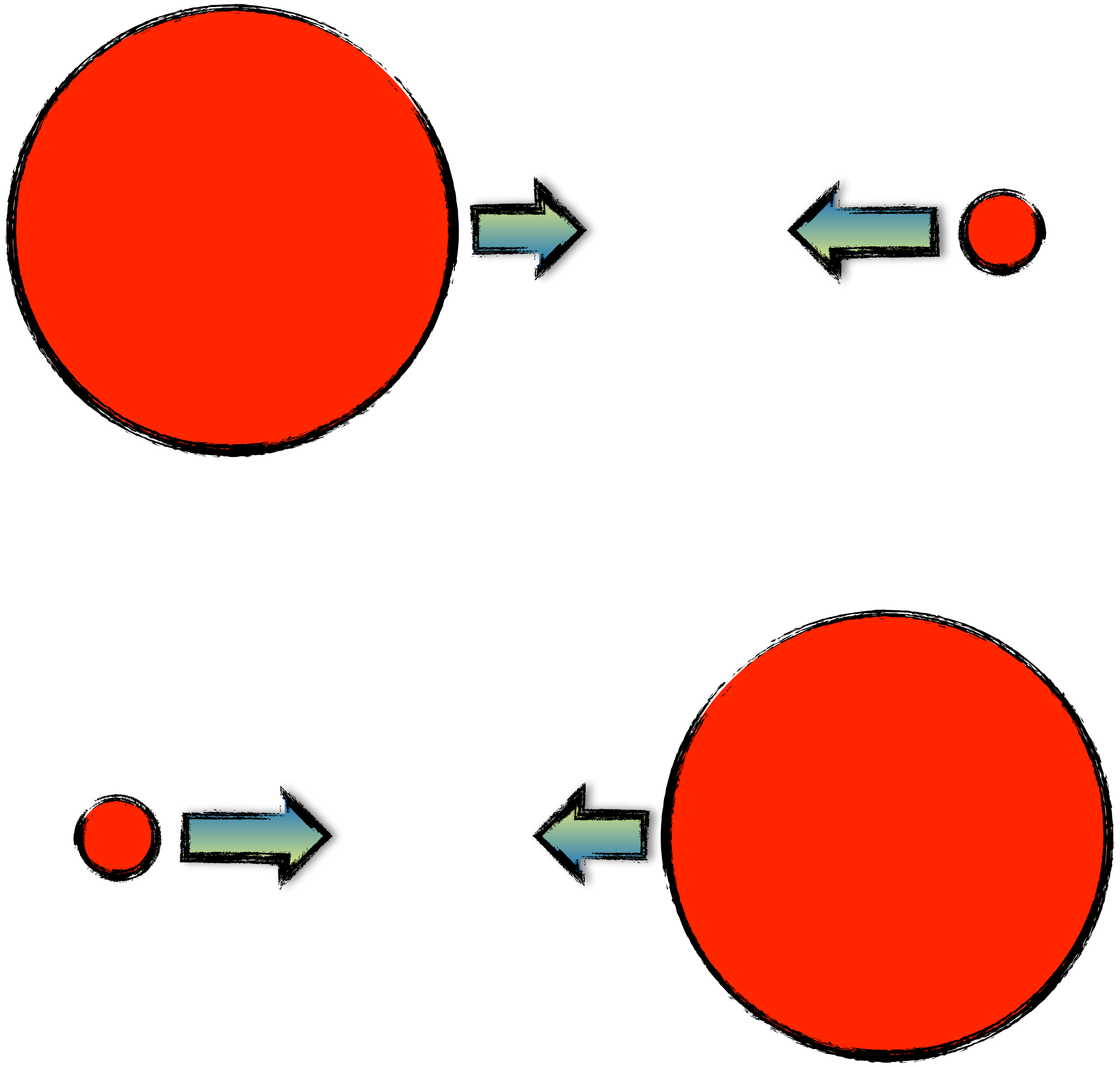
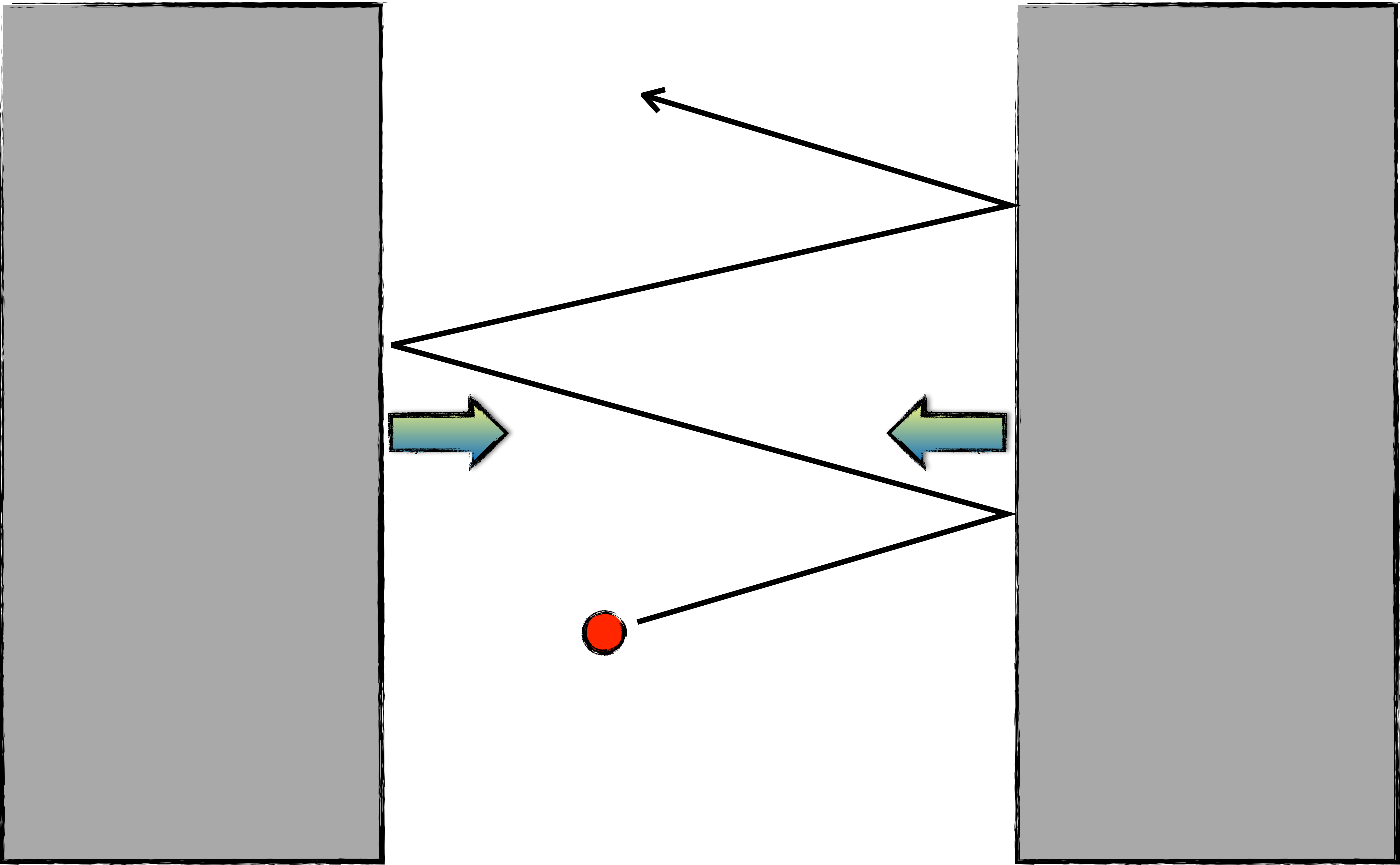


head-on

head-on

Converging walls



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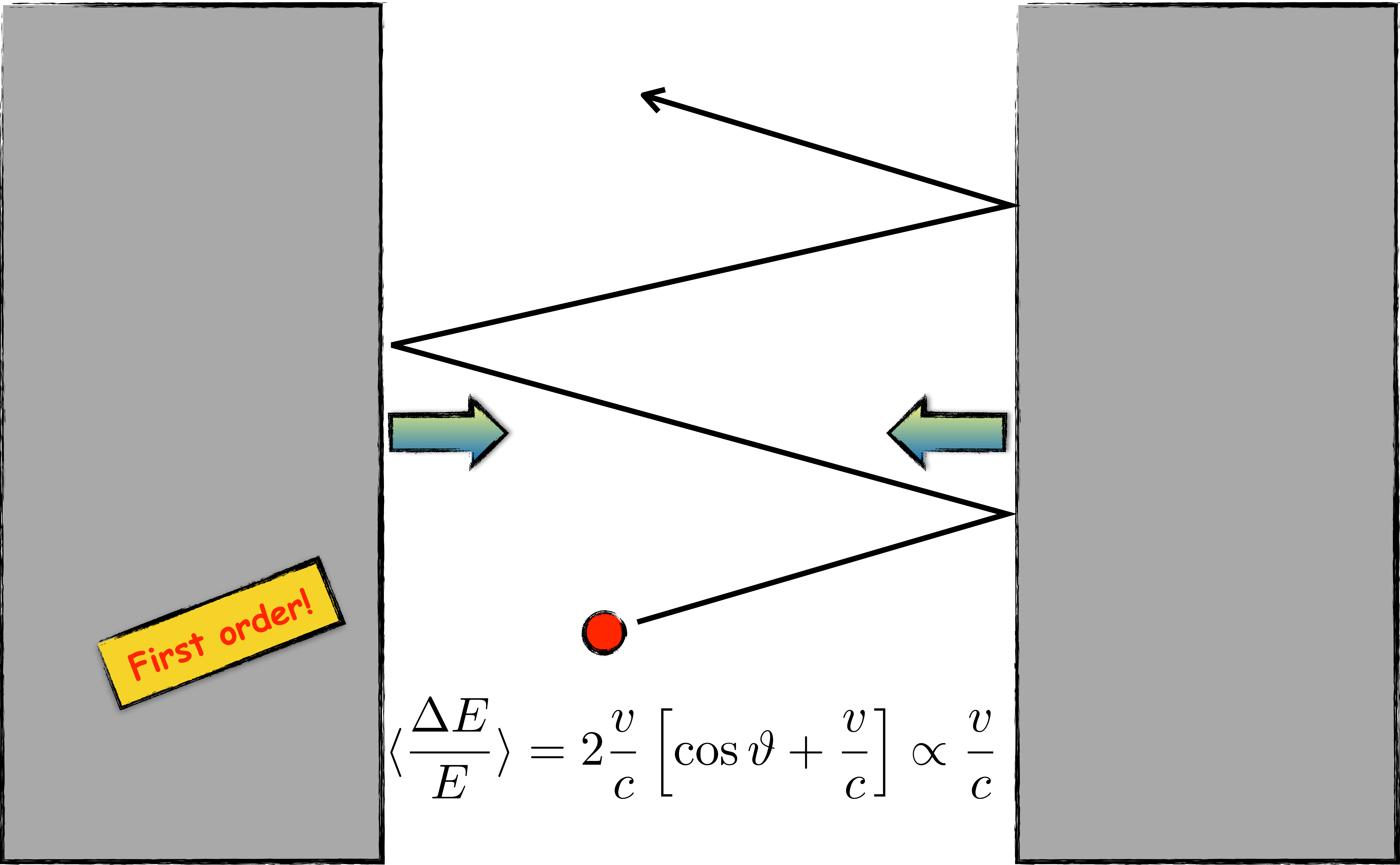


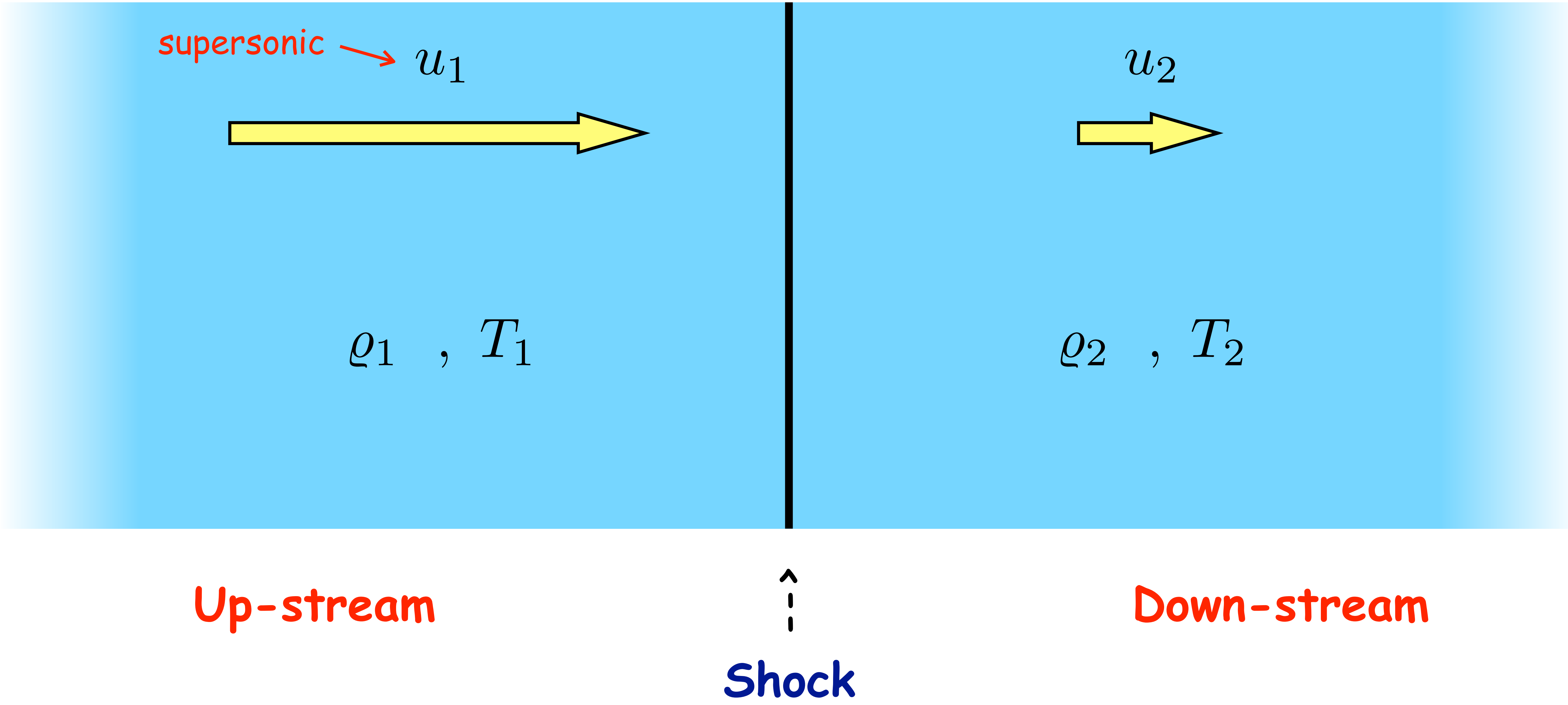
First order!

$$\langle \frac{\Delta E}{E} \rangle = 2\frac{v}{c}\left[\cos\vartheta + \frac{v}{c}\right] \propto \frac{v}{c}$$

(Strong) Shock waves
(in 1 slide)

Shock waves in one slide

Shock rest frame



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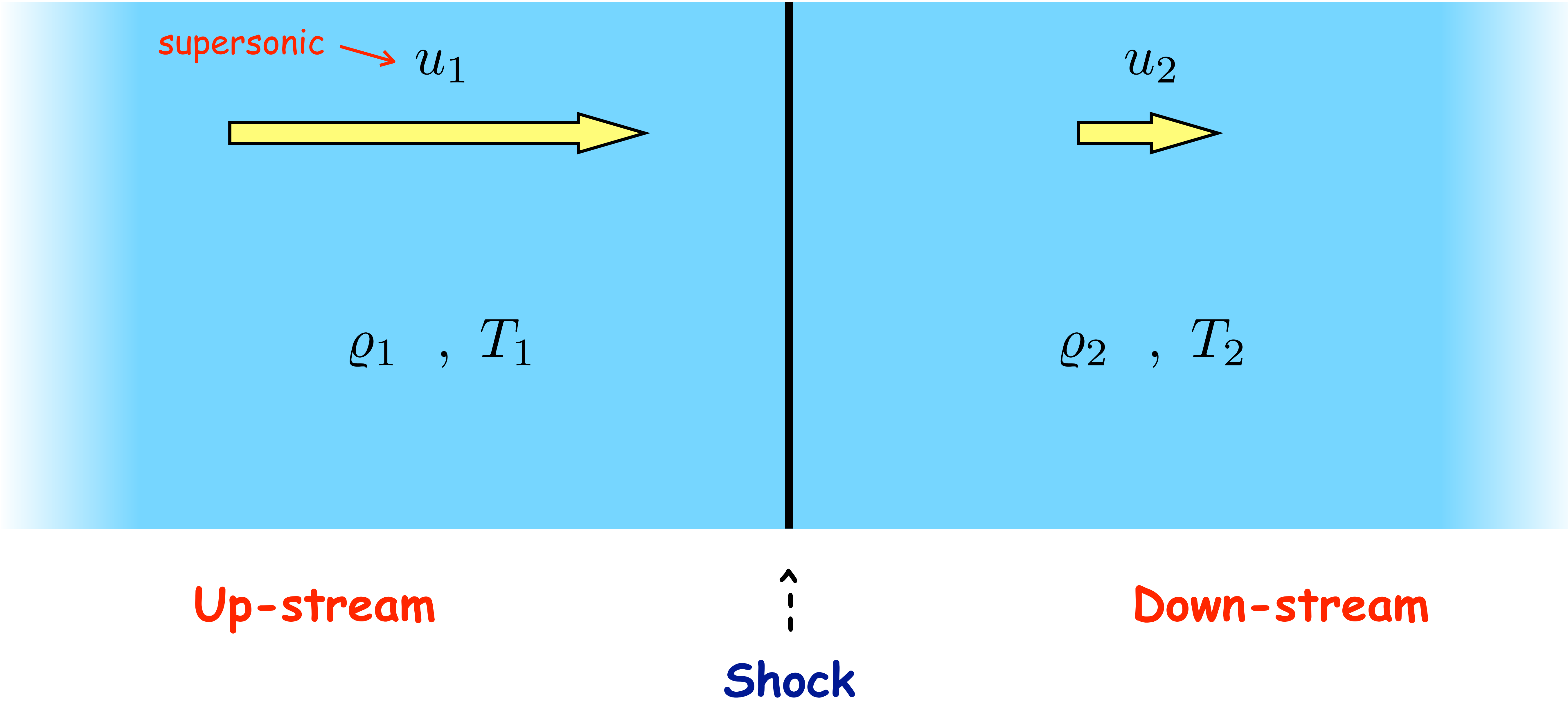
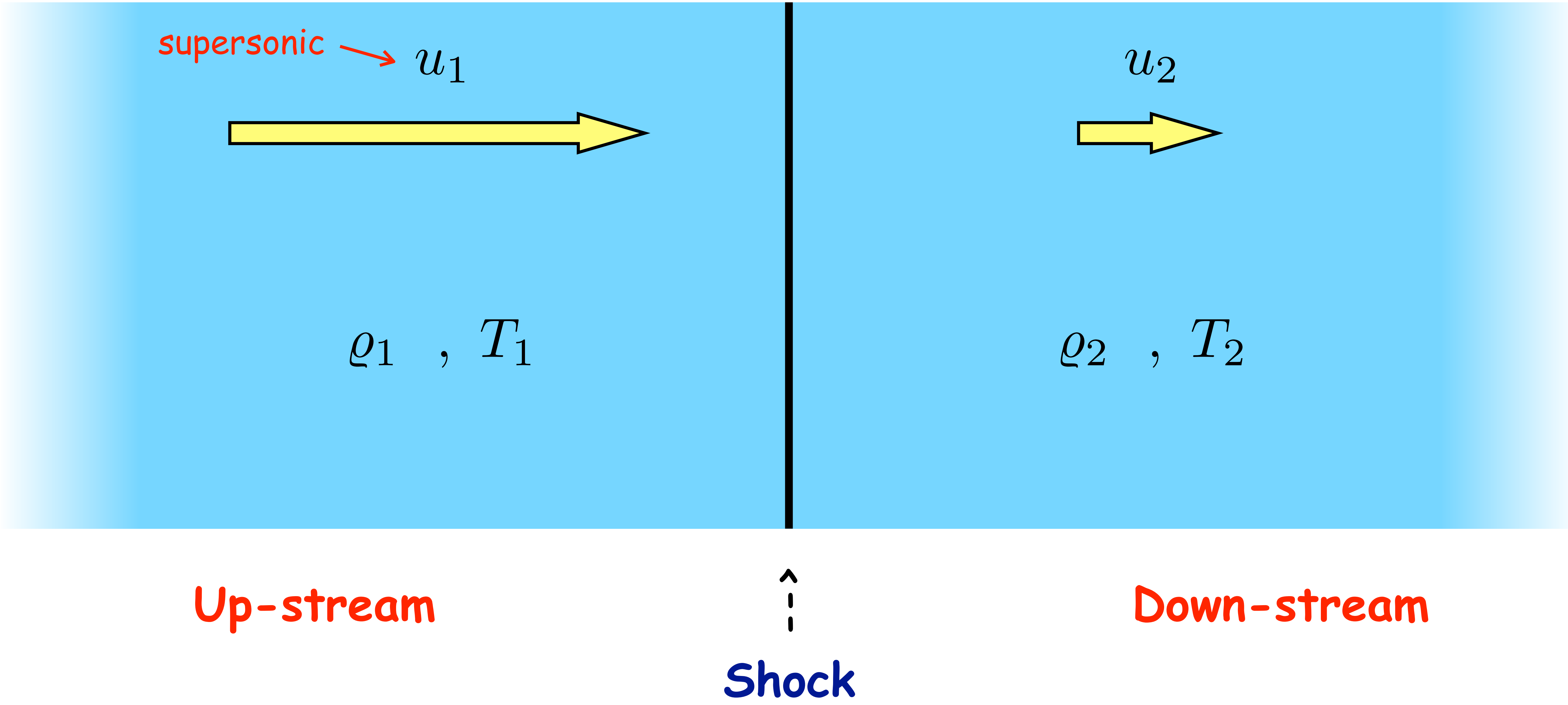


In 1960 (!) F. Hoyle (first?) suggests that shocks accelerate CRs

$$\frac{\varrho_2}{\varrho_1} = \frac{u_1}{u_2} = \frac{\gamma + 1}{\gamma - 1} = 4$$

strong shock

$$p_2 = \frac{2}{\gamma + 1} \; \varrho_1 u_1^2$$



supersonic u_1

u_2

ϱ_1 , T_1

ϱ_2 , T_2

Up-stream

Down-stream

Shock

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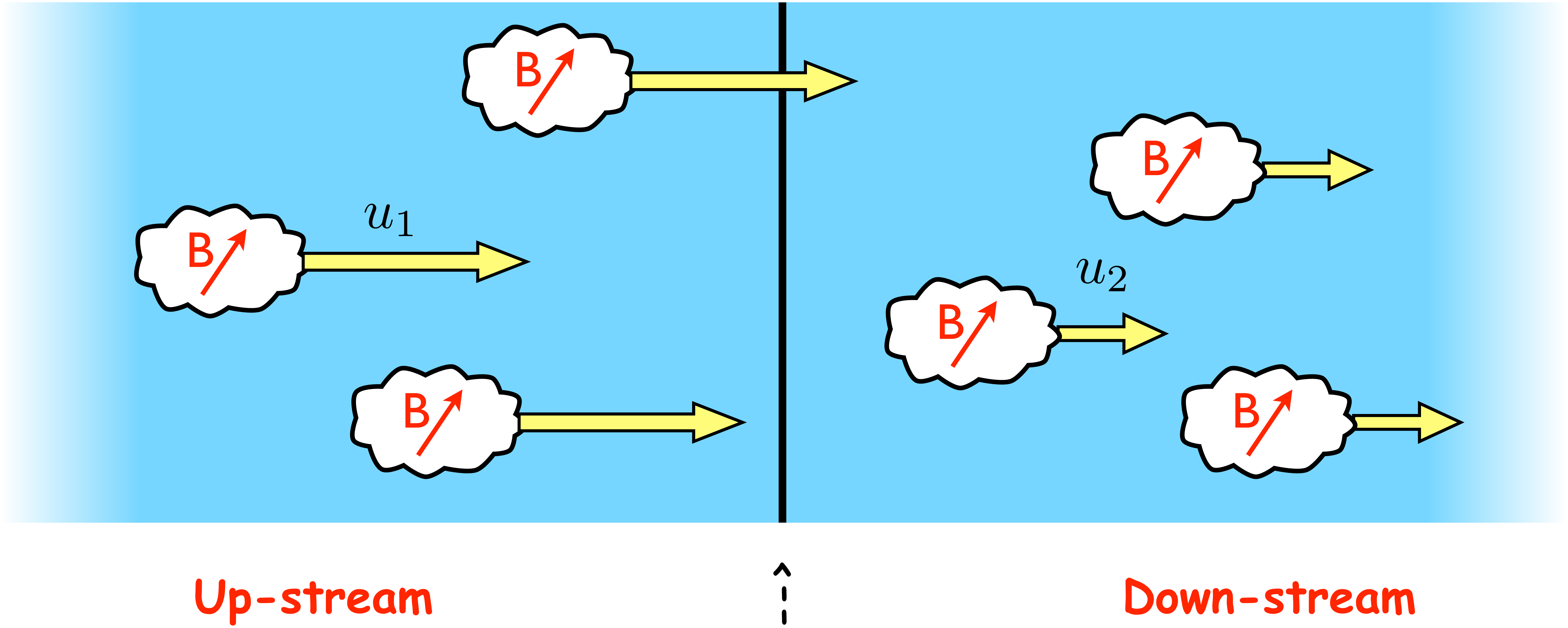
Diffusive shock acceleration

Diffusive Shock Acceleration

Shock rest frame



Up-stream

Down-stream

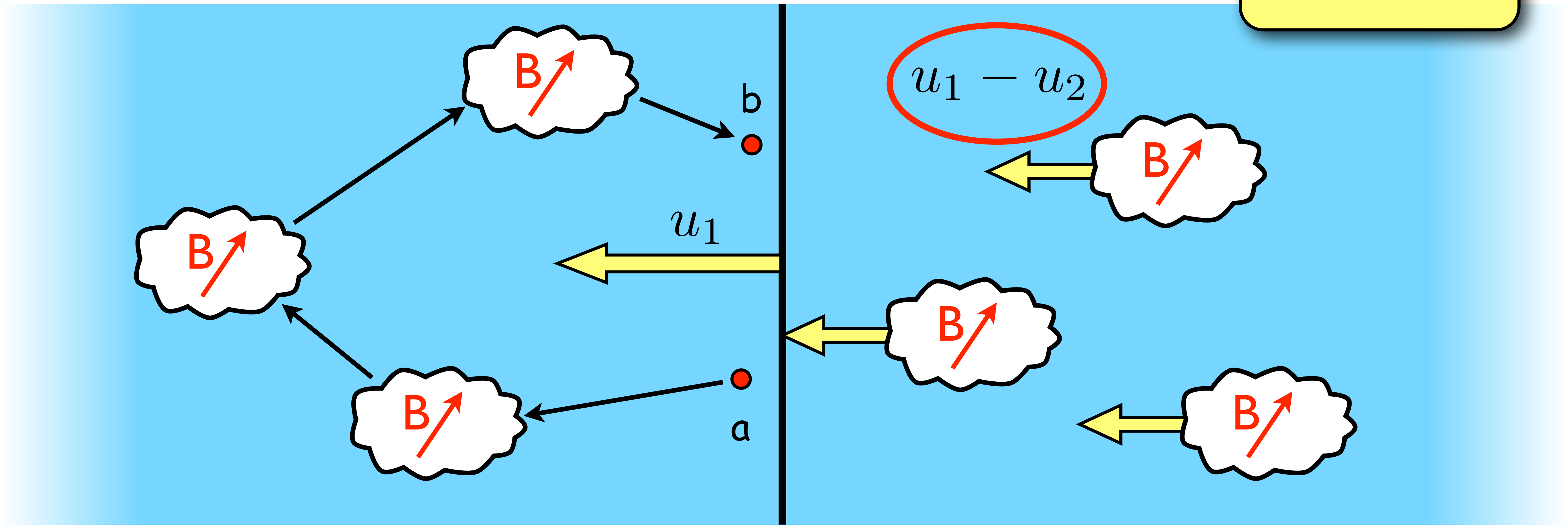
Shock

Krymskii 1977, Axford et al. 1977, Blandford & Ostriker 1978, Bell 1978

Diffusive Shock Acceleration

Up-stream rest frame

$E_a = E_b$



Up-stream

Down-stream

Shock

Diffusive Shock Acceleration

Down-stream rest frame

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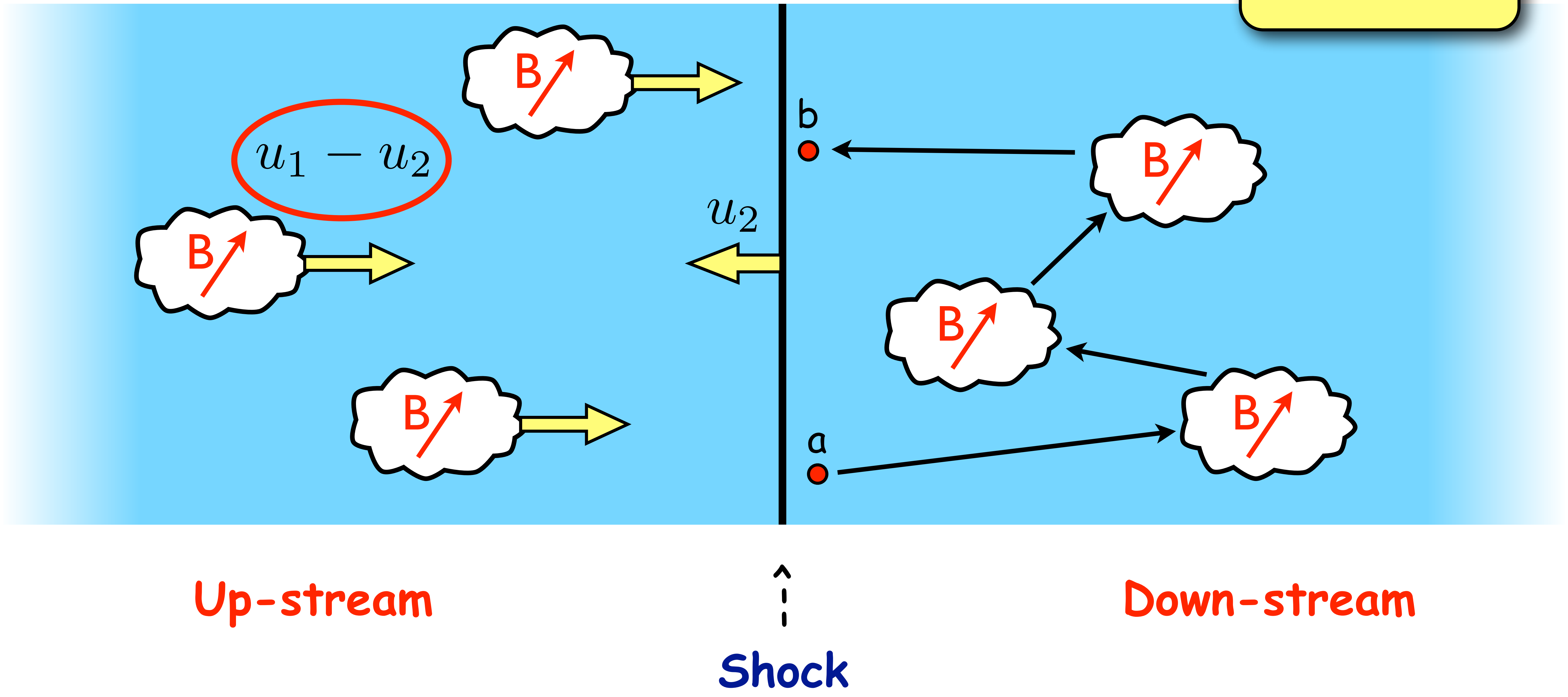

Diffusive Shock Acceleration

Symmetry

Every time the particle crosses the shock (up -> down or down -> up), it undergoes an head-on collision with a plasma moving with velocity u_1-u_2

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Asymmetry

(Infinite and plane shock:) Upstream particles always return the shock, while downstream particles may be advected and never come back to the shock

Universality
of diffusive shock acceleration

Let’s search for a test-particle solution

Assumption: scattering is so effective at shocks that the distribution of particles is very close to isotropy

-> an universal solution of the problem can be found

ΔE/E for converging walls

$$\boxed{\frac{\Delta E}{E} = \beta\left[\cos(\vartheta'_{out}) - \cos(\vartheta_{in})\right] + \beta^2\left[1 - \cos(\vartheta_{in})\cos(\vartheta'_{out})\right]}$$

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averaging over both angles —>

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$$\left\langle \frac{\Delta E}{E} \right\rangle = \frac{4}{3}\beta = \frac{4}{3}\,\frac{u_1 - u_2}{c} \xrightarrow{u_2 = \frac{u_1}{4}} \frac{u_1}{c}$$

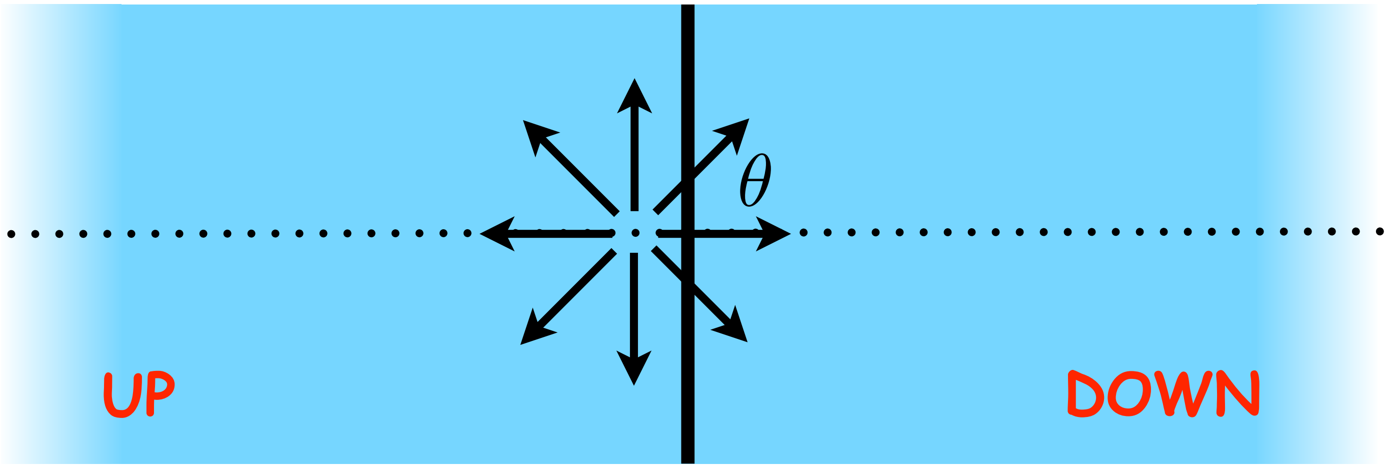
Rate at which particles cross the shock

Let's calculate R_{in}...

n -> density of accelerated particles close to the shock

n is isotropic: $\mathrm{d}n = \frac{n}{4\pi} \mathrm{d}\Omega$

velocity across the shock: $c \cos(\theta)$



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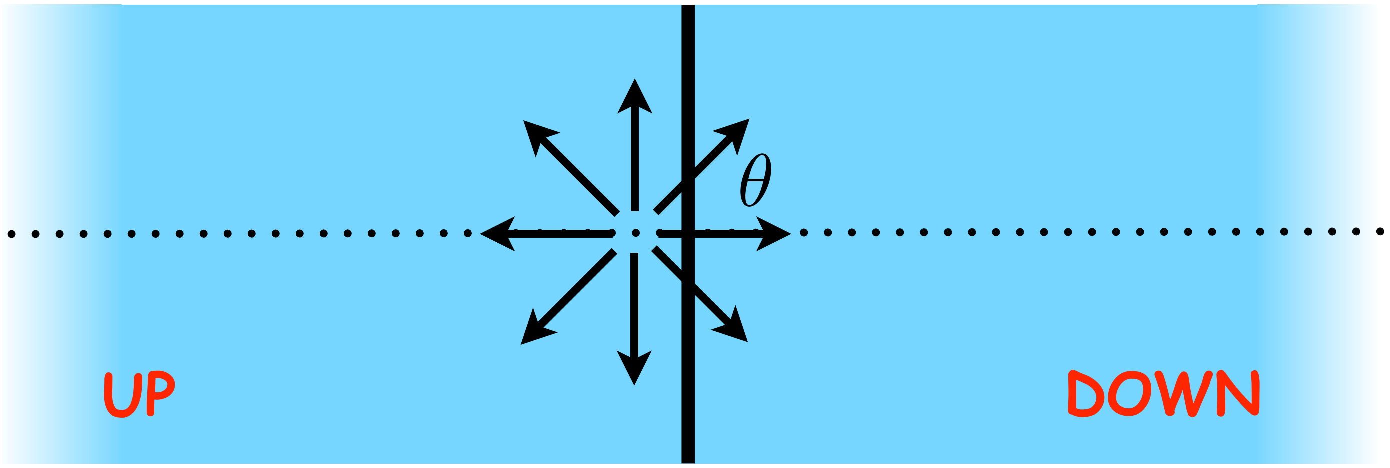

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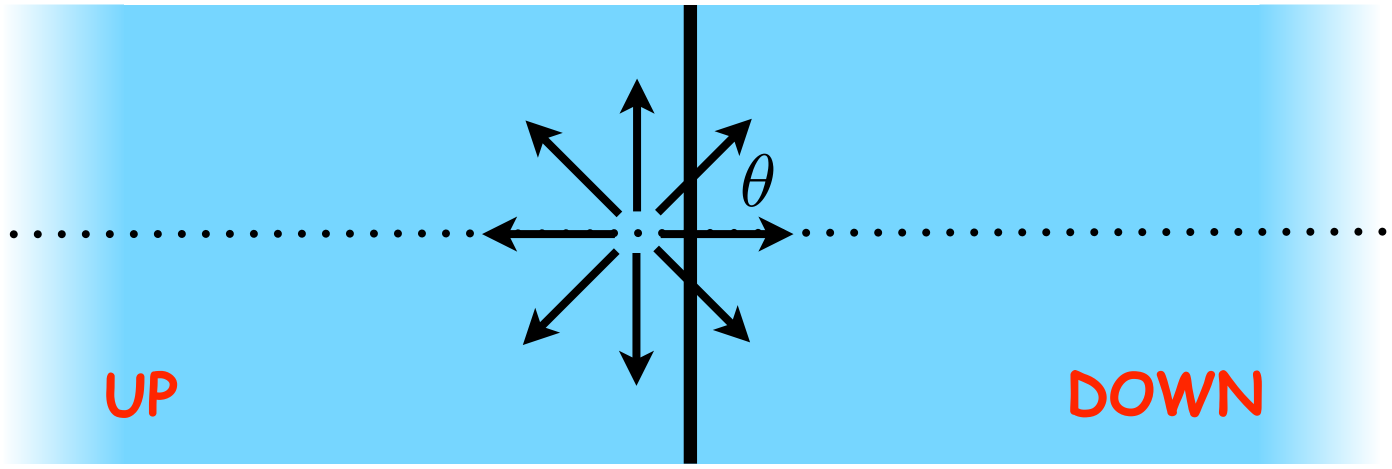

$$R_{in} = \int_{up \to down} \mathrm{d}n\ c\ \cos(\theta) = \frac{n\ c}{4\pi} \int_0^{\frac{\pi}{2}} \cos(\theta)\sin(\theta)\mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\psi = \boxed{\frac{1}{4}\ n\ c}$$

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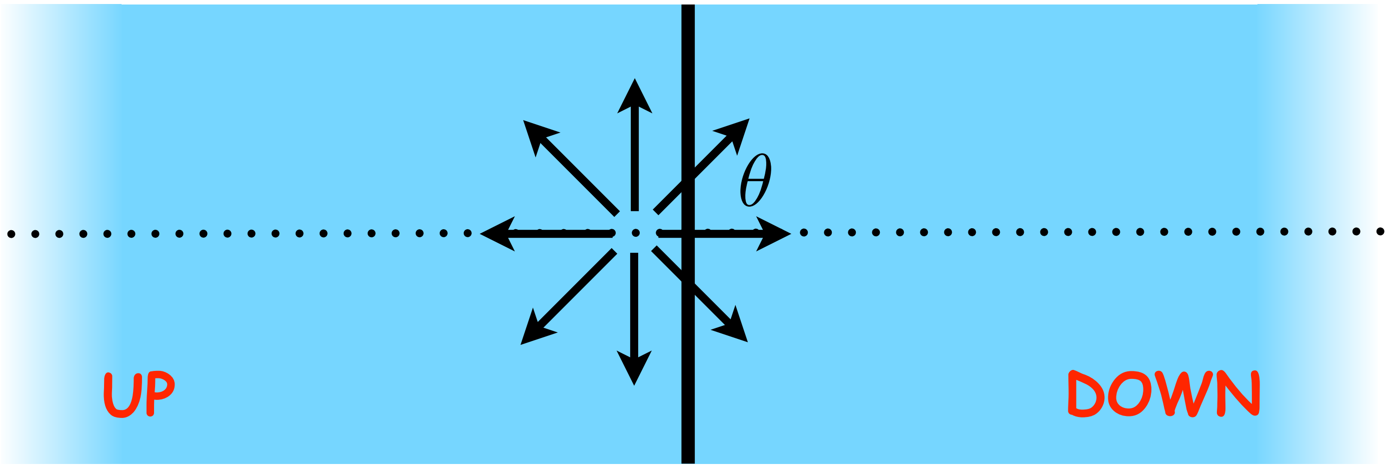

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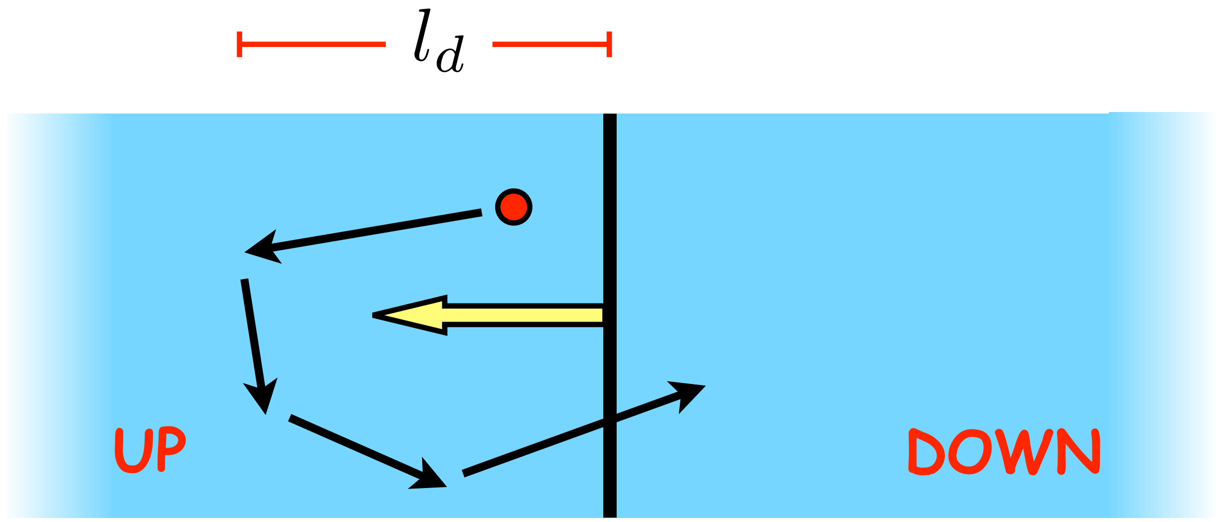
-> the same result is obtained for down -> up

Residence time upstream

-> let's find the STEADY STATE solution upstream of the shock



behavior of particles is diffusive
D(E) -> diffusion coefficient

very poorly constrained (from
both observations and theory)

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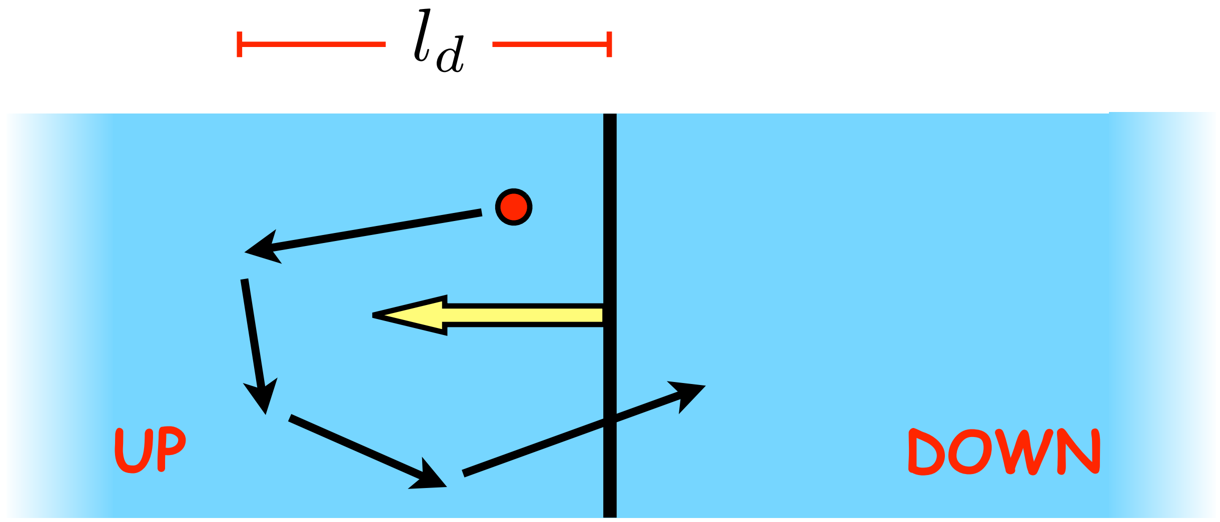

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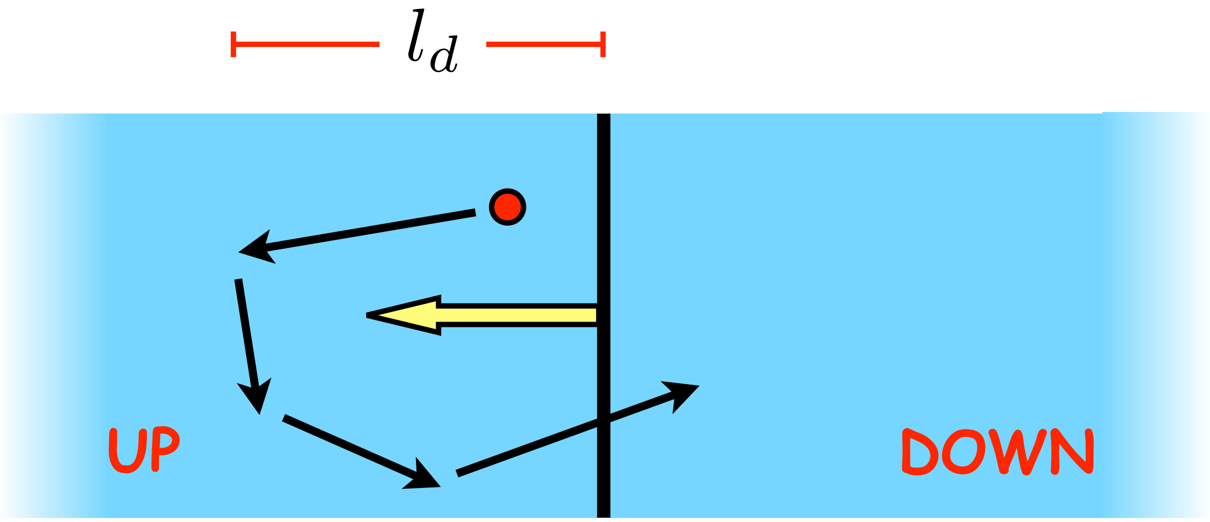

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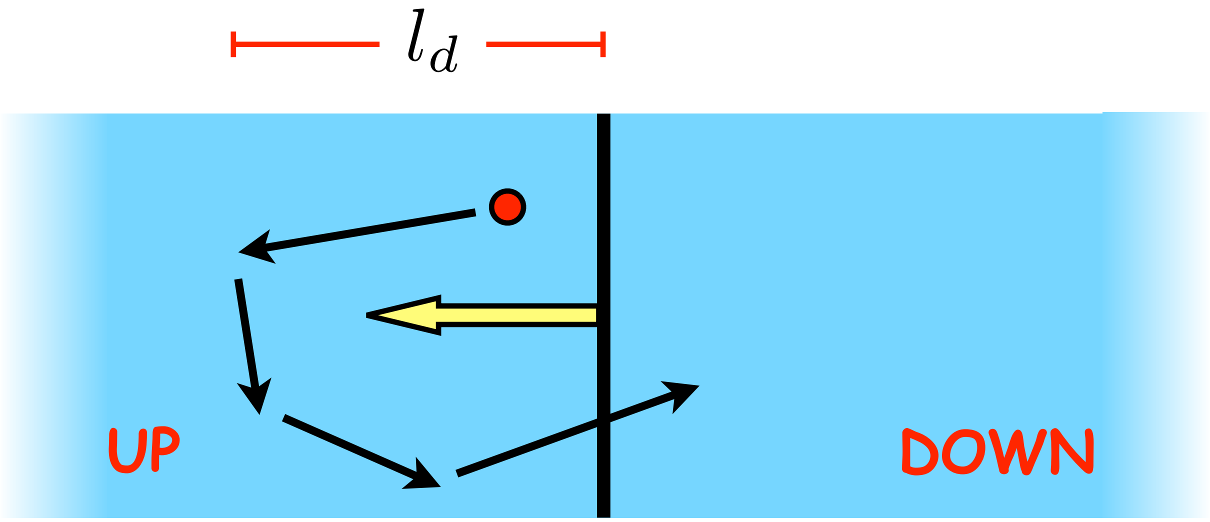

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$$\boxed{l_d \approx \frac{D}{u_1}}$$

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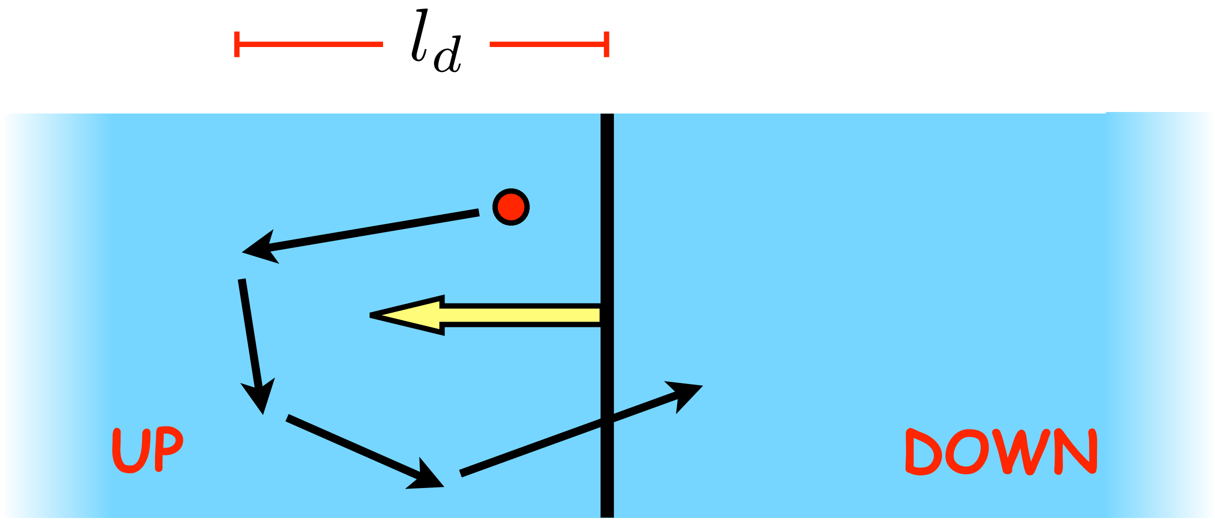

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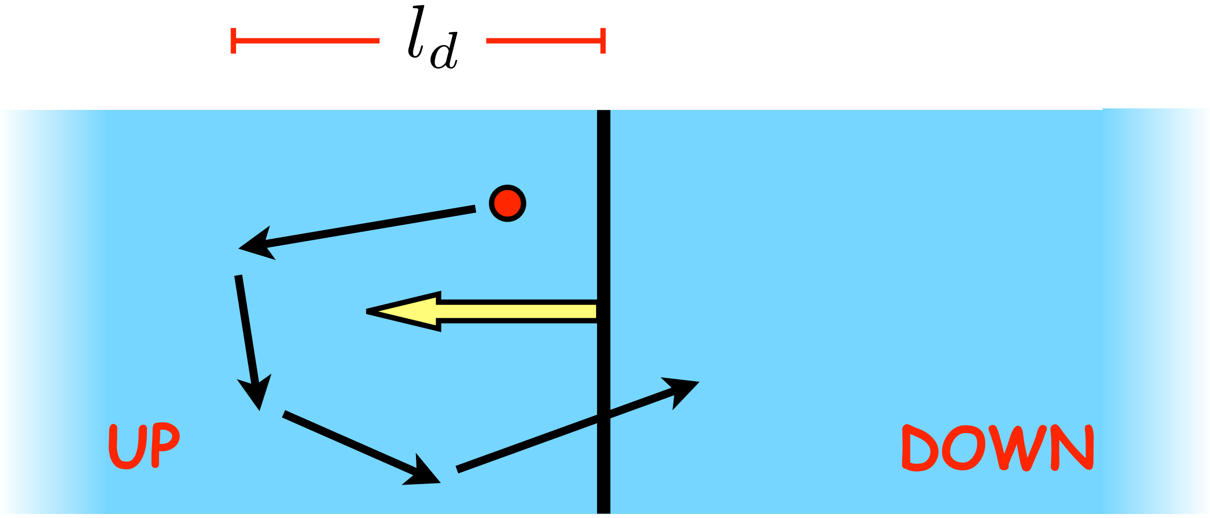

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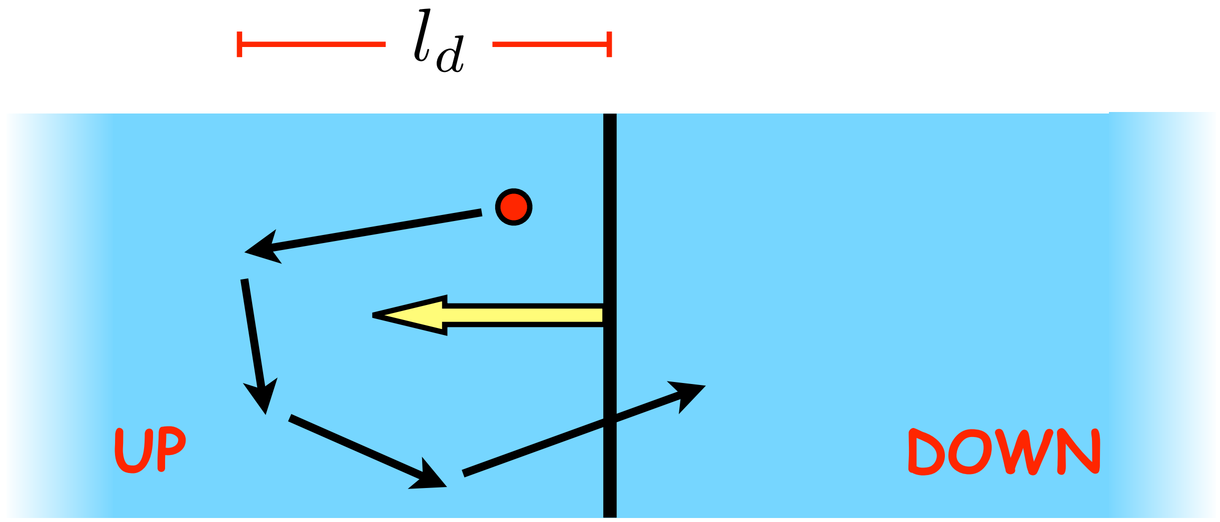

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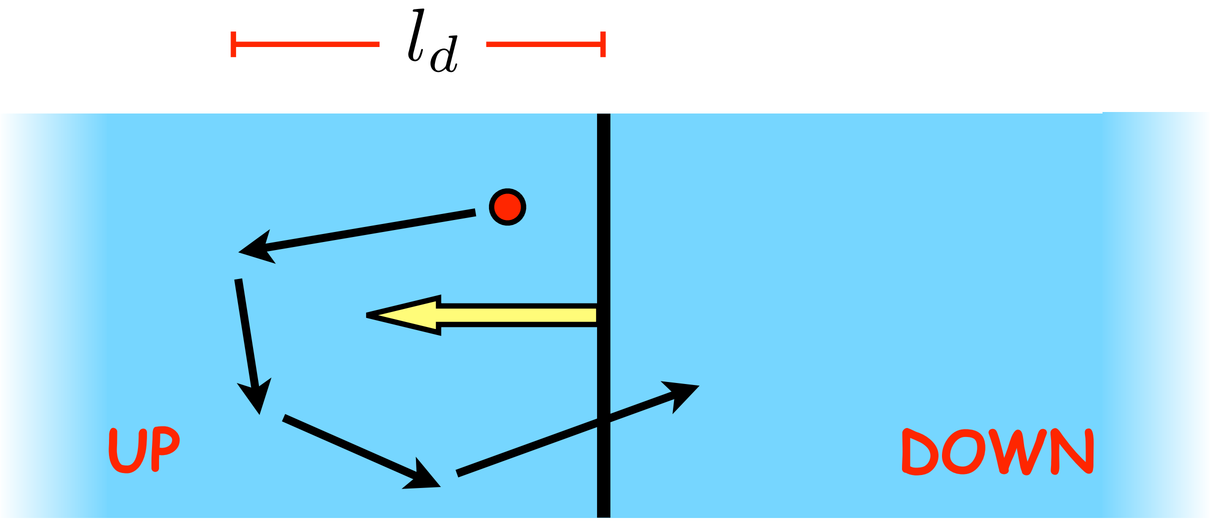

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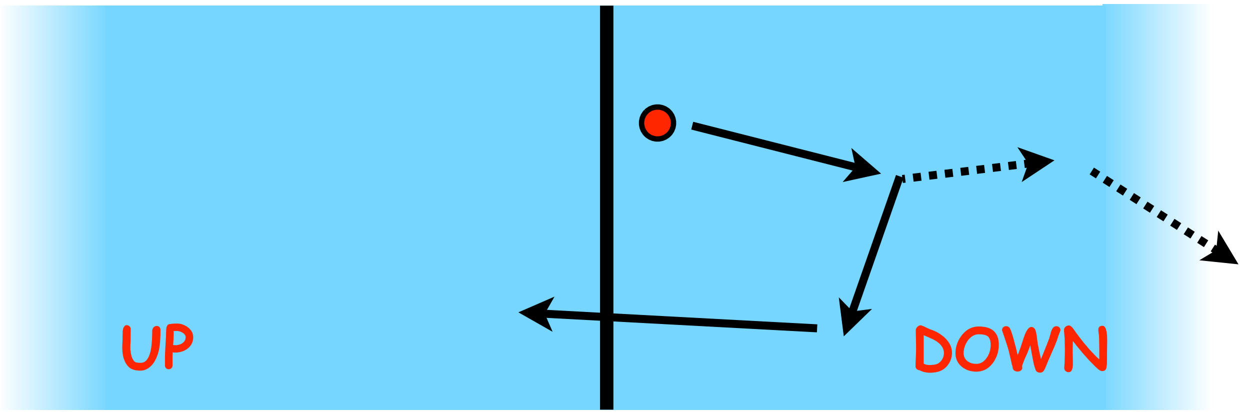
Residence time downstream

-> a bit more subtle...



n is constant downstream of the shock

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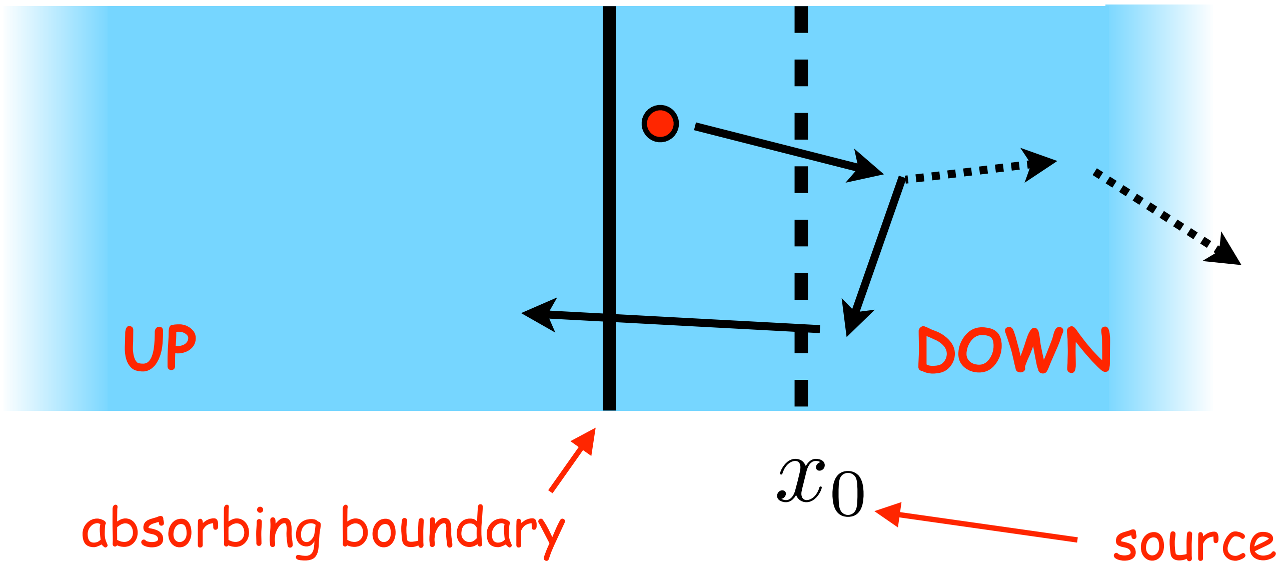

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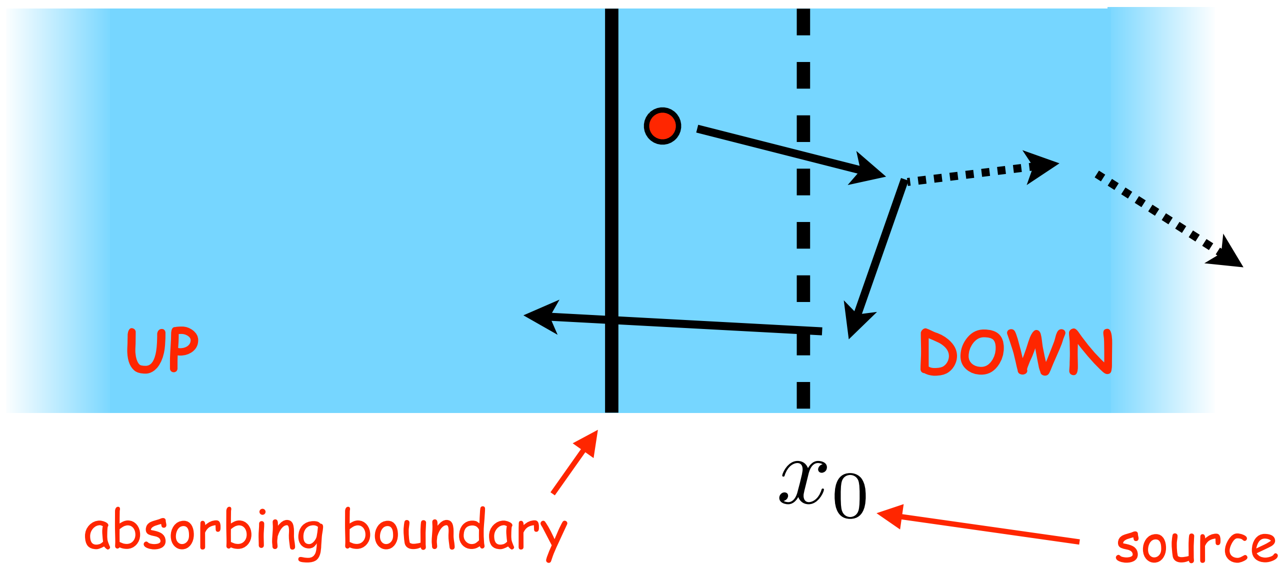

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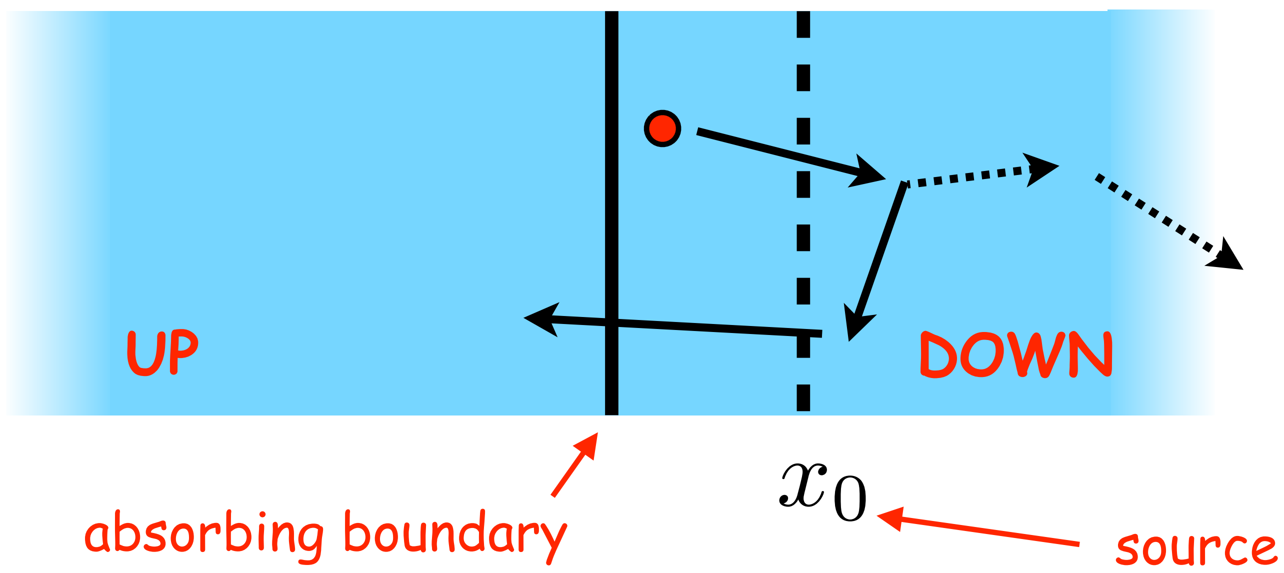
we need to know the returning flux $D \frac{\partial n}{\partial x}\Big|_{x=0}$

Residence time downstream

-> a bit more subtle...



n is constant downstream of the shock

$$u_2 \frac{\partial n}{\partial x} = D \frac{\partial^2 n}{\partial x^2} + Q\ \delta(x - x_0)$$

$$n(0) = 0$$

we need to know the returning flux $D \frac{\partial n}{\partial x}\Big|_{x=0}$ $\longrightarrow P_{ret} = \frac{D \frac{\partial n}{\partial x}\Big|_{x=0}}{Q}$

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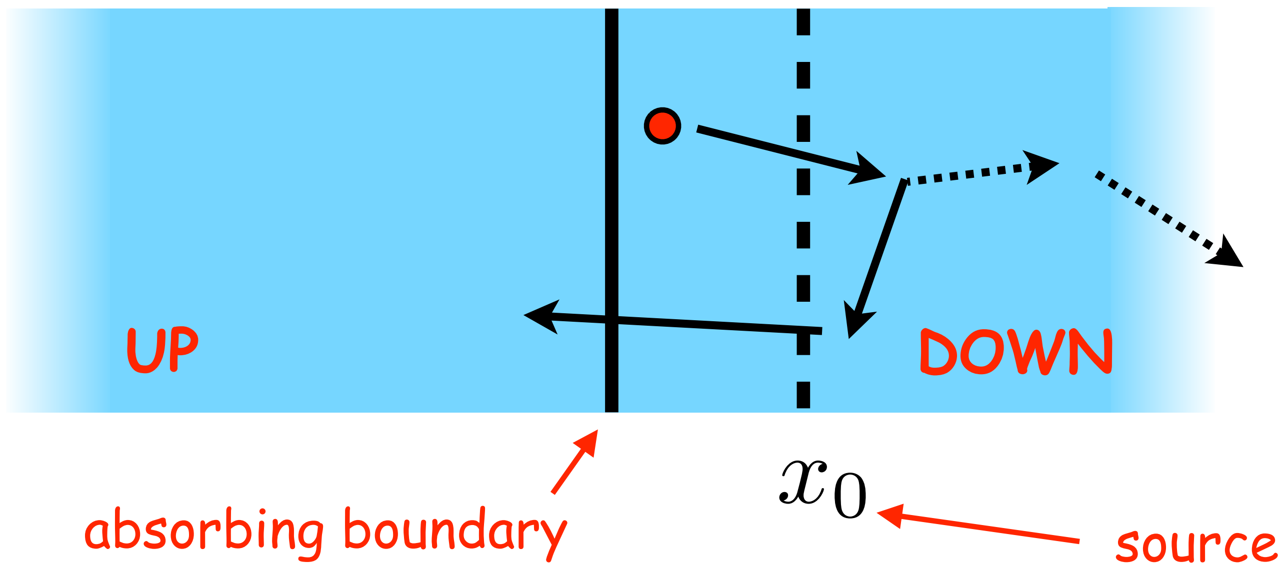

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$$\boxed{P_{ret} = \exp\left(-\frac{x_0 u_2}{D}\right)}$$

Residence time downstream

number of downstream particles that will return to the shock:

$$\int_0^\infty \mathrm{d}x \; P_{ret}(x) \; n \;=\; \frac{D \; n}{u_2}$$

same expression upstream!

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mean residence time upstream <-> mean residence time downstream

$$\frac{4D}{u_1 c} \qquad\qquad \frac{4D}{u_2 c}$$

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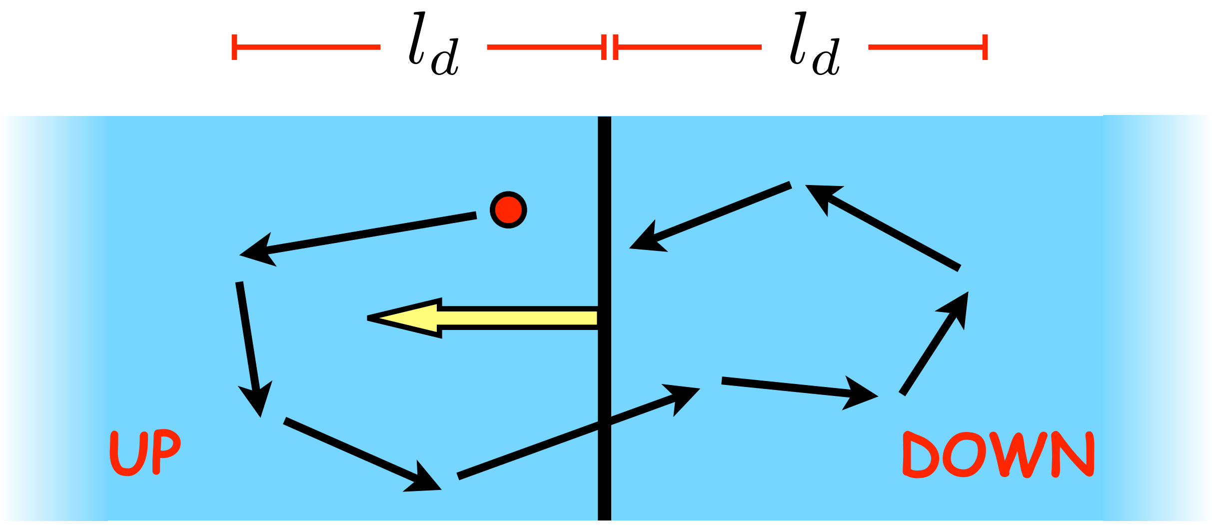
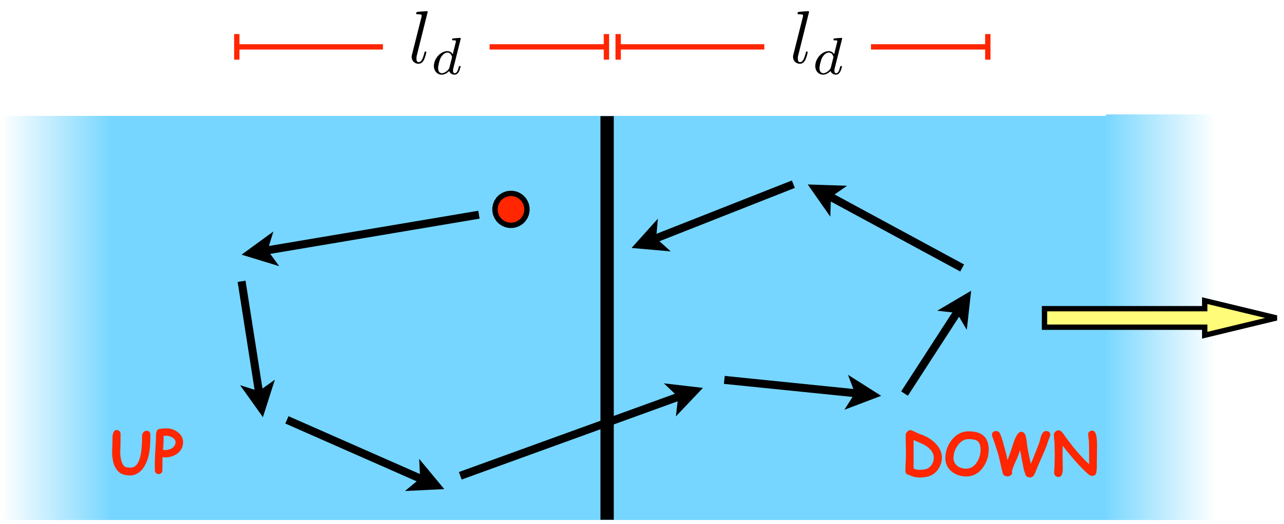


Rate at which particles leave the system



cosmic ray density n is constant downstream...

$$\boxed{R_{out} = n u_2}$$

Bell's approach

Bell (1978)

Let's start with N_0 particles of energy E_0...

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-> # of particles performing at least k cycles: $N_k = N_0 \left(1 - \frac{u_1}{c}\right)^k$

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Let's start with N_0 particles of energy E_0...

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— divide —>

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-> have an energy larger than: $E_k = E_0 \left(1 + \langle \frac{\Delta E}{E} \rangle\right)^k = E_0 \left(1 + \frac{u_1}{c}\right)^k$

Bell's approach

Bell (1978)

Let's start with N₀ particles of energy E₀...

-> # of particles starting a cycle per second: $nc/4$

-> # of particles leaving the system per second: $nu_2 = nu_1/4$

– divide –>

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Universal solution: Bell's approach

$$\log\left(\frac{N}{N_0}\right) = k\log\left(1 - \frac{u_1}{c}\right)$$

$$\log\left(\frac{E}{E_0}\right) = k\log\left(1 + \frac{u_1}{c}\right)$$

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– divide –>

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UNIVERSAL!!!

Weak shocks

weak ---> strong

Mach number

$$\mathscr{M} = \frac{u_1}{c_s} \longrightarrow \infty$$

sound speed

Weak shocks

weak ——> strong

Mach number

$$\mathscr{M} = \frac{u_1}{c_s} \longrightarrow \infty$$

sound speed

$$\frac{\varrho_2}{\varrho_1} = \frac{u_1}{u_2} = r \longrightarrow 4$$

$$\frac{\mathrm{d}N}{\mathrm{d}E} \propto E^{-\frac{3r}{r-1}} \longrightarrow -2$$

How fast?

let's compute the acceleration rate

energy gain
in a cycle

$$r_{acc} = \frac{\left\langle \frac{\Delta E}{E} \right\rangle}{\tau_{up} + \tau_{down}}$$

duration
of a cycle

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$$r_{acc} = \frac{\left\langle \frac{\Delta E}{E} \right\rangle}{\tau_{up} + \tau_{down}} = \frac{\frac{u_1}{c}}{\frac{4D}{u_1 c} + \frac{4D}{u_2 c}}$$

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$$D = \frac{1}{3}\lambda c$$

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the "elementary" process is resonant scattering

$$D = \frac{1}{3}\lambda c \longrightarrow \frac{1}{3} R_L c$$

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most optimistic choice for D

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$$\boxed{\tau_{acc} = \frac{1}{r_{acc}} = \frac{20 R_L c}{3 u_1^2}}$$

most optimistic choice for D

Comparison with Hillas

Hillas acceleration time —>

$$\tau_{acc}^{H} = \left(\frac{v}{c}\right)^{-1} \frac{R_L}{c}$$

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shock acceleration —>

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shock acceleration —>

$$\tau_{acc} = a \left(\frac{v}{c}\right)^{-1} \tau_{acc}^{H}$$

factor of several

>>1

DSA is faster than Fermi II but still (obviously) slower than Hillas

Particle acceleration at relativistic shocks

Achterberg et al. (2001)
Achterberg, Lecture notes, Les Houches School (2004)

Acceleration @non-relativistic/relativistic shocks: most important difference

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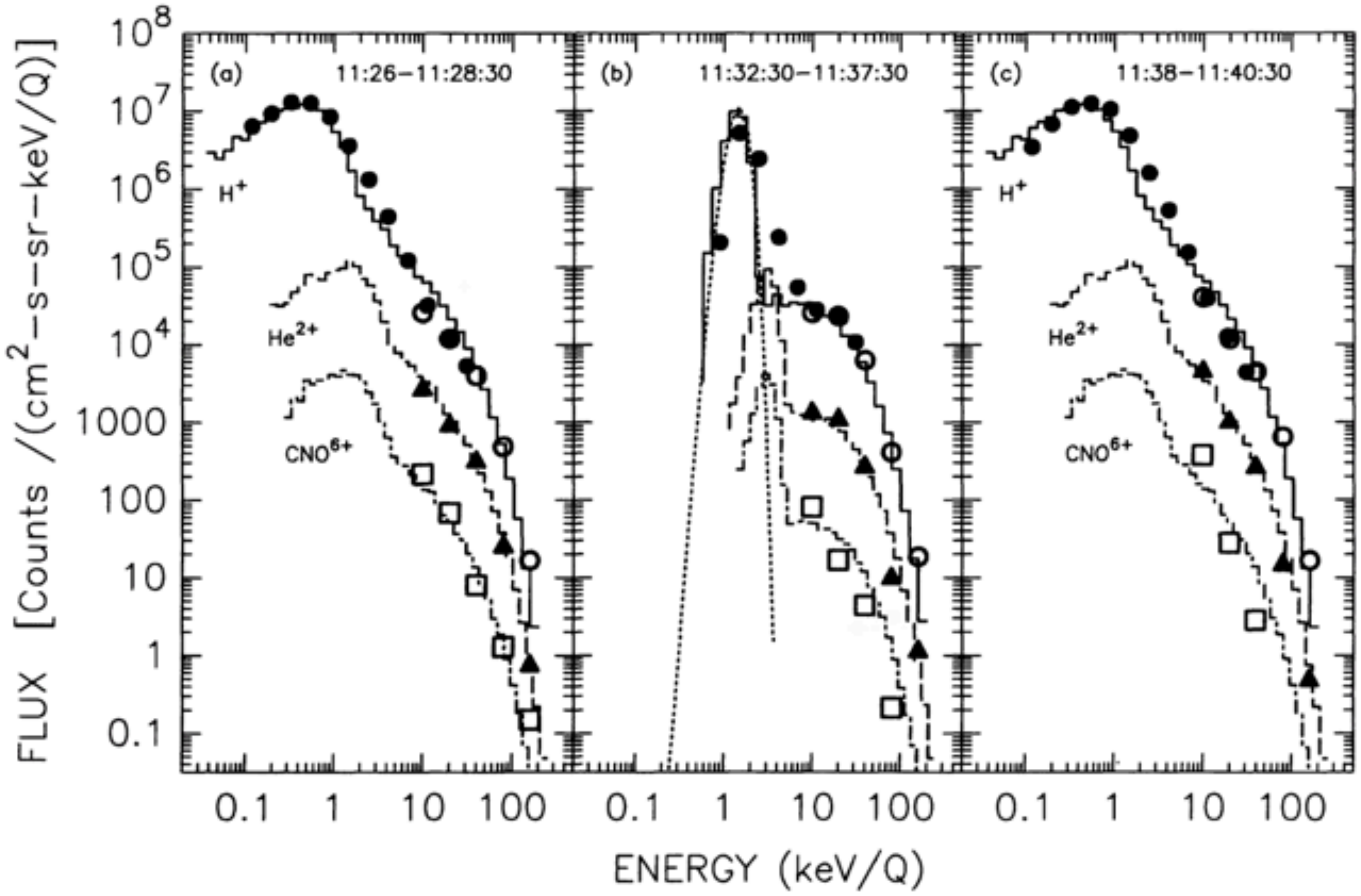
DSA at non relativistic shock EXISTS!



Ellison+ 1990

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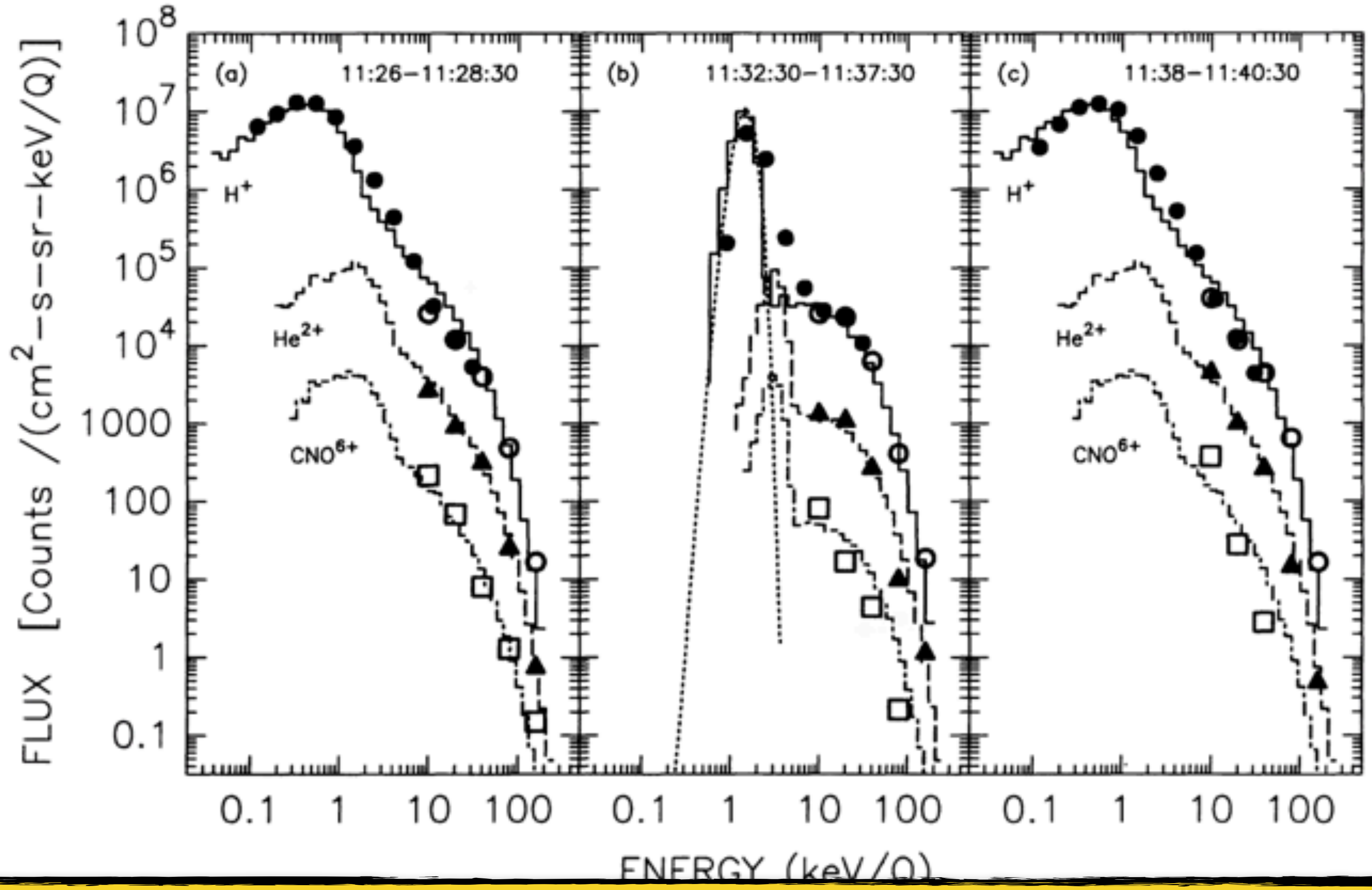
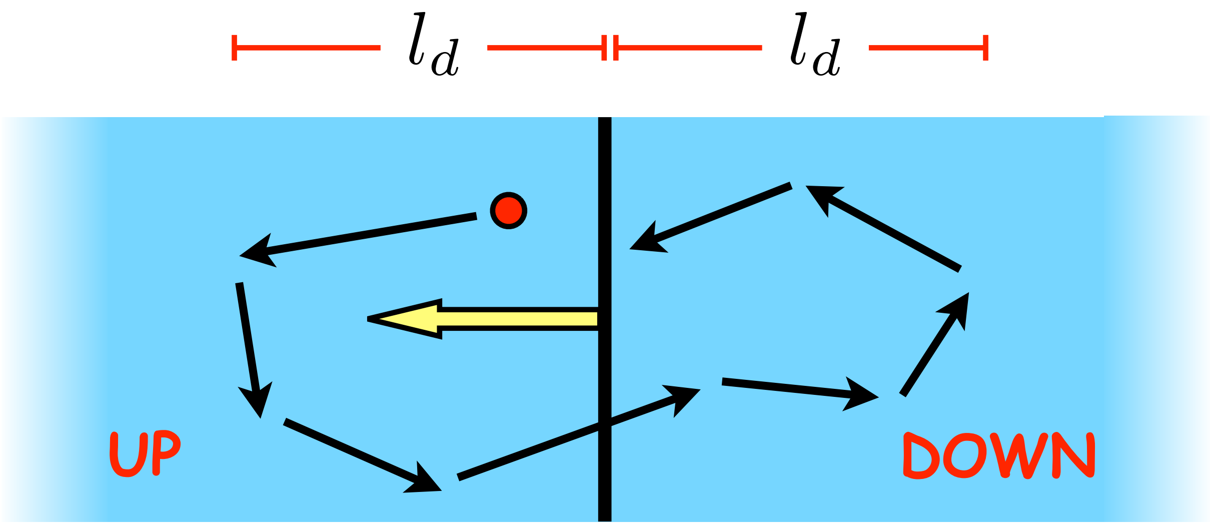


space probes “saw” it! (down-up-downstream passage at the Earth bow shock)

Ellison+ 1990

Key aspects of DSA at non-relativistic shocks

particles are accelerated through a series of cycles up—>down—>up stream



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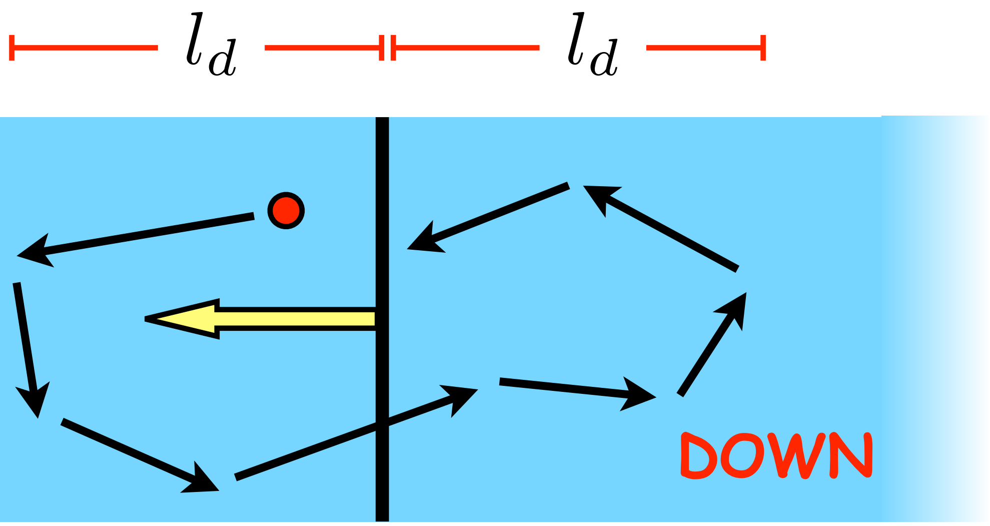

- diffusive transport —> isotropy

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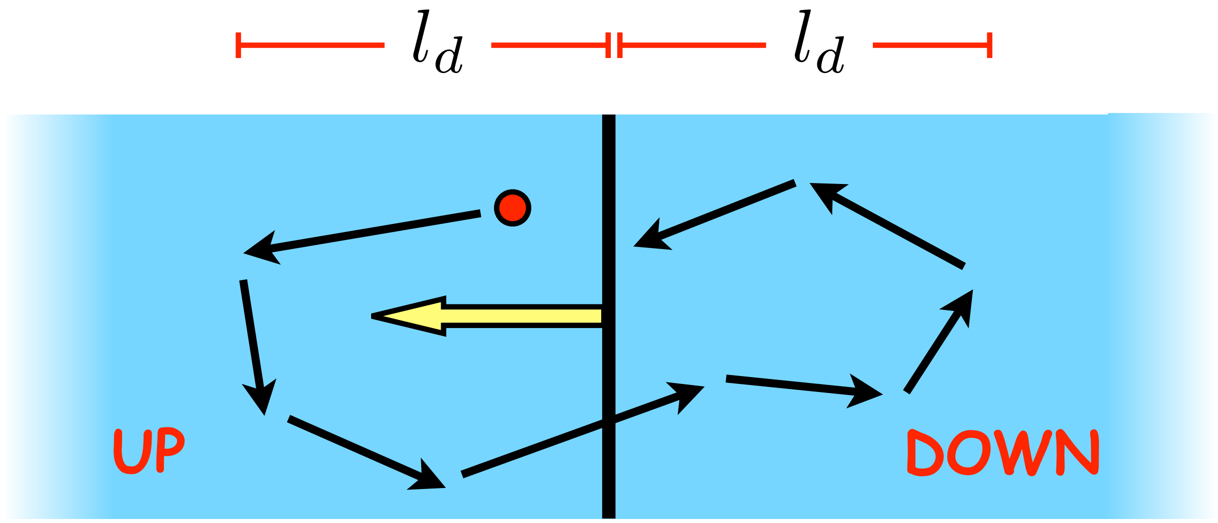

- diffusive transport —> isotropy
- energy gain per cycle —> small

$$\Delta_{acc} = \Delta E/E = u_1/c$$

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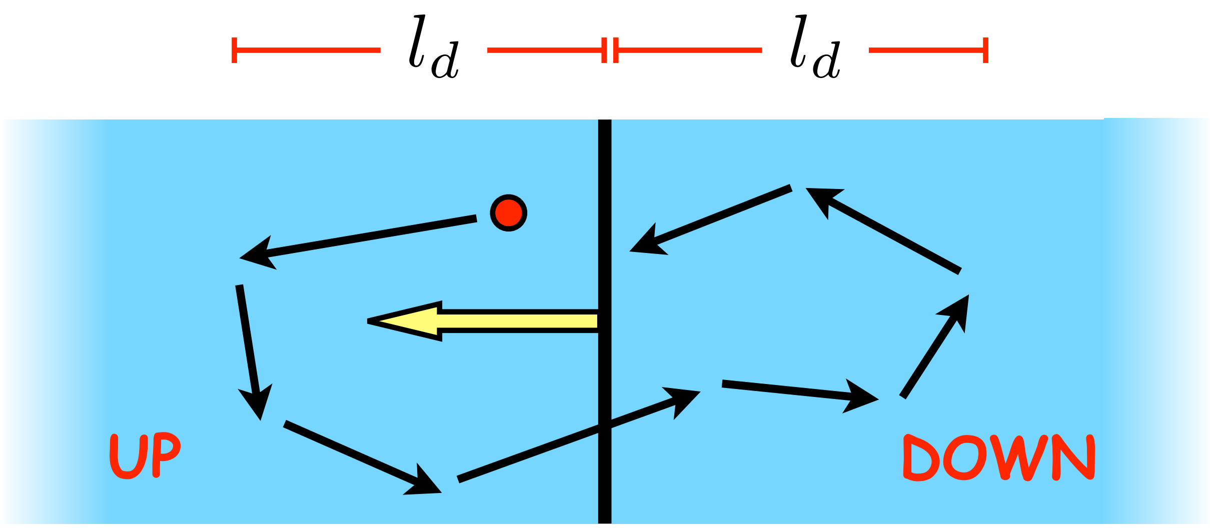

- diffusive transport —> **isotropy**
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- escape probability per cycle —> **small**

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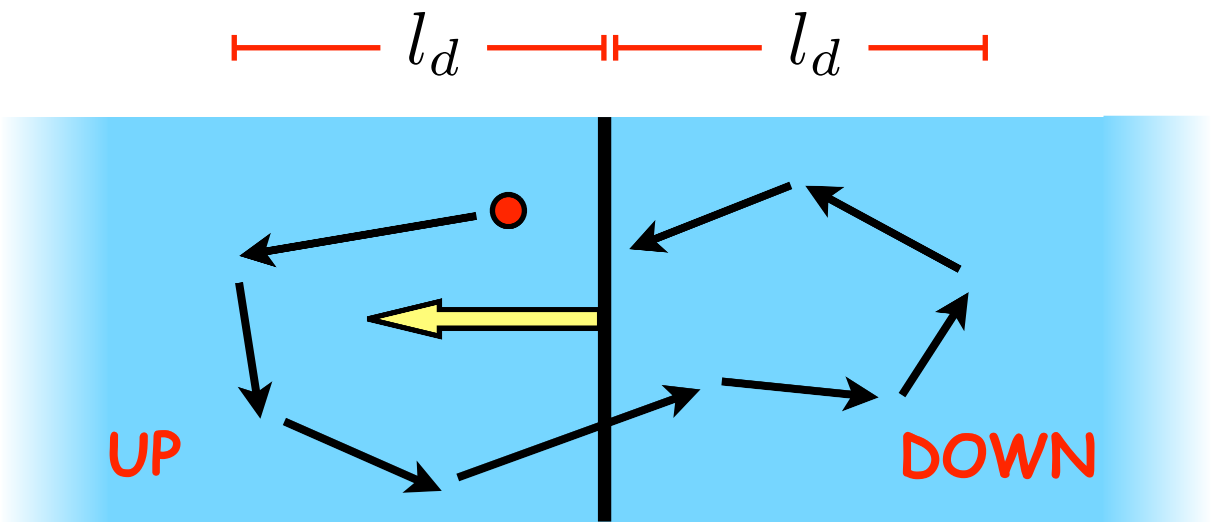

- diffusive transport —> **isotropy**
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- spectral slope —> **E^{-s}**

$$\Delta_{acc} = \Delta E/E = u_1/c$$

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$$s = 1 + \frac{P_{esc}}{\Delta_{acc}}$$

What happens if we consider a relativistic shock

c=1

The velocity of the shock is comparable to the velocity of accelerated particles

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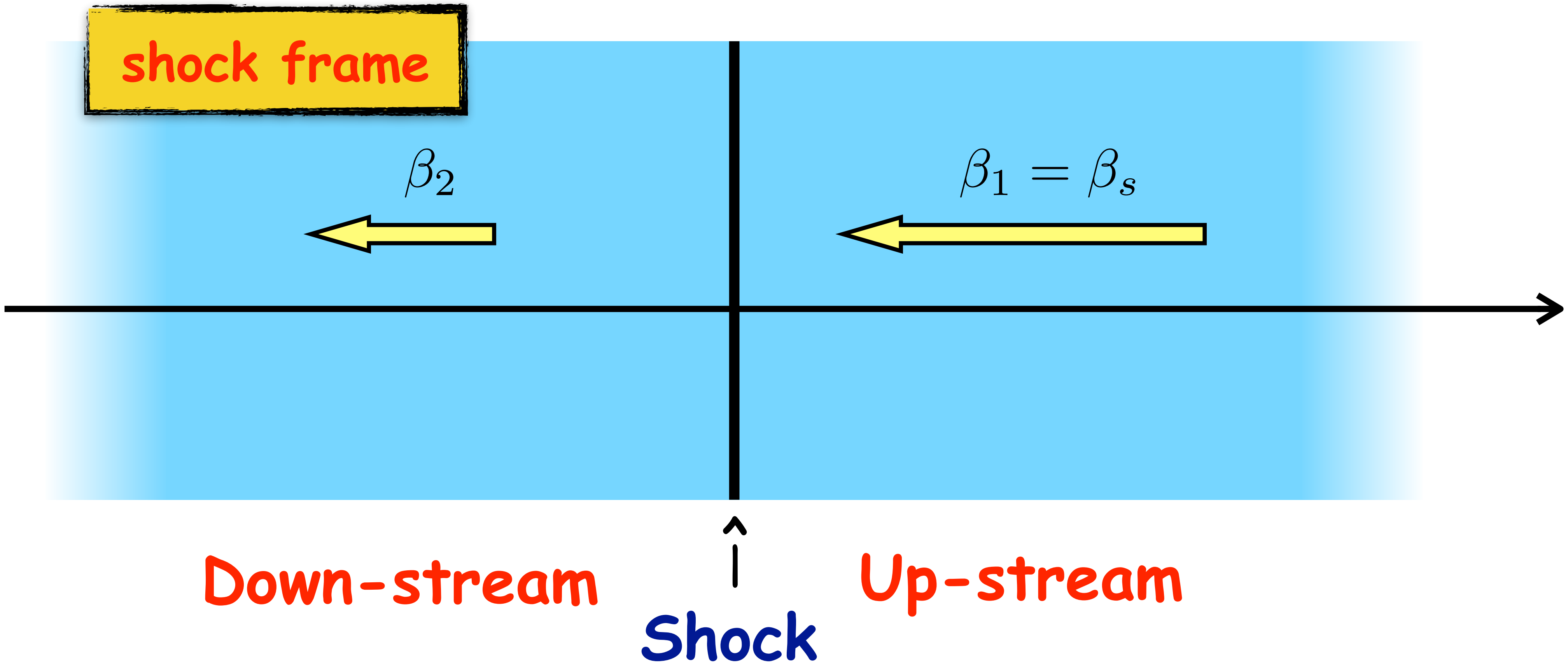
$$\boxed{\beta_s \approx 1 - \frac{1}{2\Gamma_s^2}}$$

Ultra-relativistic shocks



shock frame

β_2

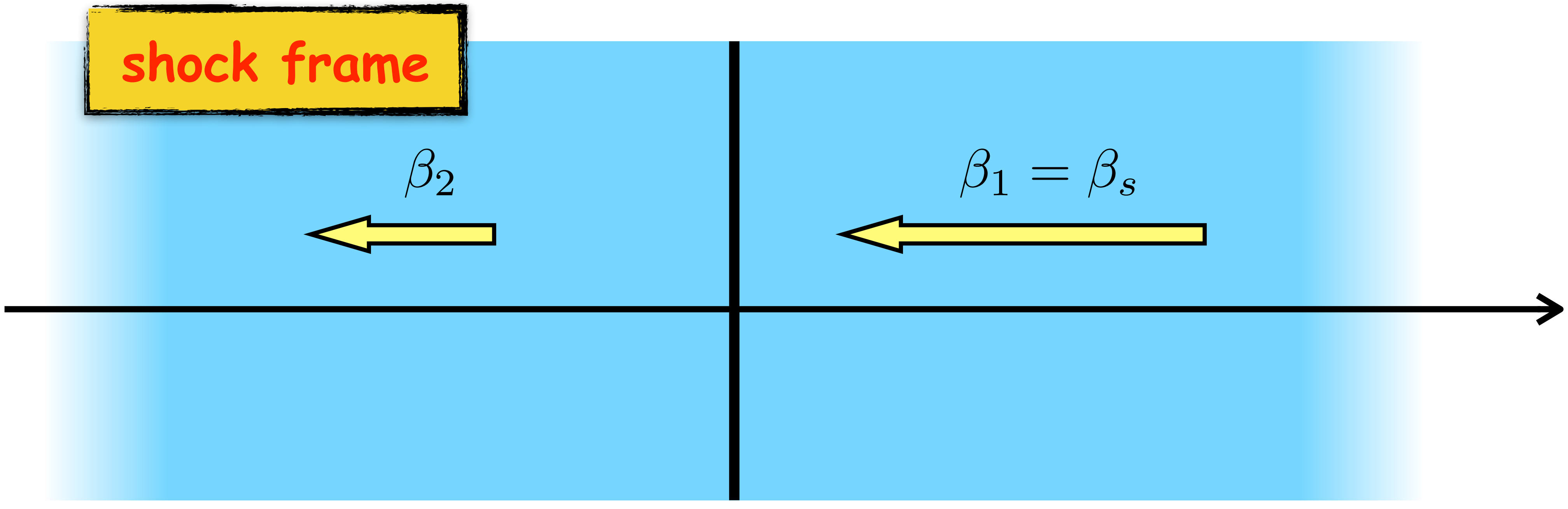
$\beta_1 = \beta_s$

Down-stream

Shock

Up-stream

Ultra-relativistic shocks



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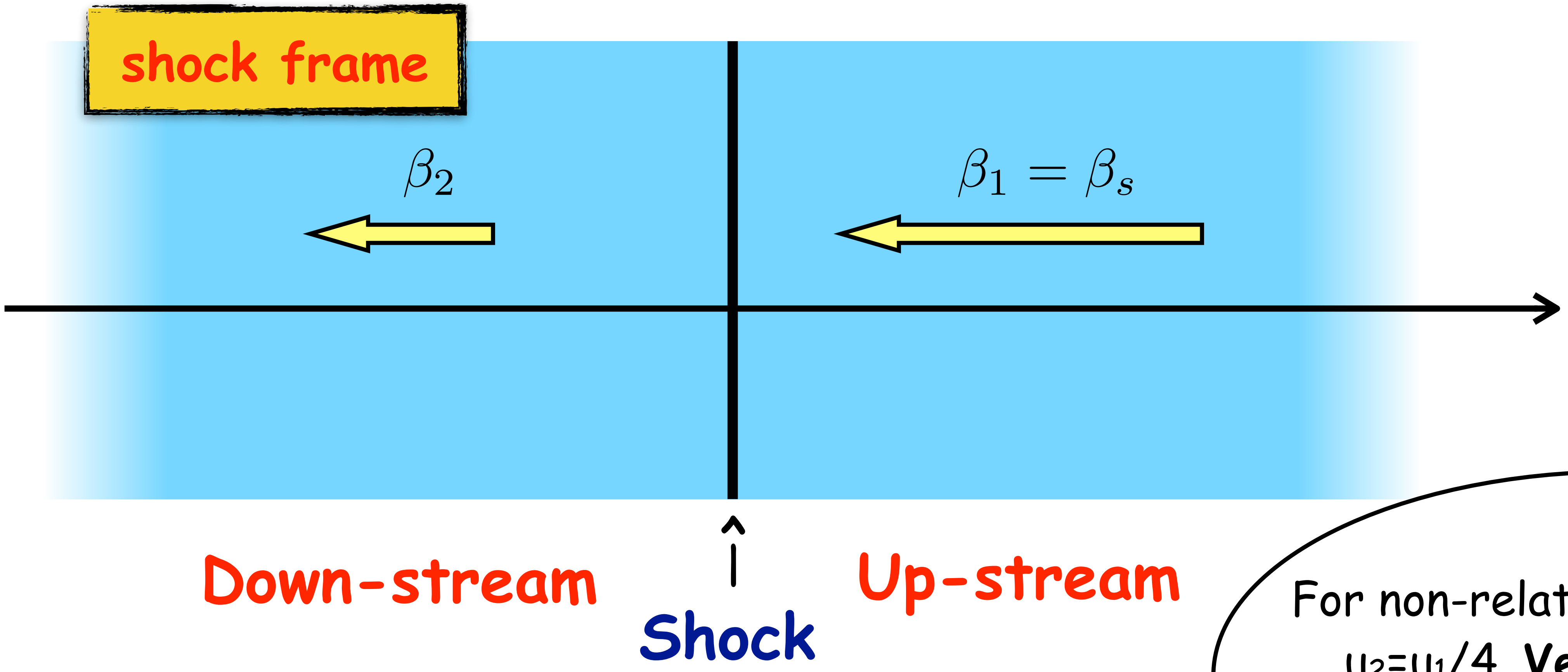

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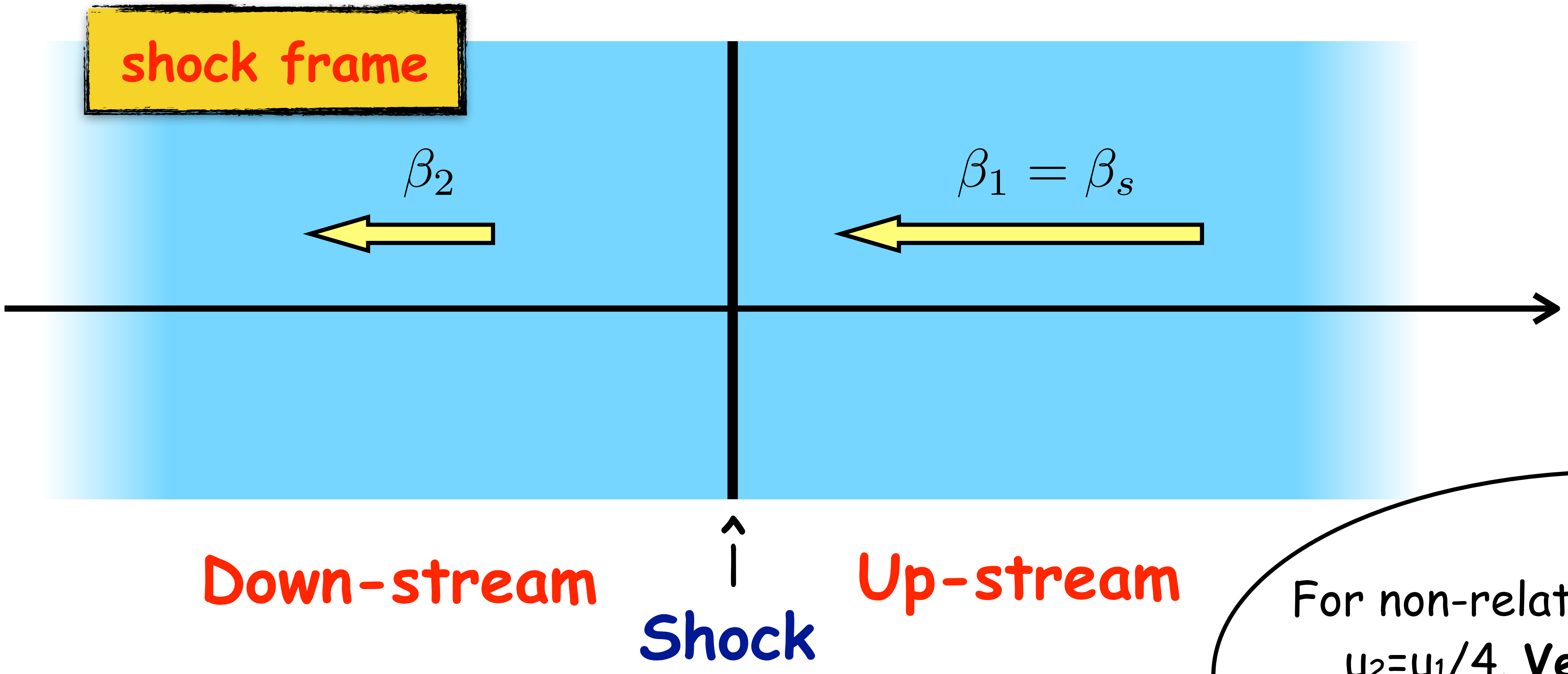
For non-relativistic shocks it is $u_2=u_1/4$. **Very naive** guess: spectra are in this case steeper than E^{-2}?

Ultra-relativistic shocks



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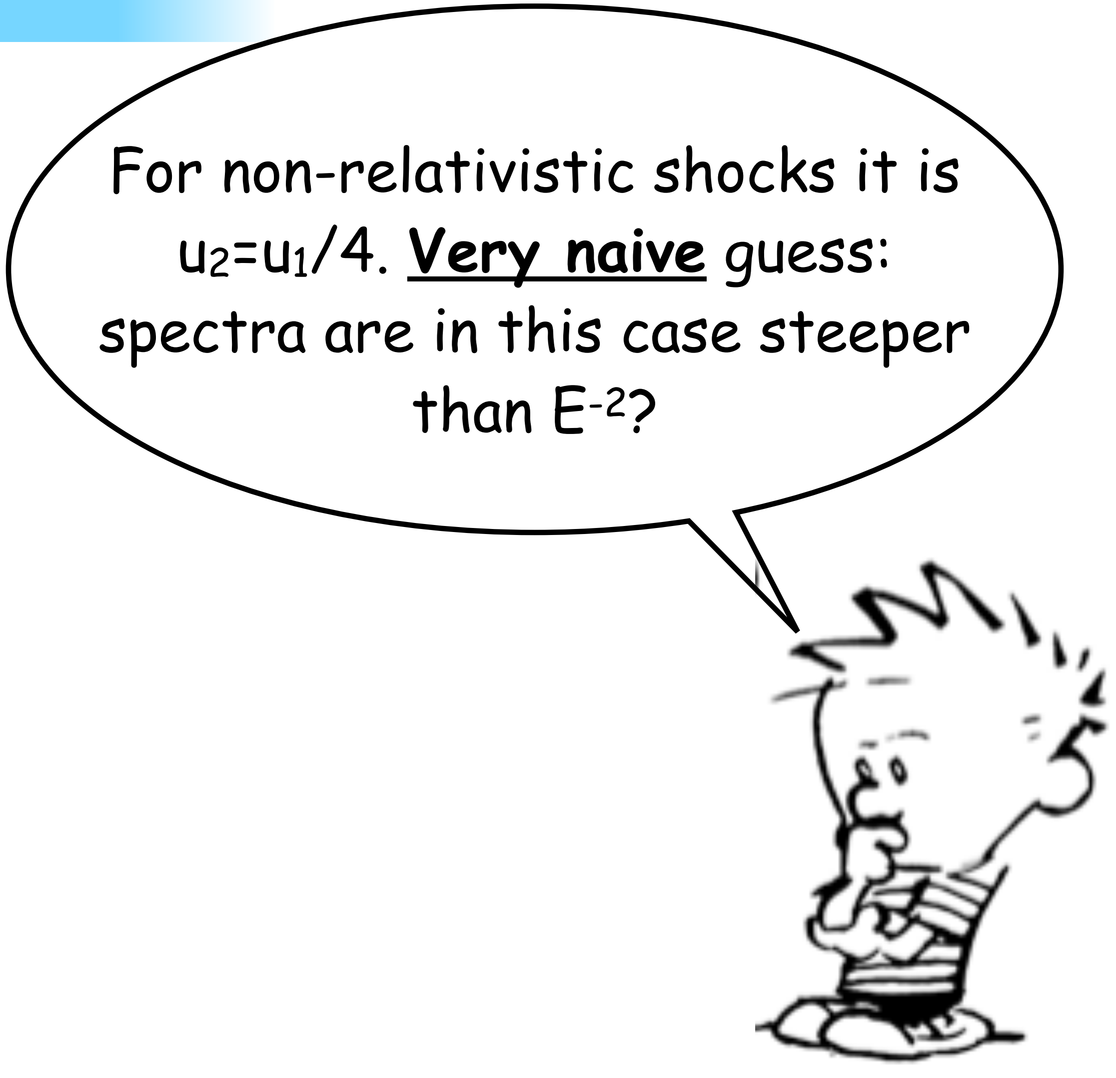
relative speed

$$\beta_{rel} = \frac{\beta_1 - \beta_2}{1 - \beta_1 \beta_2}$$



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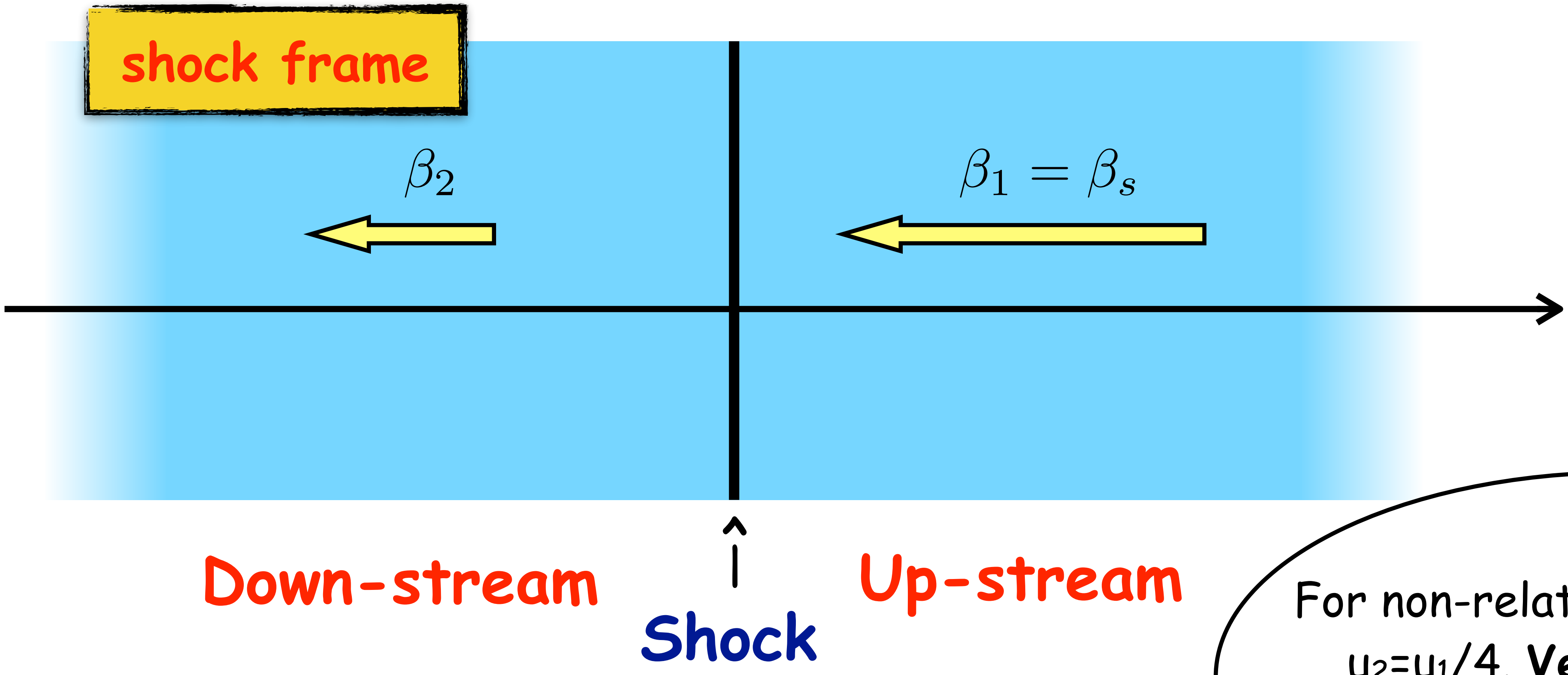

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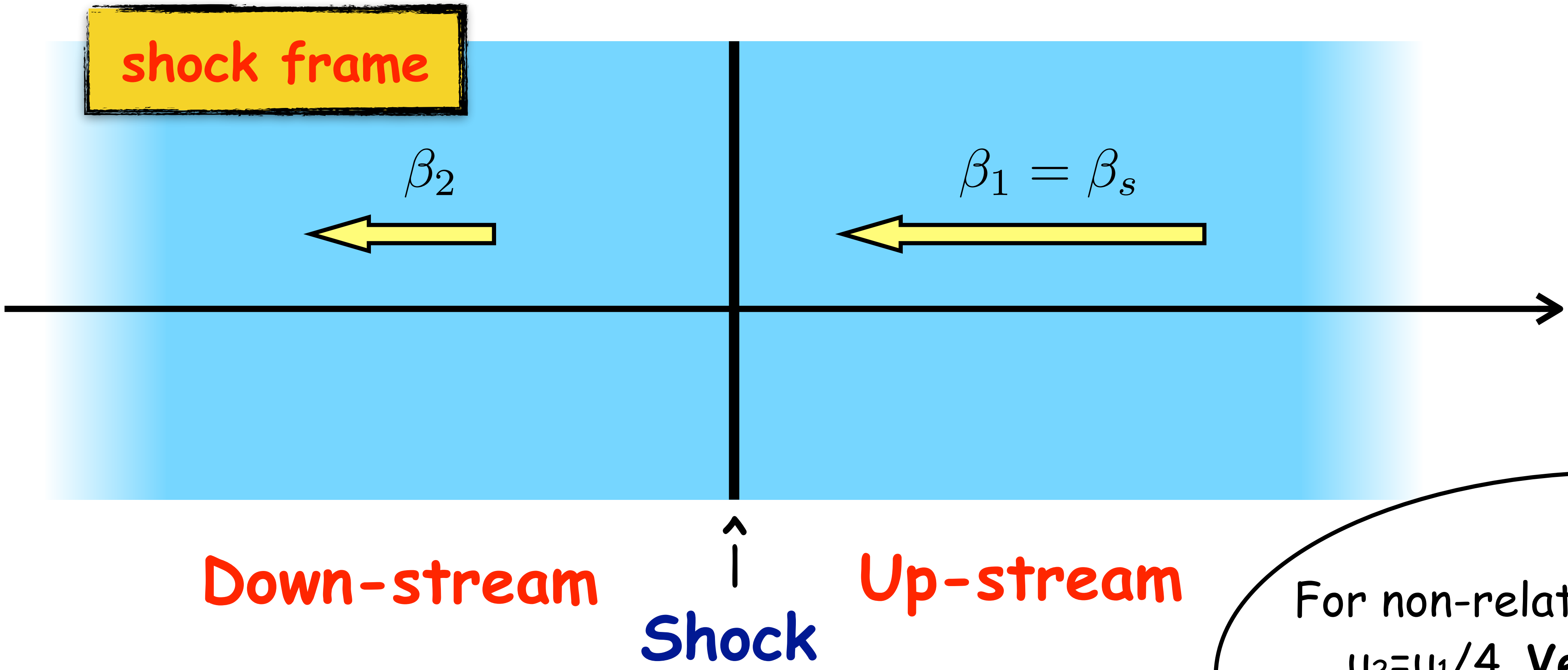

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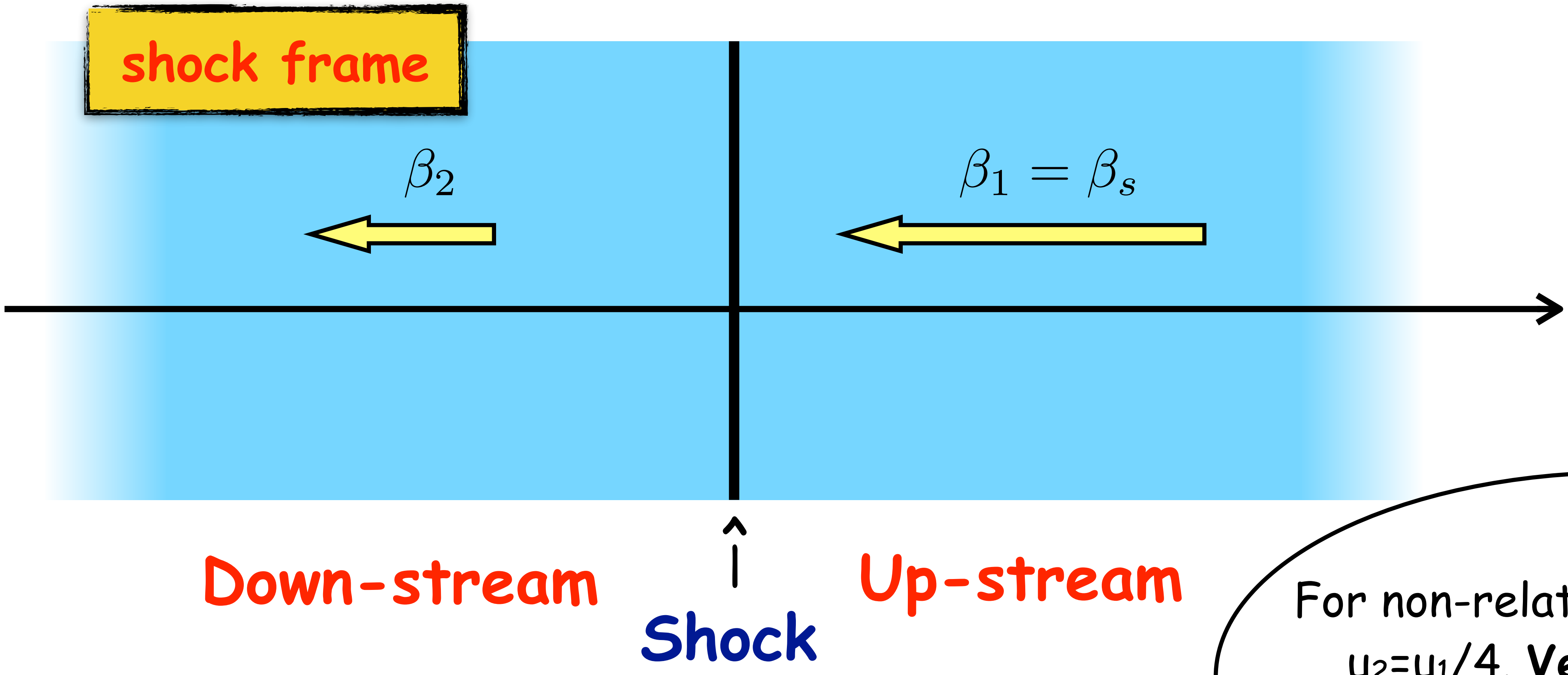

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$$= \frac{\left(1 - \frac{1}{\Gamma_s^2}\right)^{1/2}}{\left(1 + \frac{1}{2\Gamma_s^2}\right)} \approx \left(1 - \frac{1}{2\Gamma_s^2}\right)\left(1 - \frac{1}{2\Gamma_s^2}\right) \approx 1 - \frac{1}{\Gamma_s^2}$$

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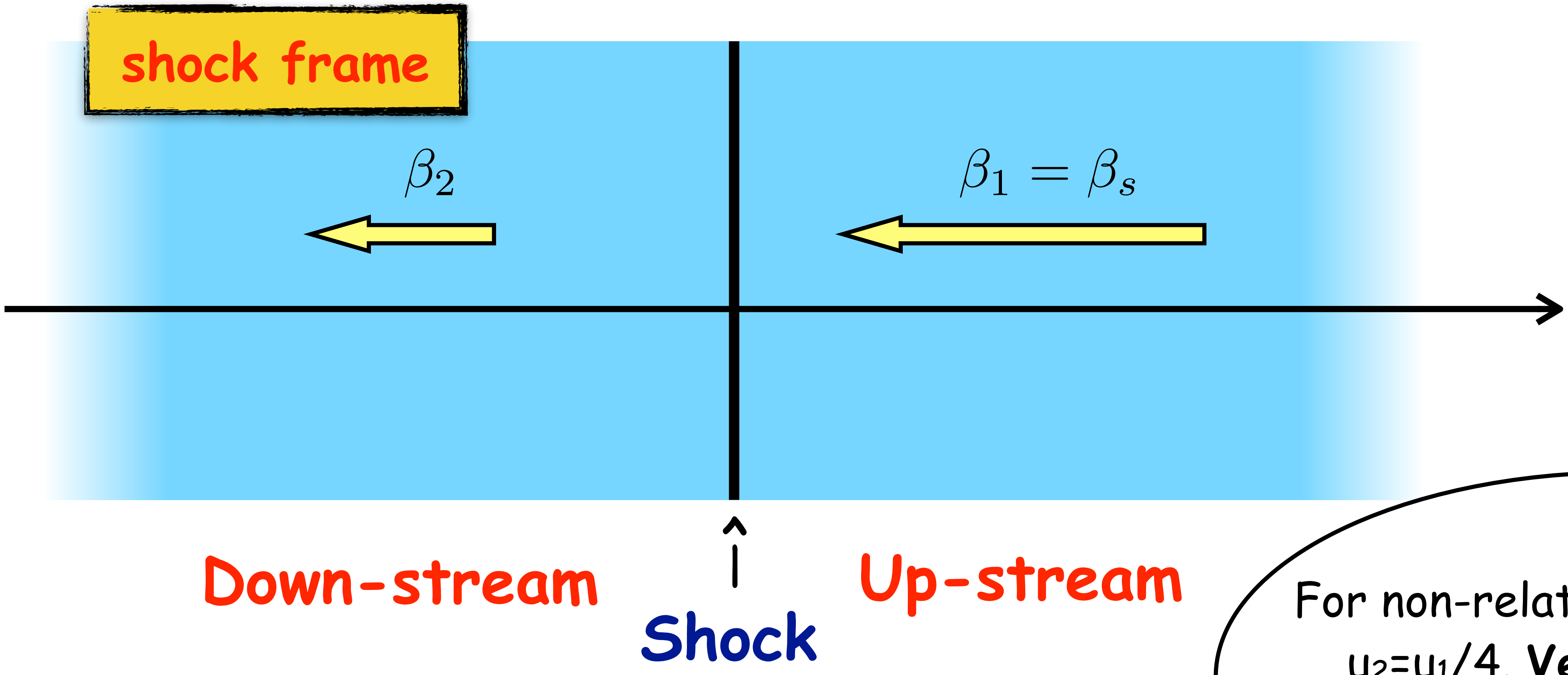
Ultra-relativistic shocks



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Ultra-relativistic shocks: summary

shock velocity

$$\beta_s \approx 1 - \frac{1}{2\Gamma_s^2}$$

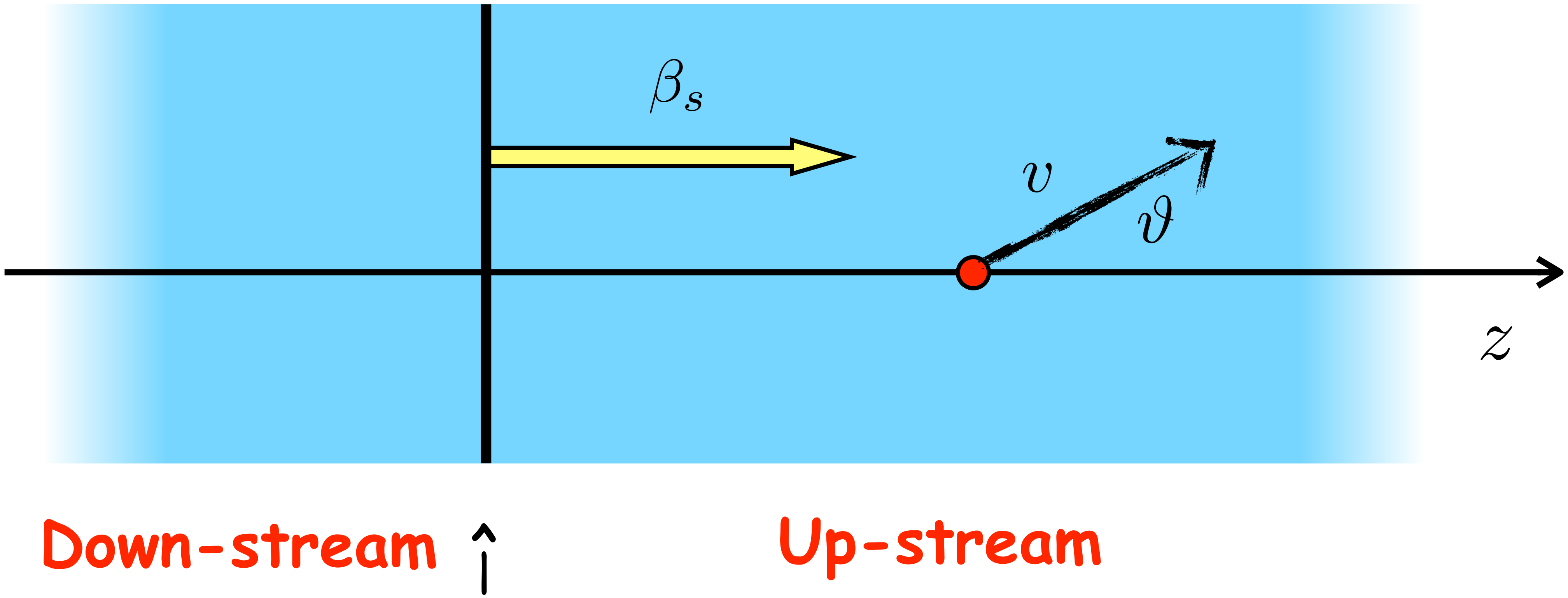
relative speed

$$\beta_{rel} \approx 1 - \frac{1}{\Gamma_s^2}$$

jump condition

$$\beta_2 = \frac{\beta_1}{3}$$

Loss cone



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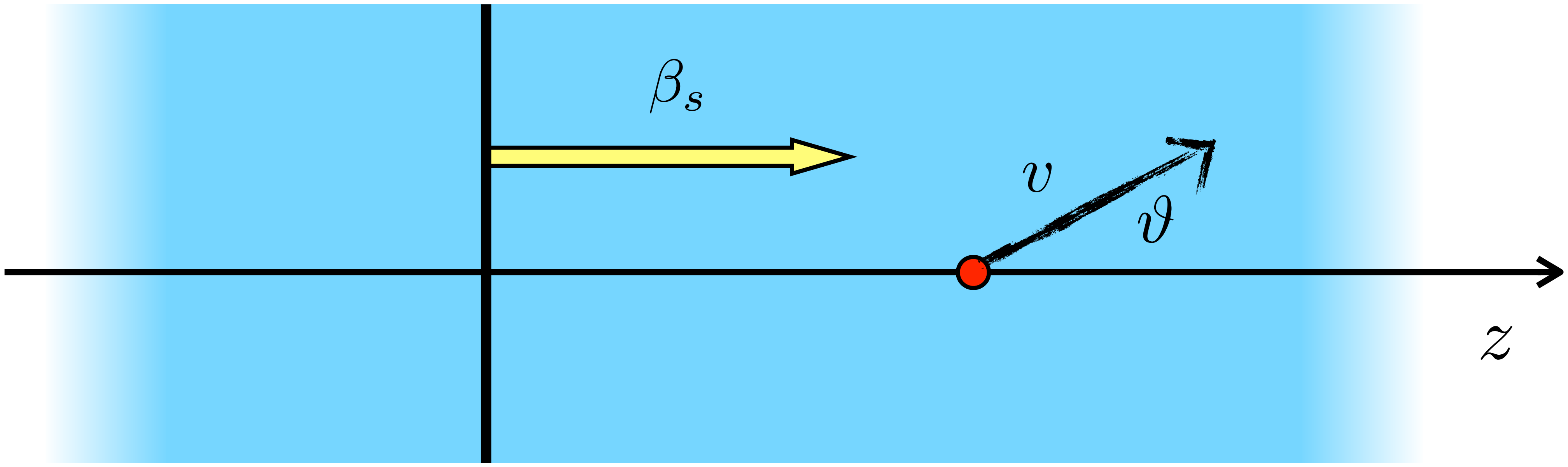
Shock

Loss cone



Down-stream ↑ Up-stream

Shock

will the shock catch the particle?

$$\beta_s > v_z \sim \cos\vartheta$$

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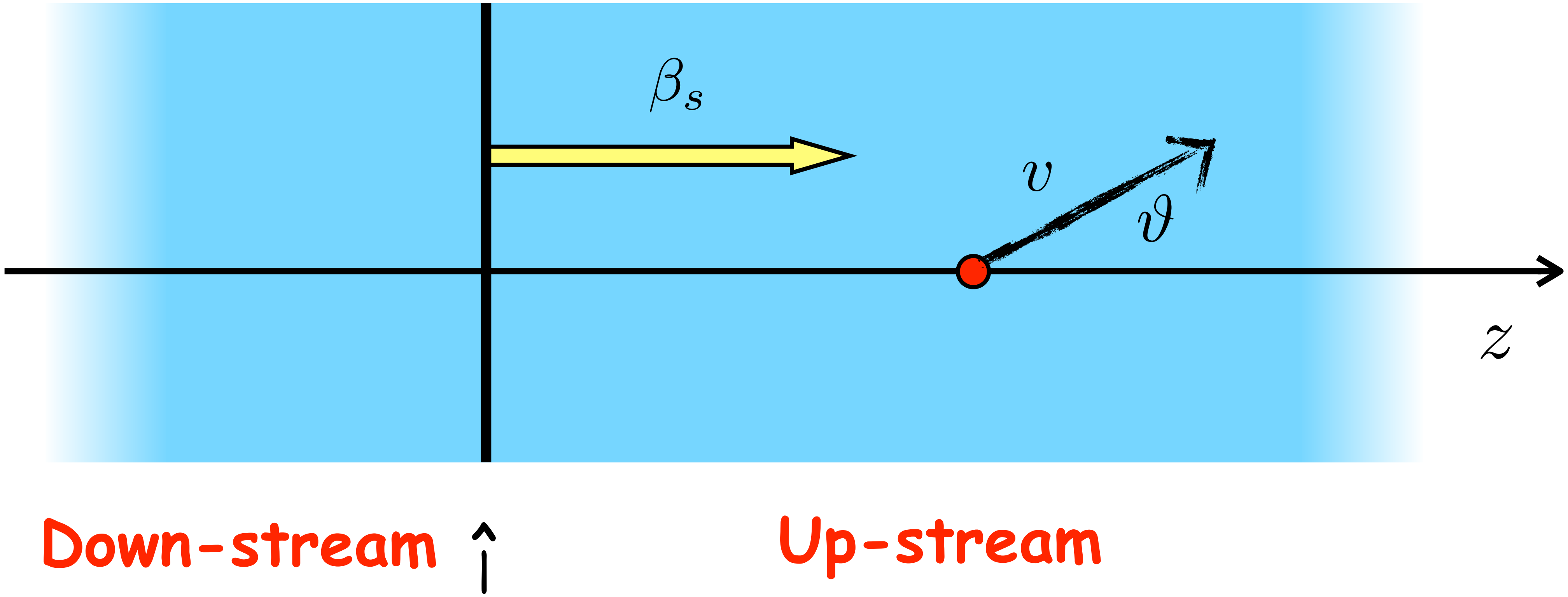

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Up-stream

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$$1 - \frac{1}{2\Gamma_s^2} \sim \beta_s > v_z \sim \cos\vartheta$$

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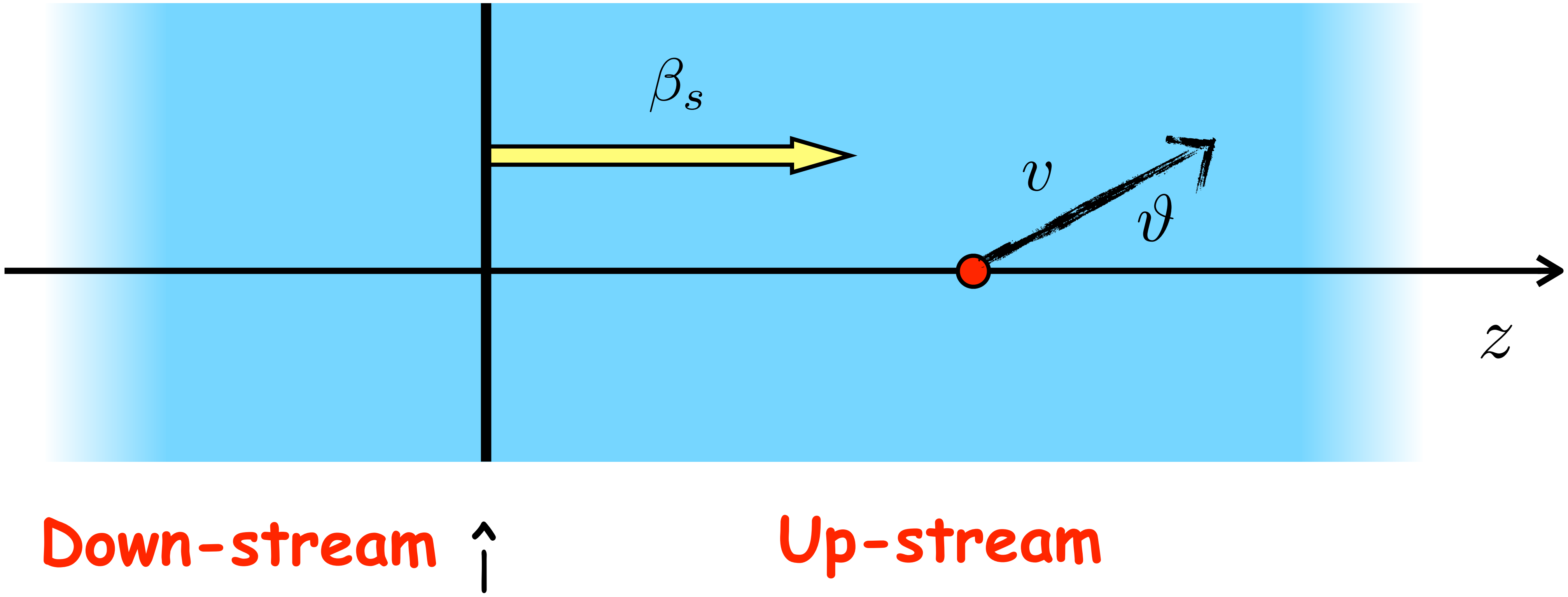

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Up-stream

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$$1 - \frac{1}{2\Gamma_s^2} \sim \beta_s > v_z \sim \cos\vartheta \longrightarrow \frac{1}{2\Gamma_s^2} < 1 - \cos\vartheta \sim \frac{\vartheta^2}{2}$$

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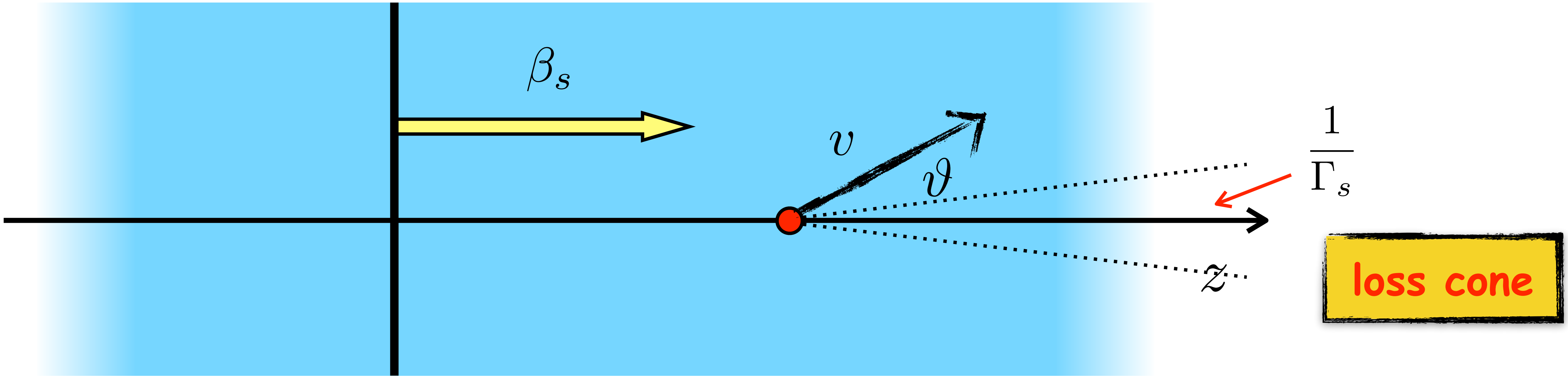

Down-stream ↑ **Up-stream**

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$$\boxed{\vartheta>\frac{1}{\Gamma_s}}$$

Anisotropy

Consequences:

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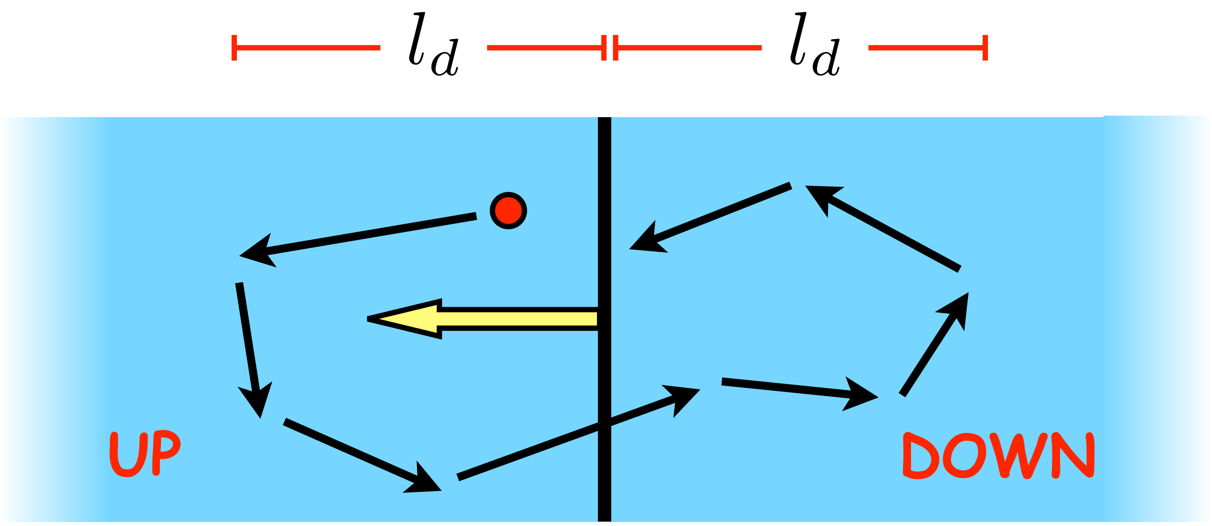
The particle distribution function upstream of the shock is very anisotropic —> the transport is NOT diffusive

it can be demonstrate that also downstream particles are highly anisotropic

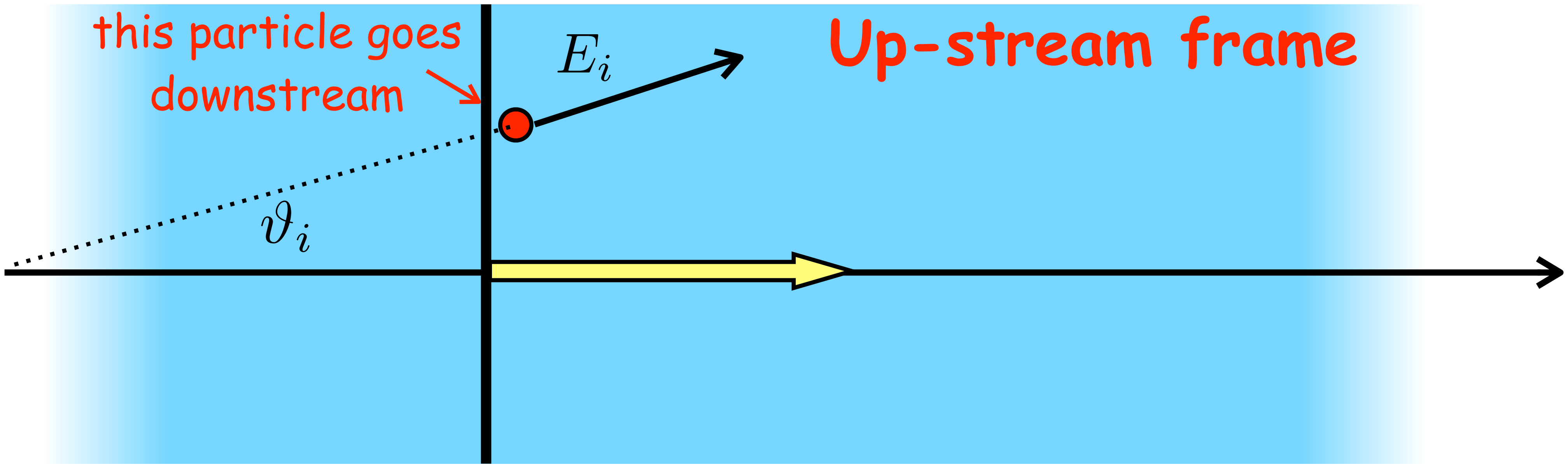
Key aspects of DSA
at ~~non~~-relativistic shocks

particles are accelerated through a series of cycles up—>down—>up stream



- non- diffusive transport —> ~~isotropy~~

Energy gain in a cycle



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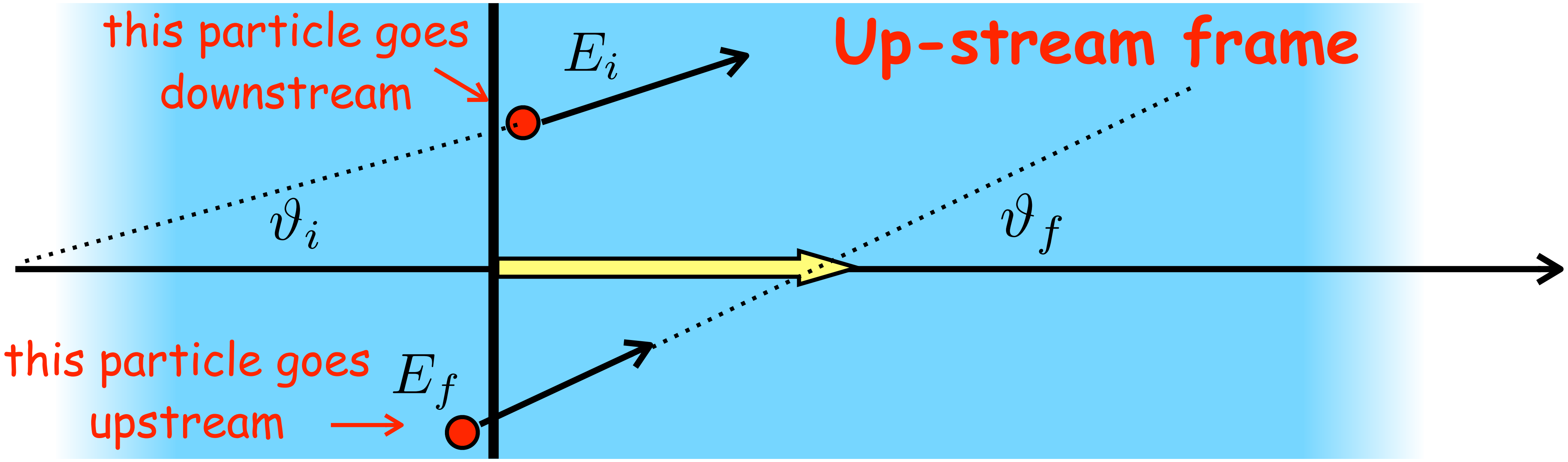

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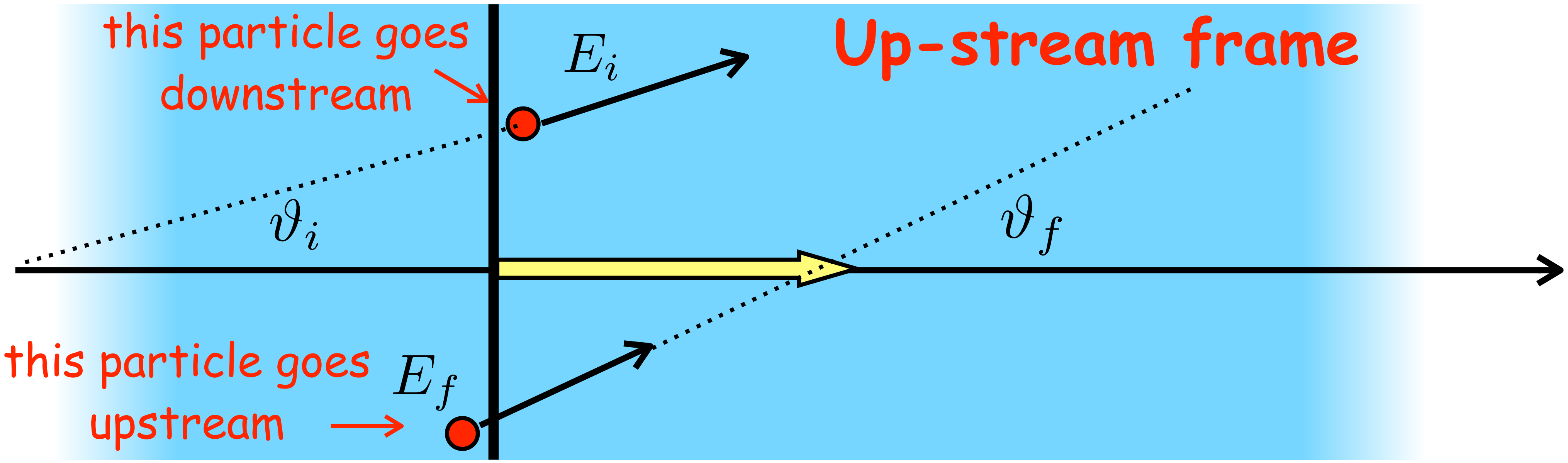
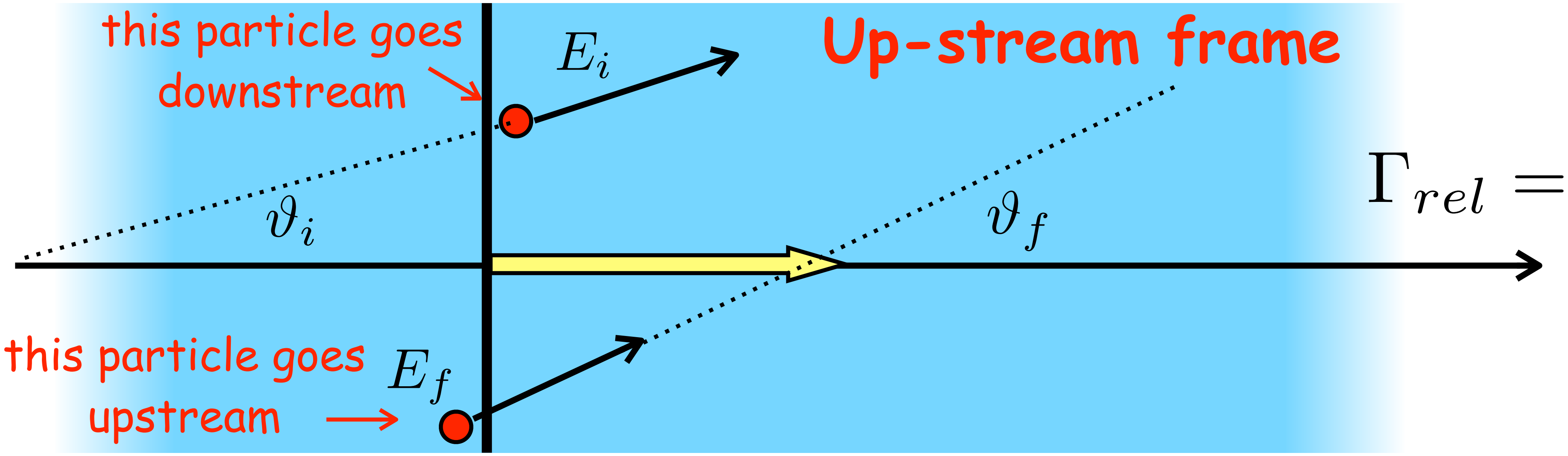


primed quantities are in the downstream frame

$$E_i' = E_f'$$

Energy gain in a cycle



$$\Gamma_{rel} = (1 - \beta_{rel}^2)^{-1/2}$$

primed quantities are in the downstream frame

$$\Gamma_{rel}\left(E_i - \beta_{rel} p_{z,i}\right) = E_i' = E_f' = \Gamma_{rel}\left(E_f - \beta_{rel} p_{z,f}\right)$$

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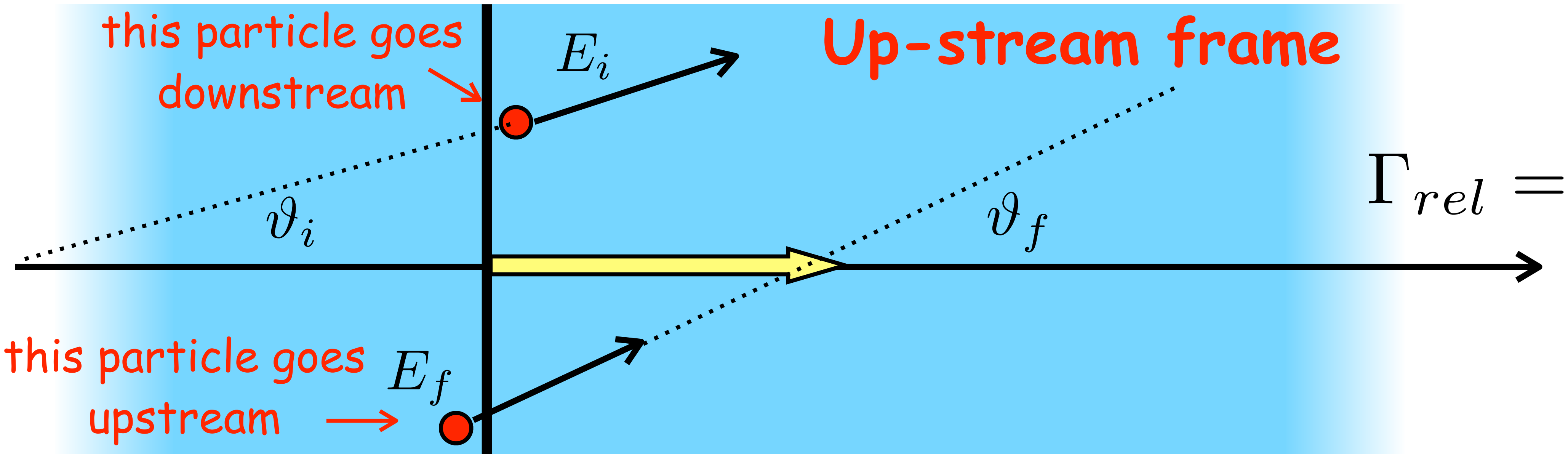

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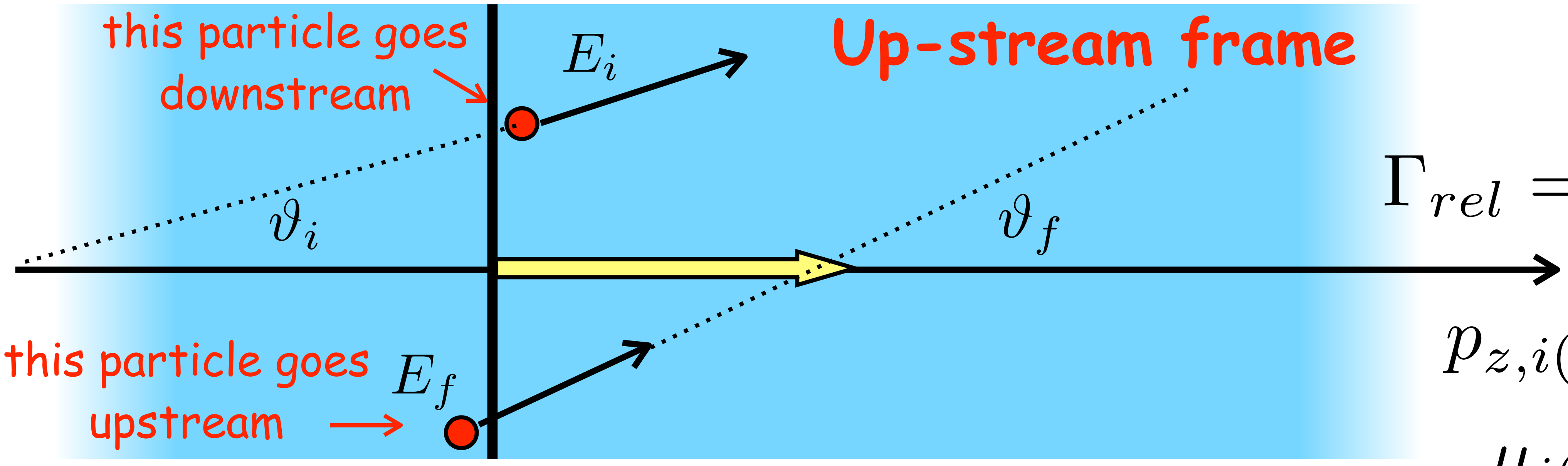
Energy gain in a cycle



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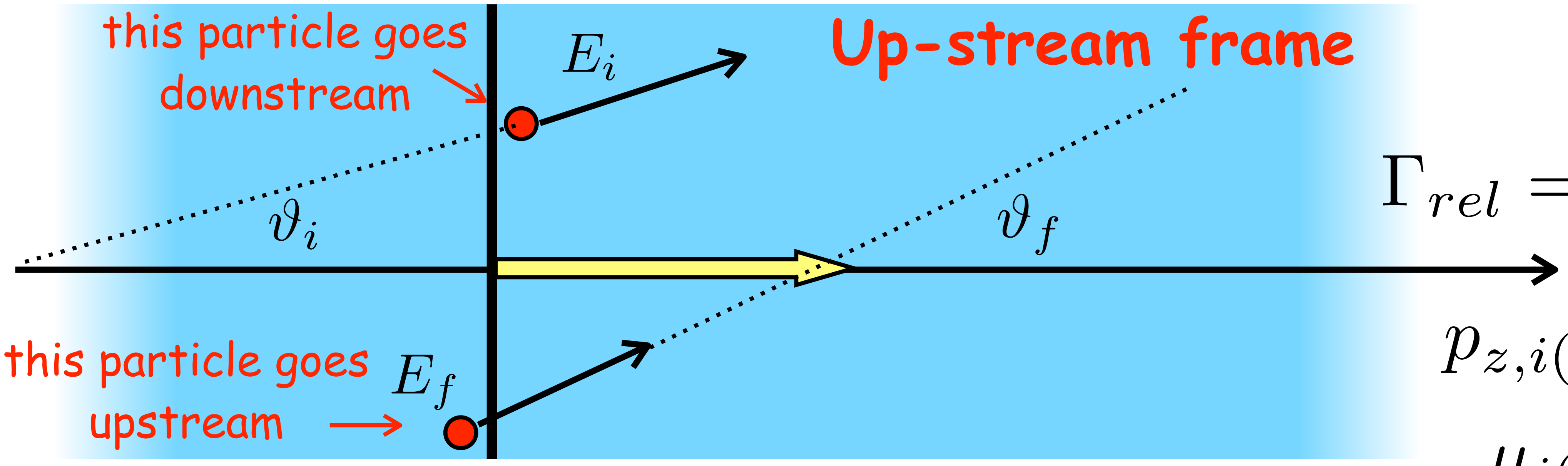

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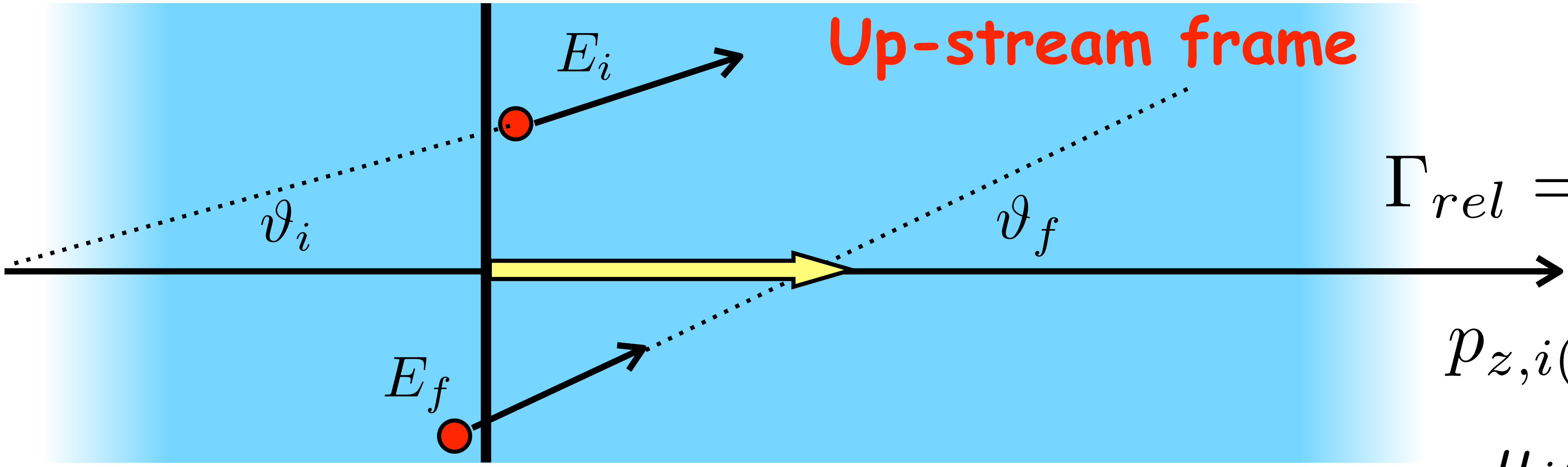

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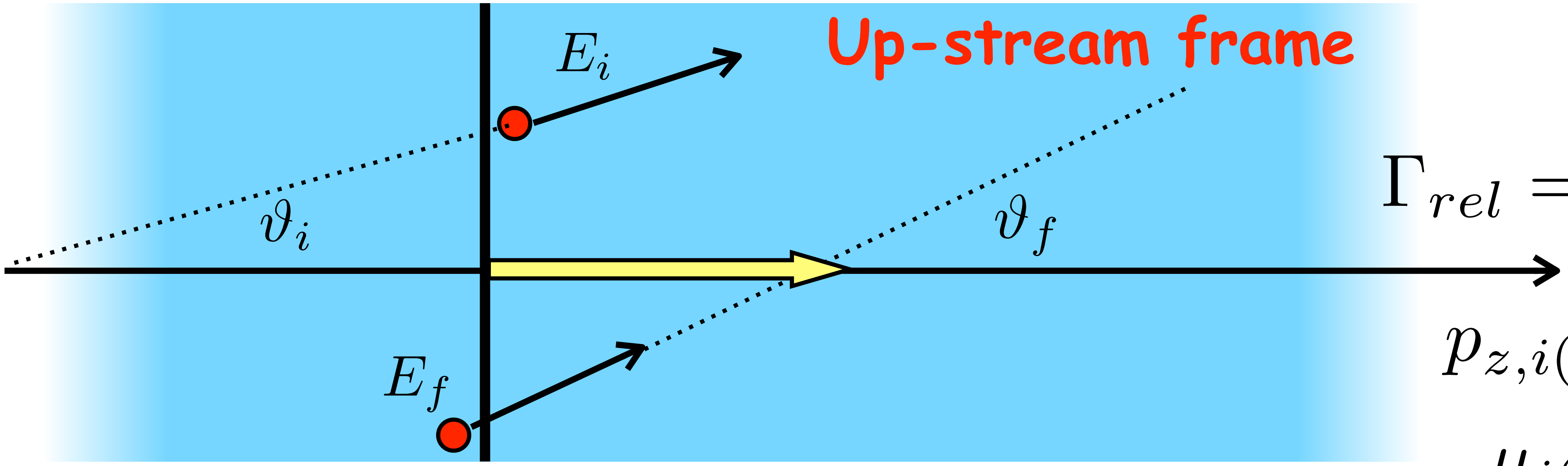

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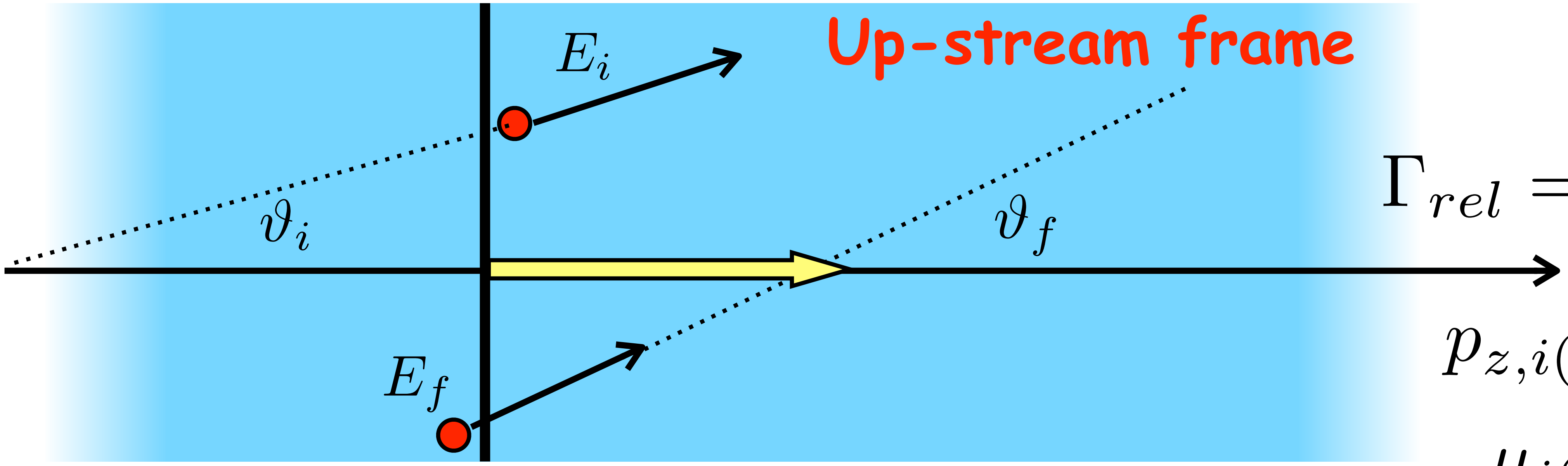

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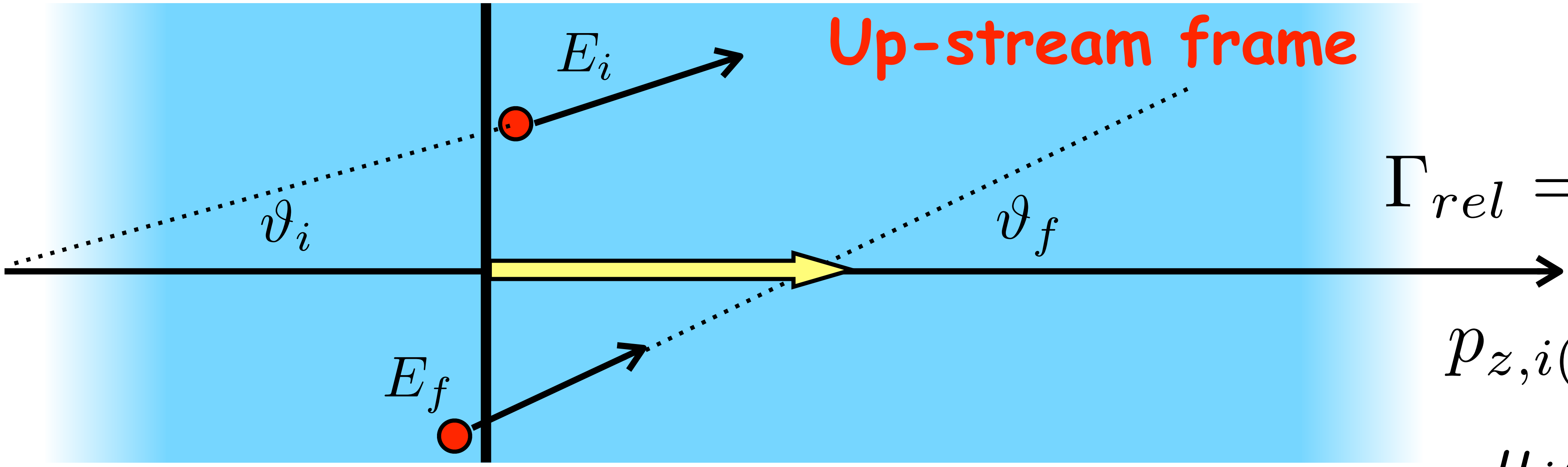

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up —> down

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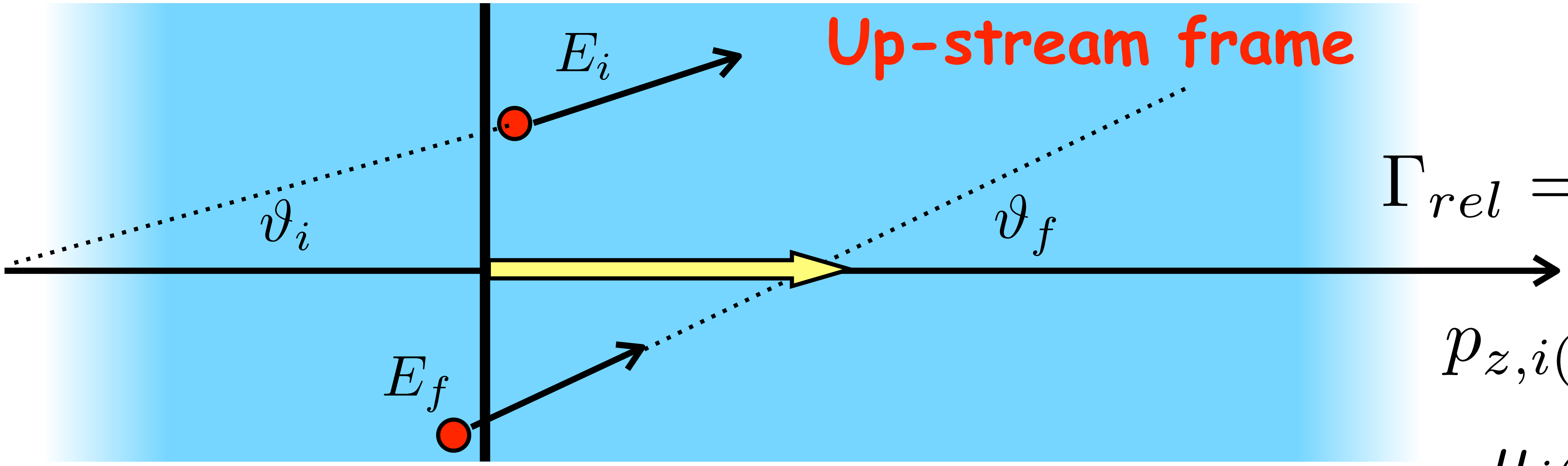

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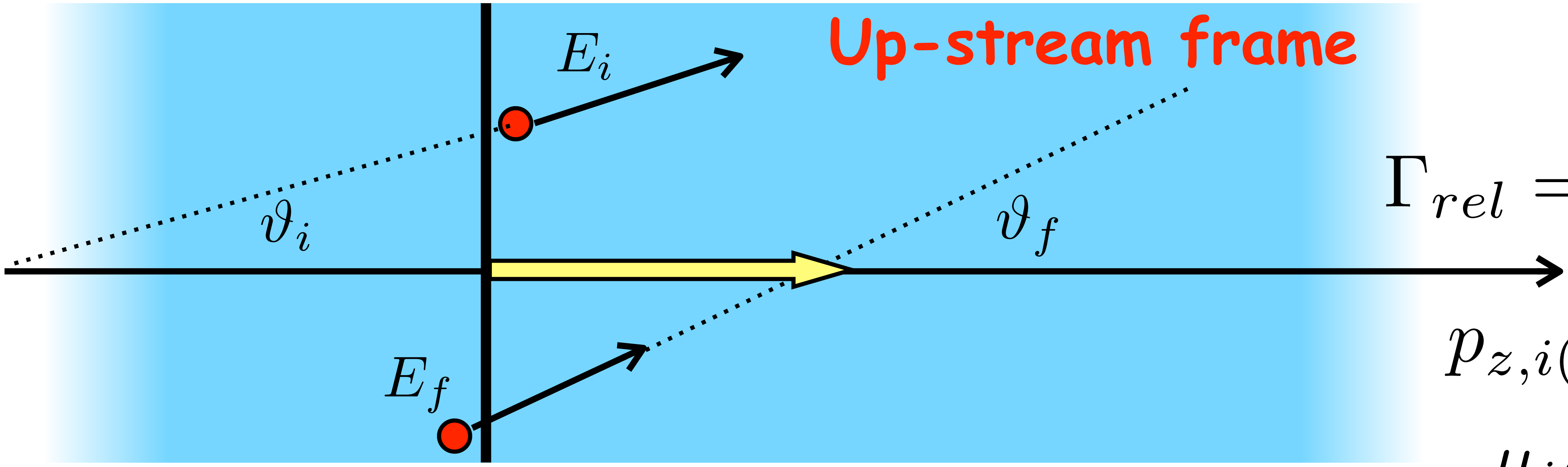

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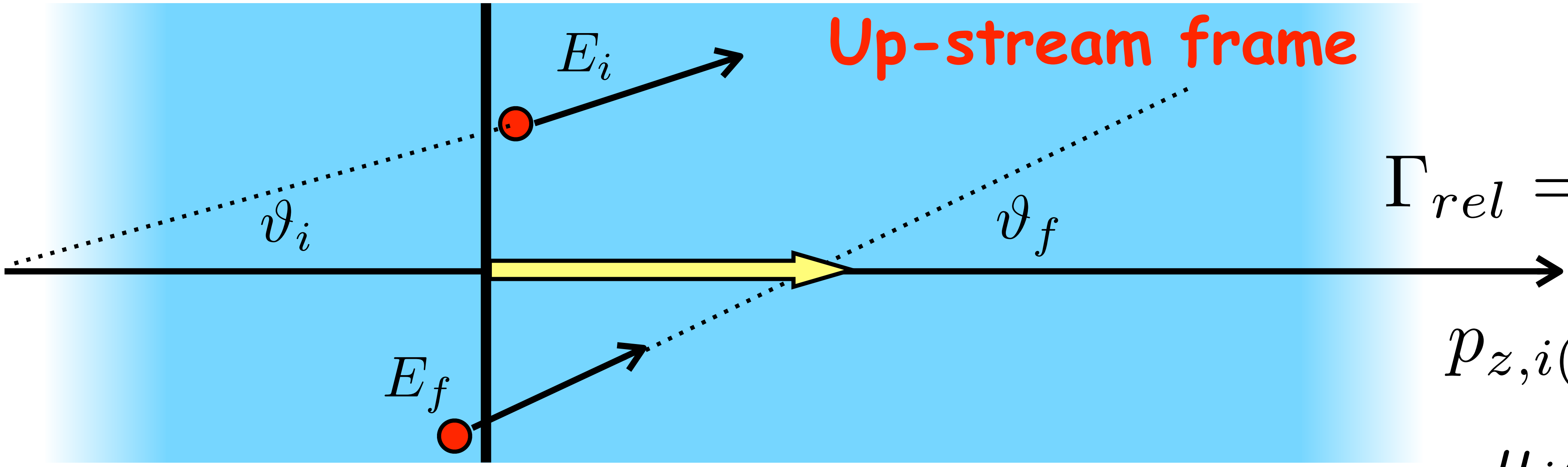

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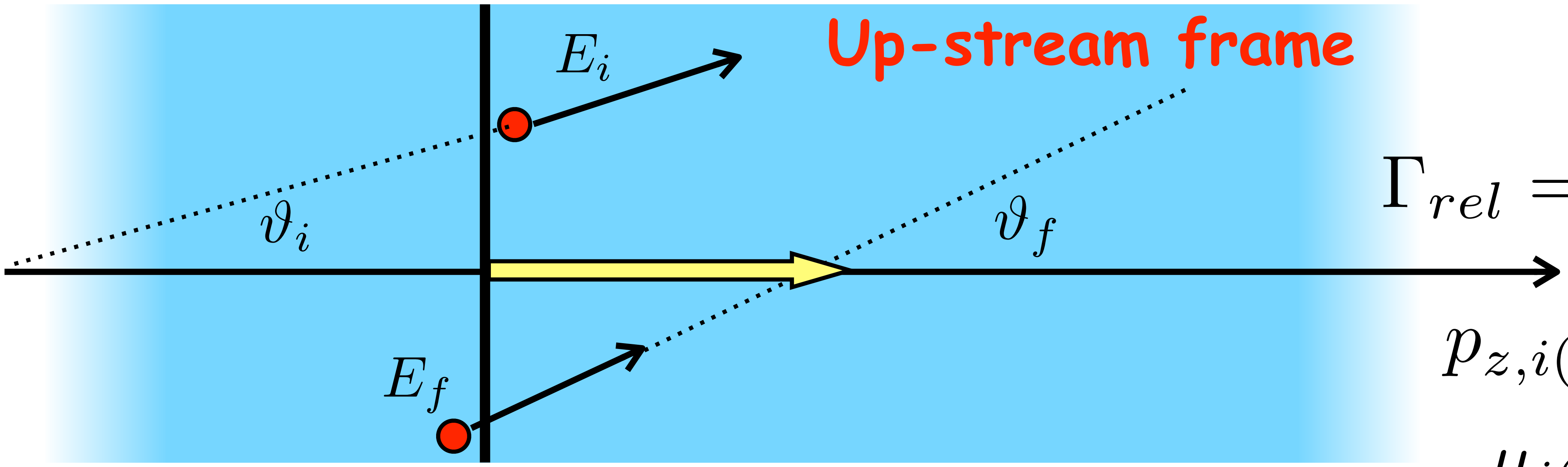

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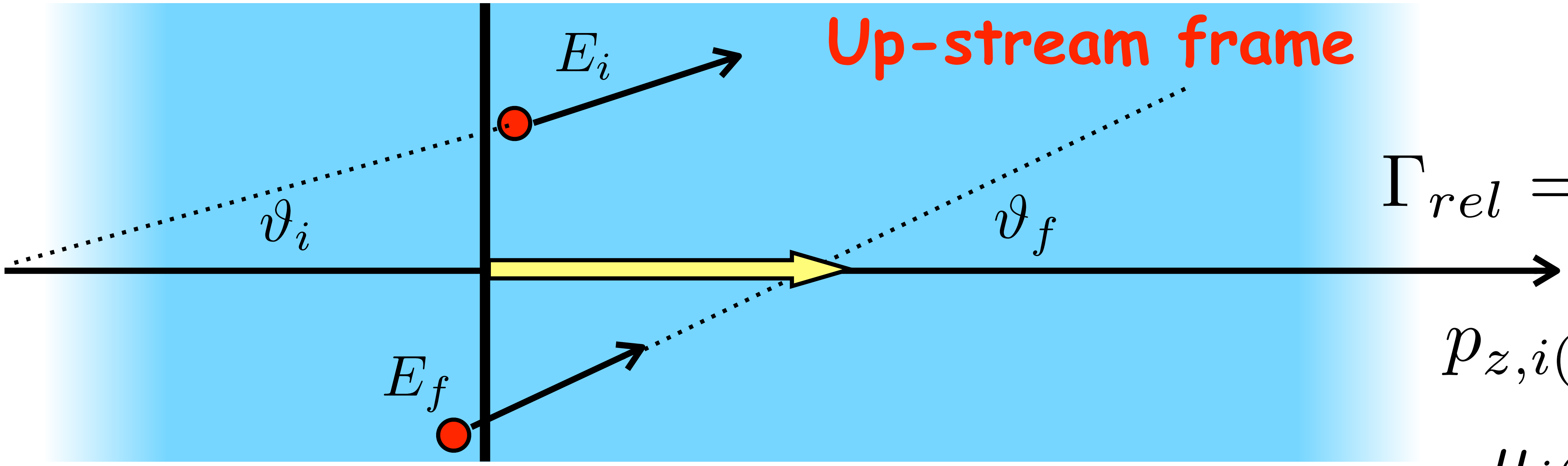
as for non-relativistic shocks, particles always gain energy, but now the gain can be large!

Energy gain in a cycle



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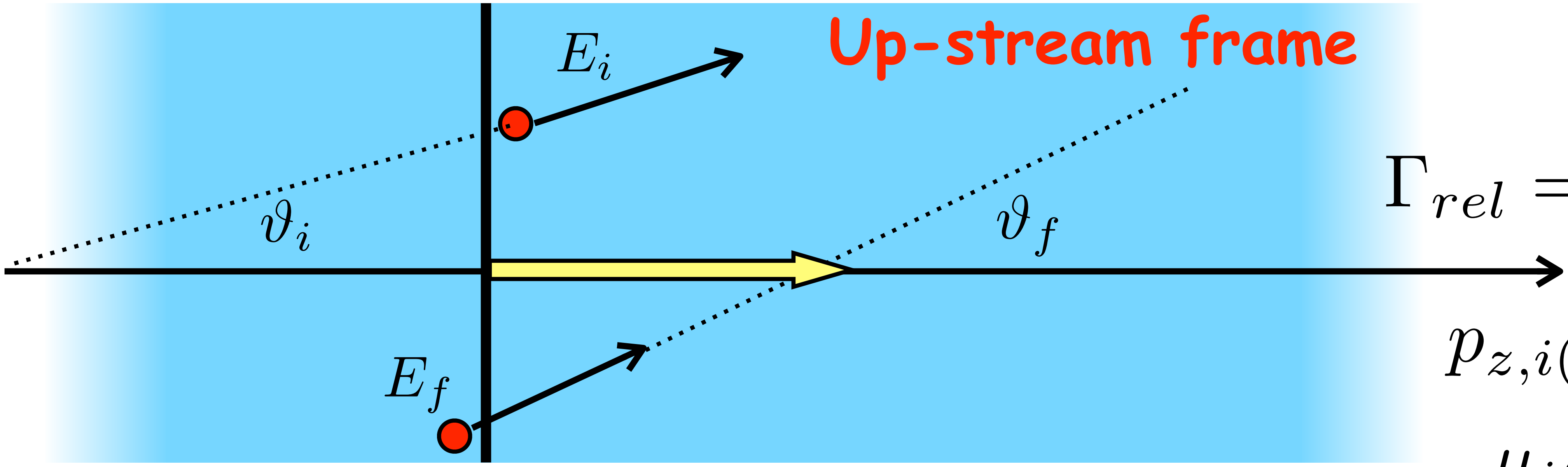

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it can be shown that:

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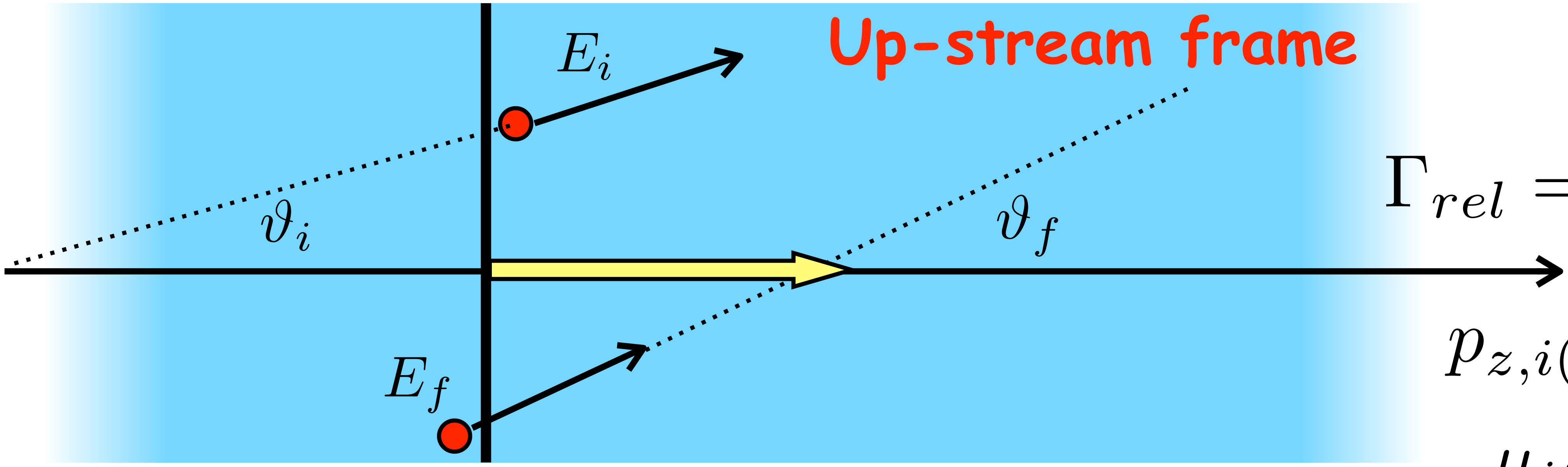
Energy gain in a cycle



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the energy gain is large!

First shock crossing: the initial boost

So far, we always considered particles entering the upstream region crossing the shock from the downstream one —> particle velocity upstream is very close to the shock normal (loss cone)

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very large boost of the energy at the first crossing!

Key aspects of DSA
at ~~non~~-relativistic shocks

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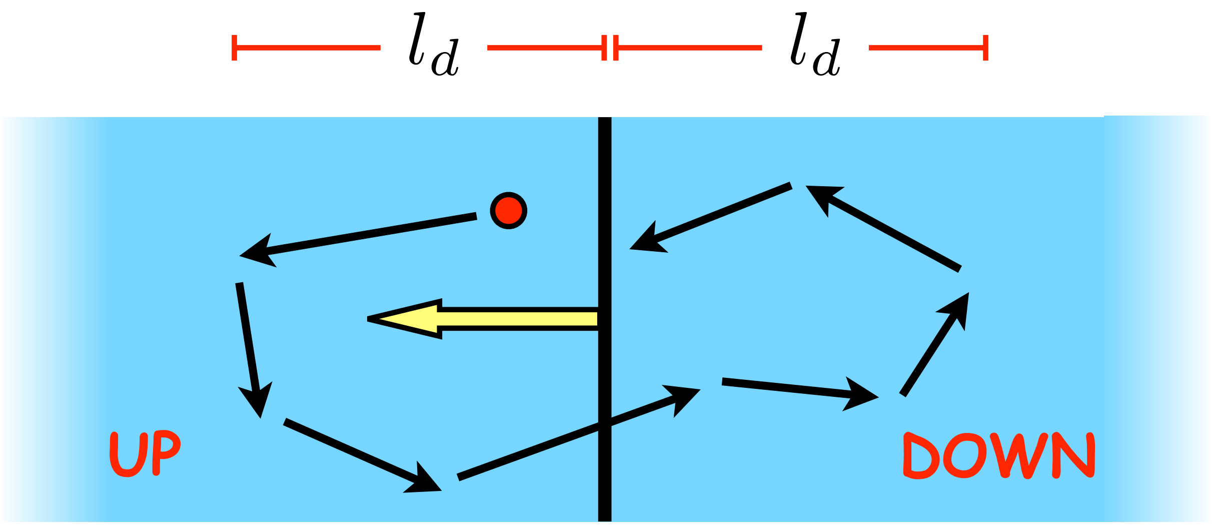
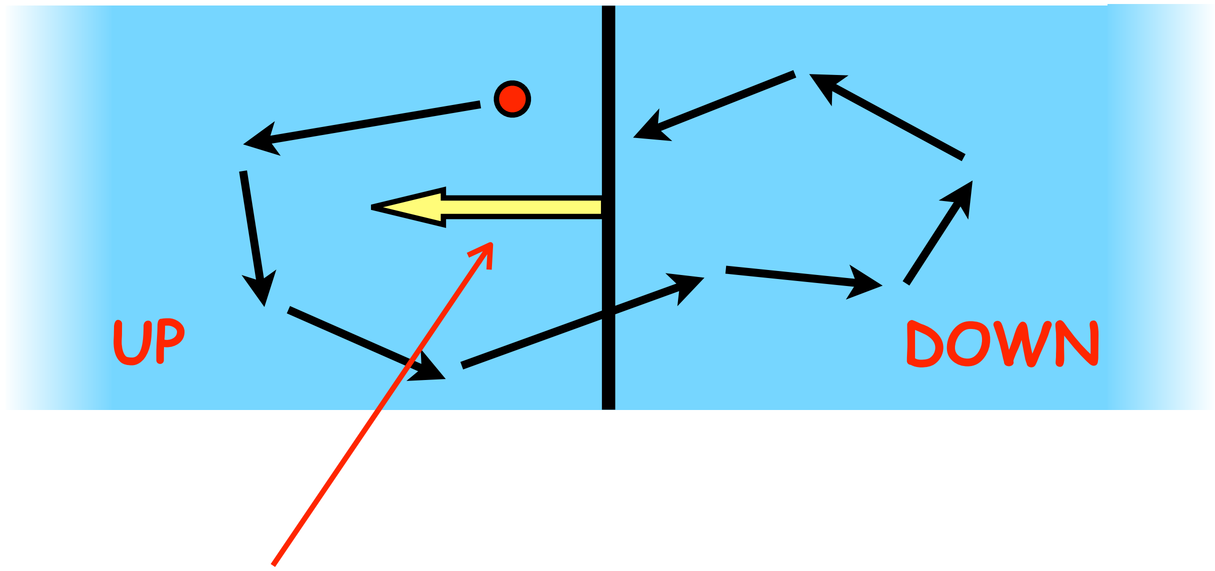


- non-diffusive transport —> ~~isotropy~~
- energy gain per cycle —> ~~small~~ large

$$\Delta_{acc} = \Delta E/E \sim \Gamma_s^2$$ 1st crossing

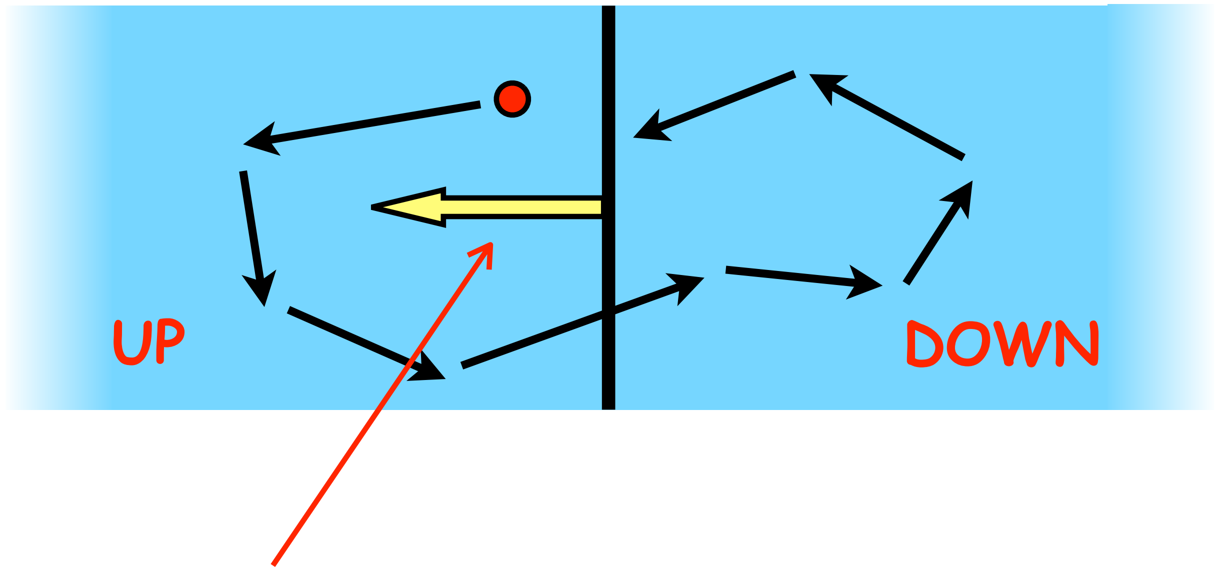
$$\Delta_{acc} = \Delta E/E \sim 1$$ all others

Escape probability per cycle



$$\beta_s' = \beta_2 = \frac{\beta_s}{3}$$

Escape probability per cycle



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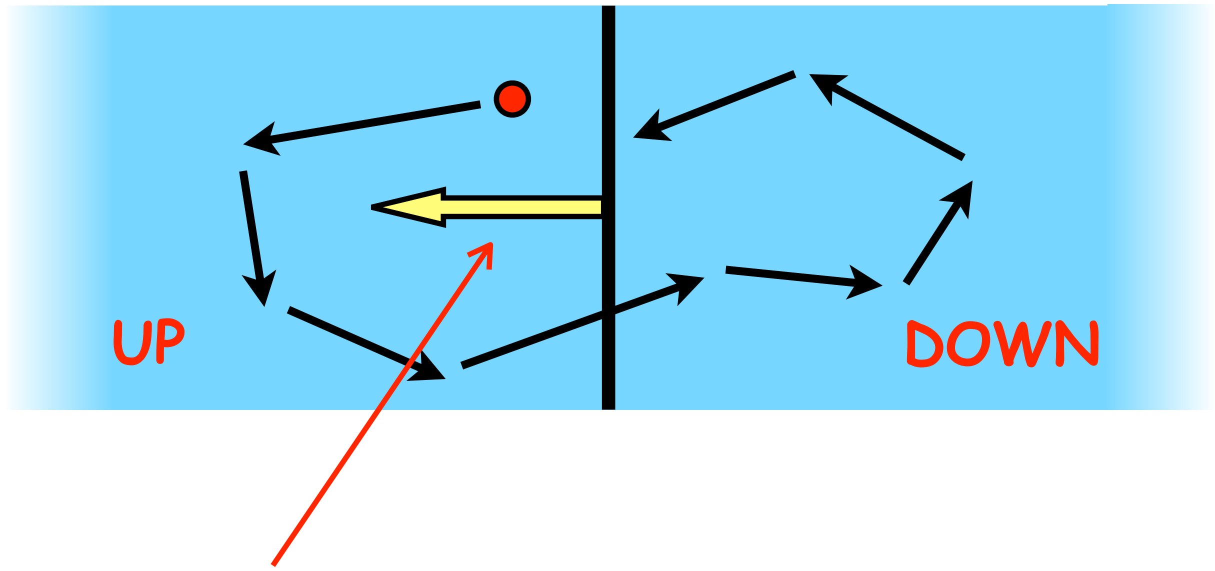
the particle downstream has to invert its motion (strong scattering required) and has to catch the shock that moves away at c/3

Escape probability per cycle



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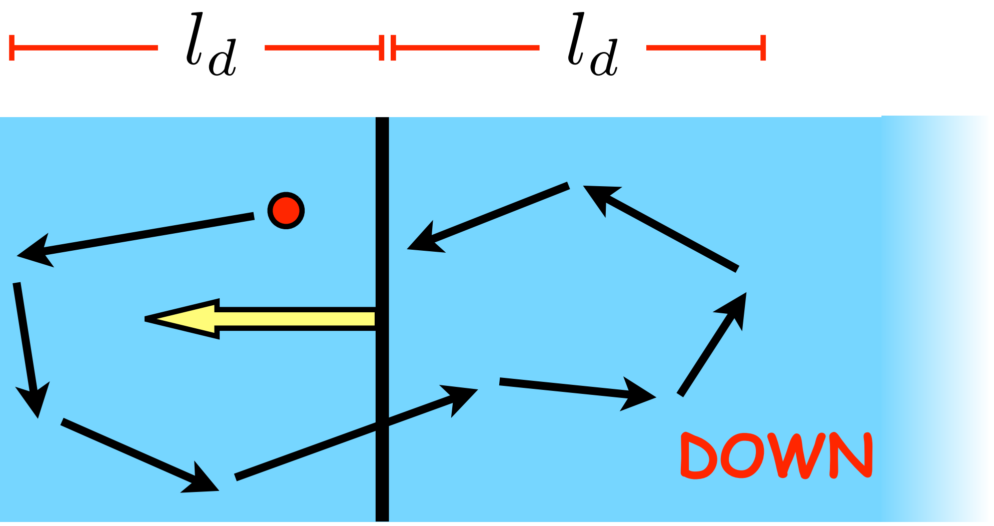

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$$\Delta_{acc} = \Delta E/E \sim 1$$ all others

$$P_{esc} \approx 0.5$$

Spectrum of accelerated particles

small

non-relativistic shocks —>

small

Spectrum of accelerated particles

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$$s = 1 + \frac{P_{esc}}{\Delta_{acc}}$$

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general expression —>

$$s = 1 - \frac{\ln P_{ret}}{\ln\left(\frac{E_f}{E_i}\right)}$$

Spectrum of accelerated particles

small

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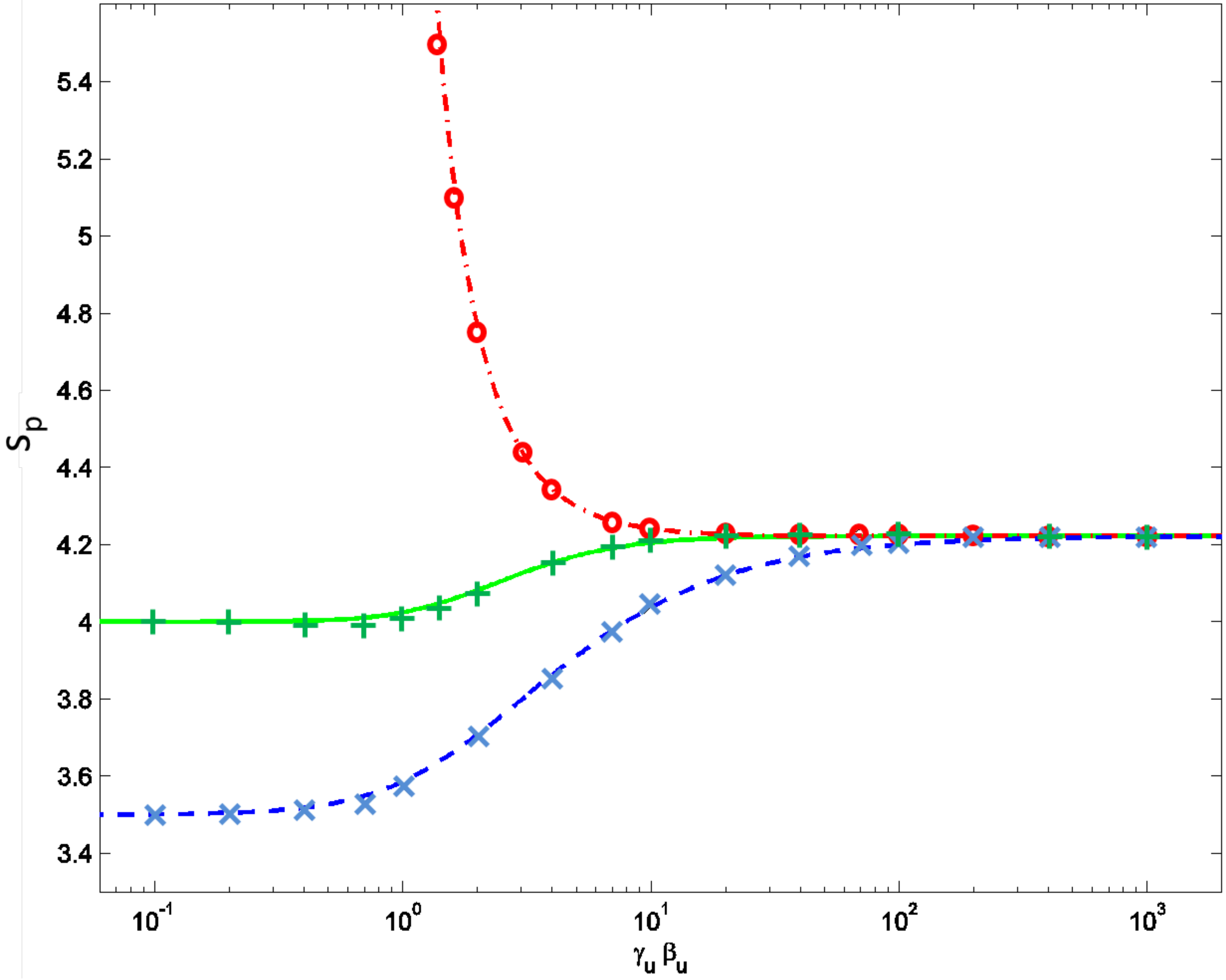
very roughly: s is of the order of 2

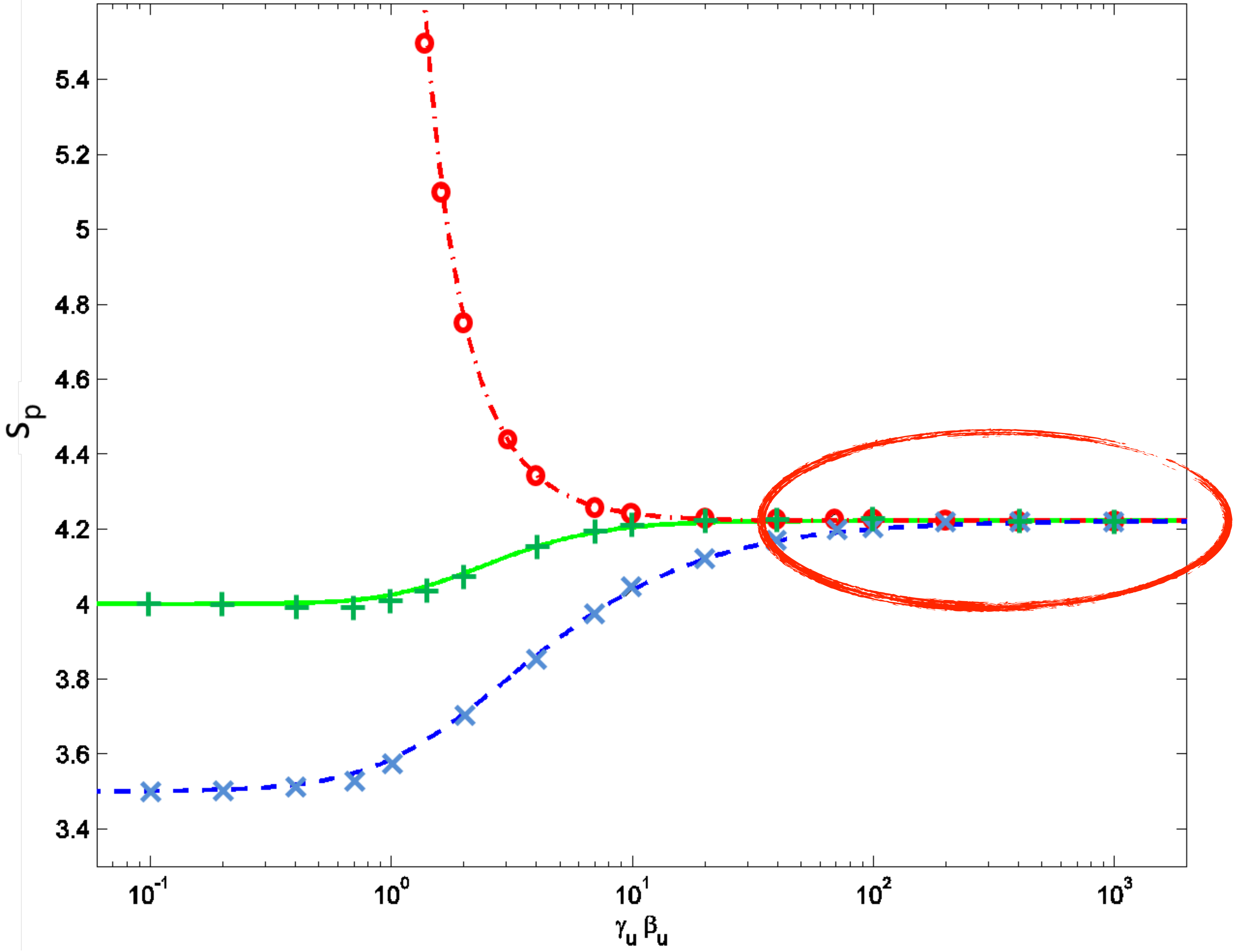
more sophisticated approaches are needed to give a more accurate answer

Spectrum of accelerated particles



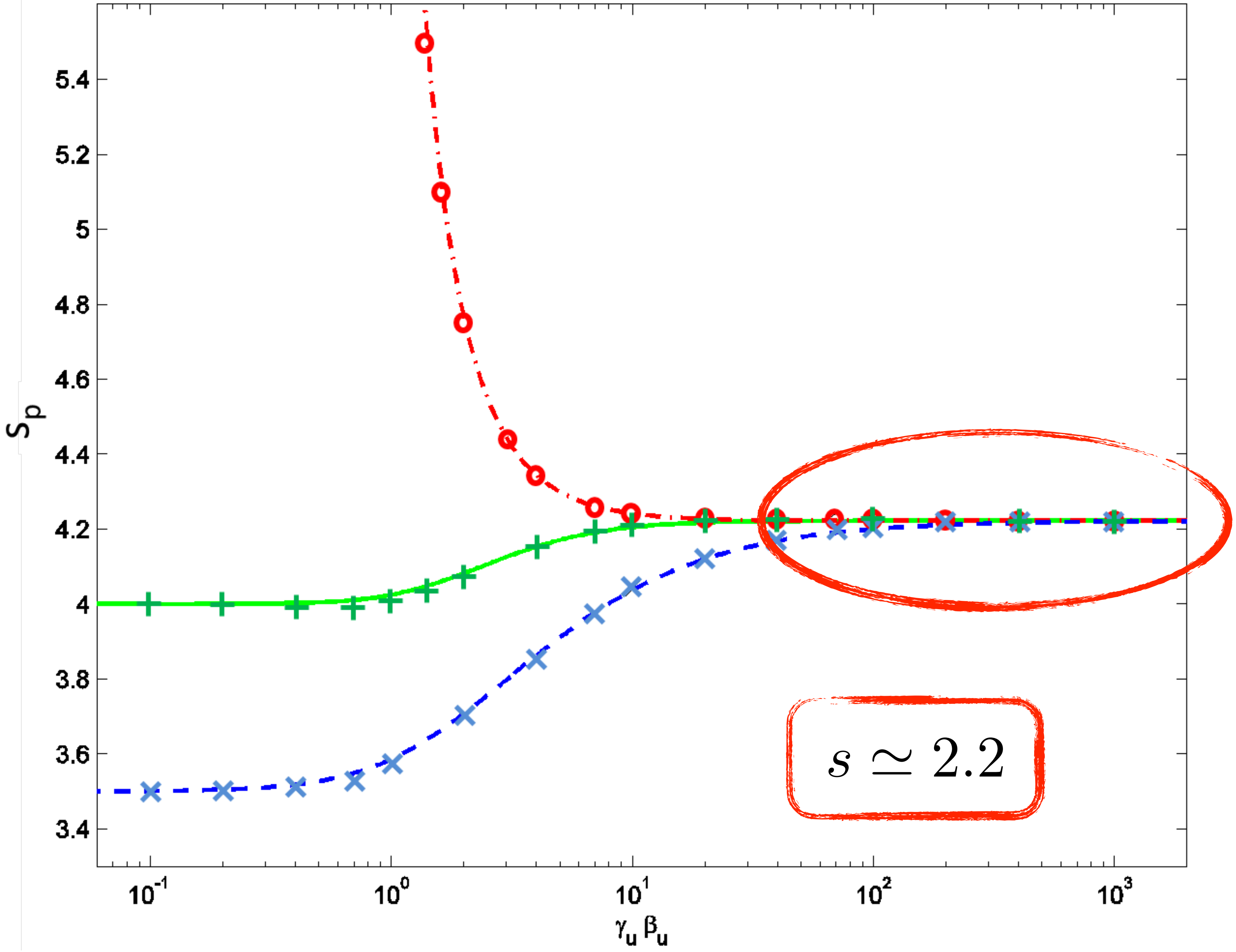
Sironi, Keshet, Lemoine (2015)

Spectrum of accelerated particles



Sironi, Keshet, Lemoine (2015)

Spectrum of accelerated particles

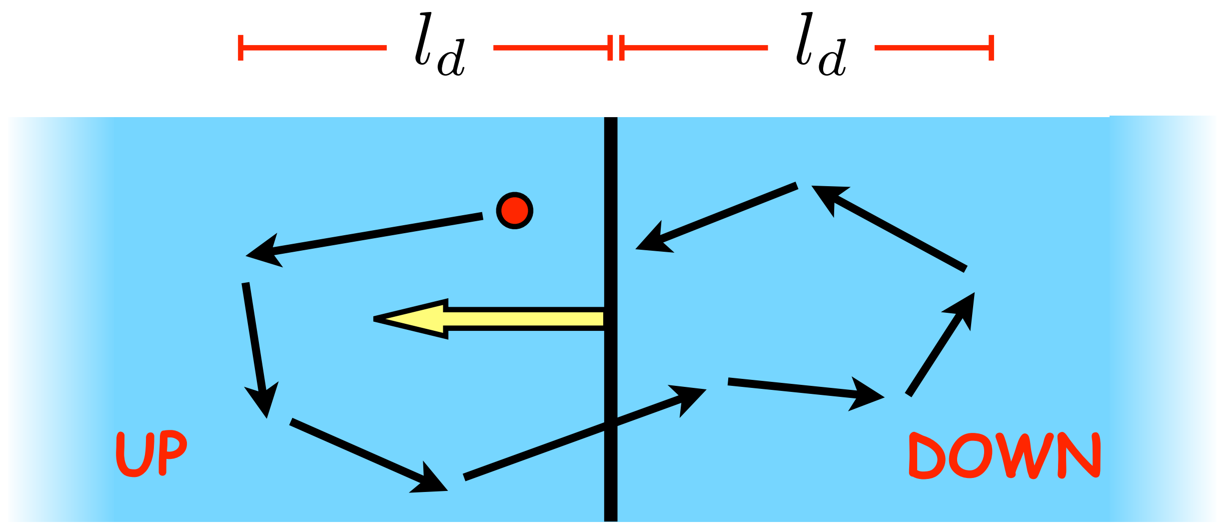


$s \simeq 2.2$

Sironi, Keshet, Lemoine (2015)

Key aspects of DSA at ~~non~~-relativistic shocks

particles are accelerated through a series of cycles up—>down—>up stream



- non-diffusive transport —> ~~isotropy~~
- energy gain per cycle —> ~~small~~ large

$$\Delta_{acc} = \Delta E/E \sim \Gamma_s^2$$ 1st crossing

$$\Delta_{acc} = \Delta E/E \sim 1$$ all others

- escape probability per cycle —> ~~small~~ large

$$P_{esc} \approx 0.5$$

- spectral slope —> E^{-s}

$$s \simeq 2.2$$

Acceleration time

simple consideration

since $\Delta_{acc} = \Delta E/E \sim 1$ then $\tau_{acc} \sim \tau_{cyc} = t_{up} + t_{down}$

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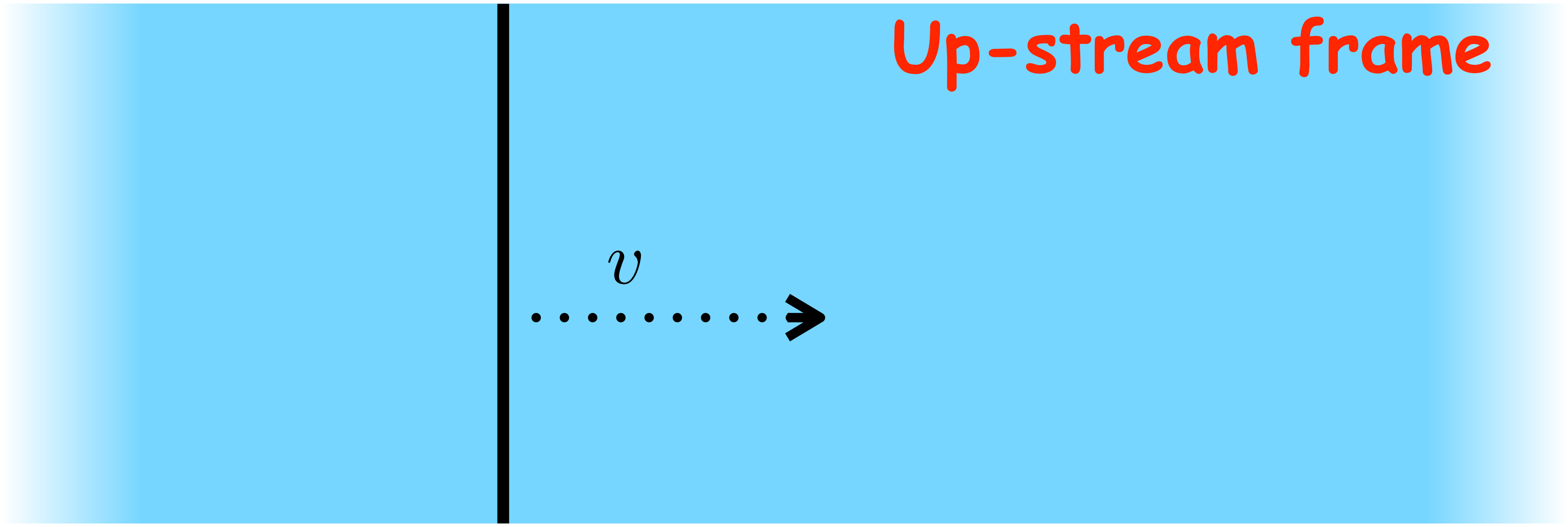
$$\boxed{\tau_{acc} \approx t_{up}}$$

Residence time upstream

Up-stream frame

Residence time upstream



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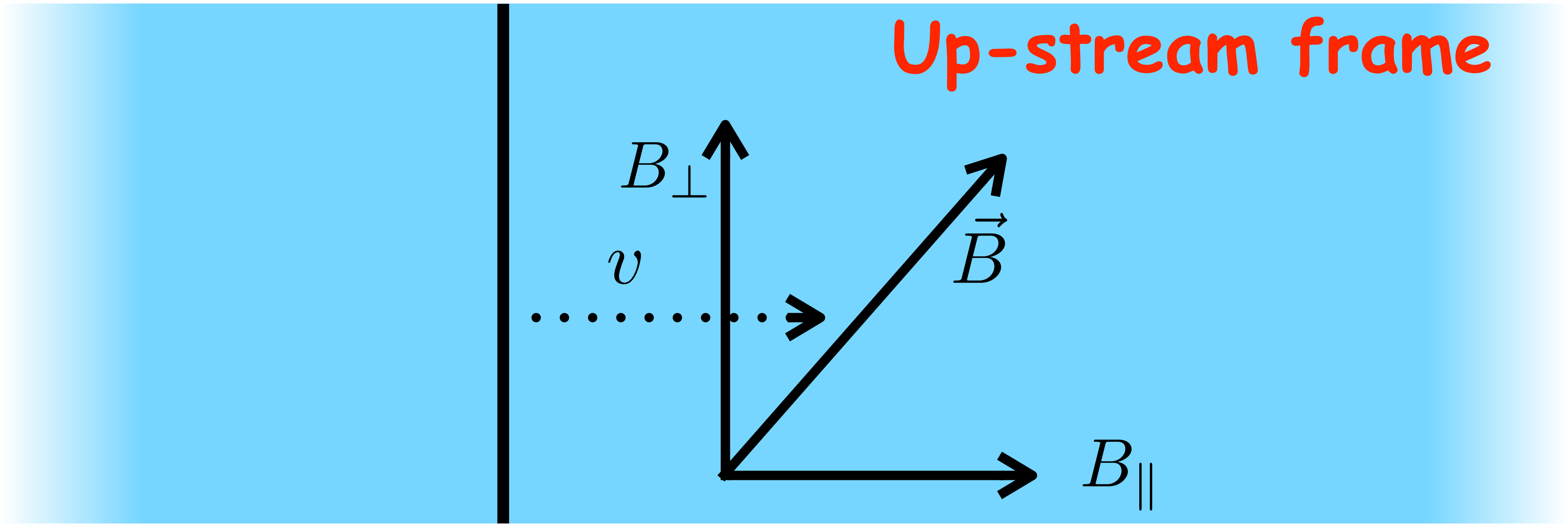

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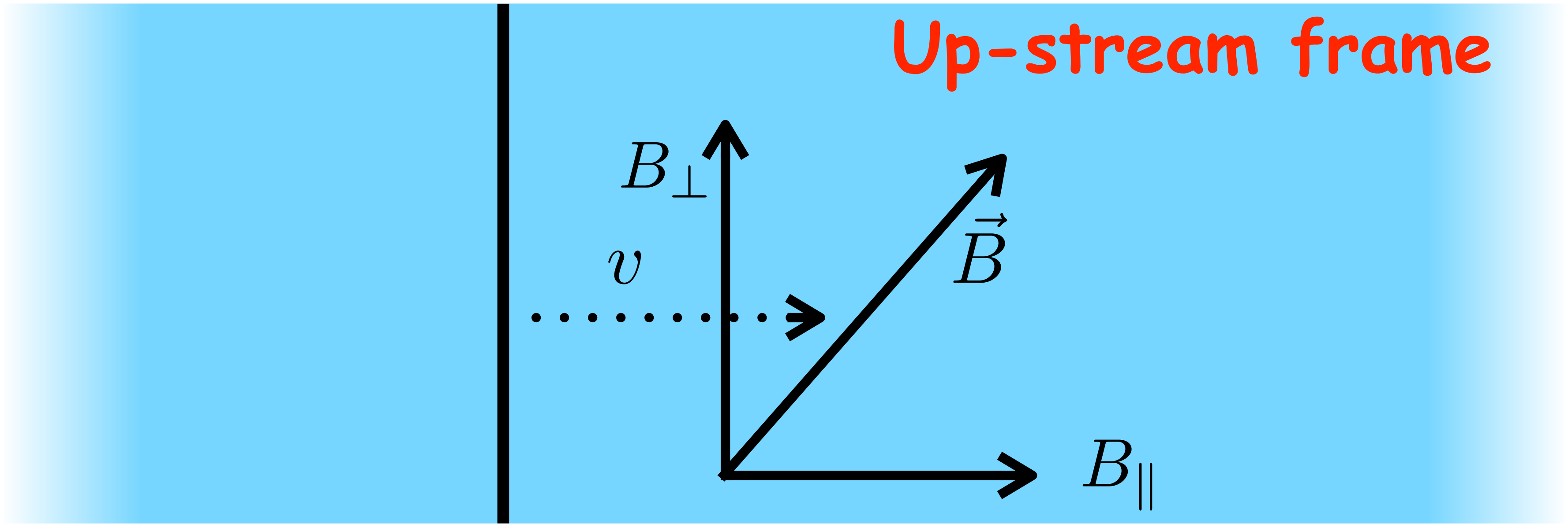

Larmor radius

$$R_L \sim \frac{p}{qB_\perp} = \frac{\gamma m v}{qB_\perp}$$

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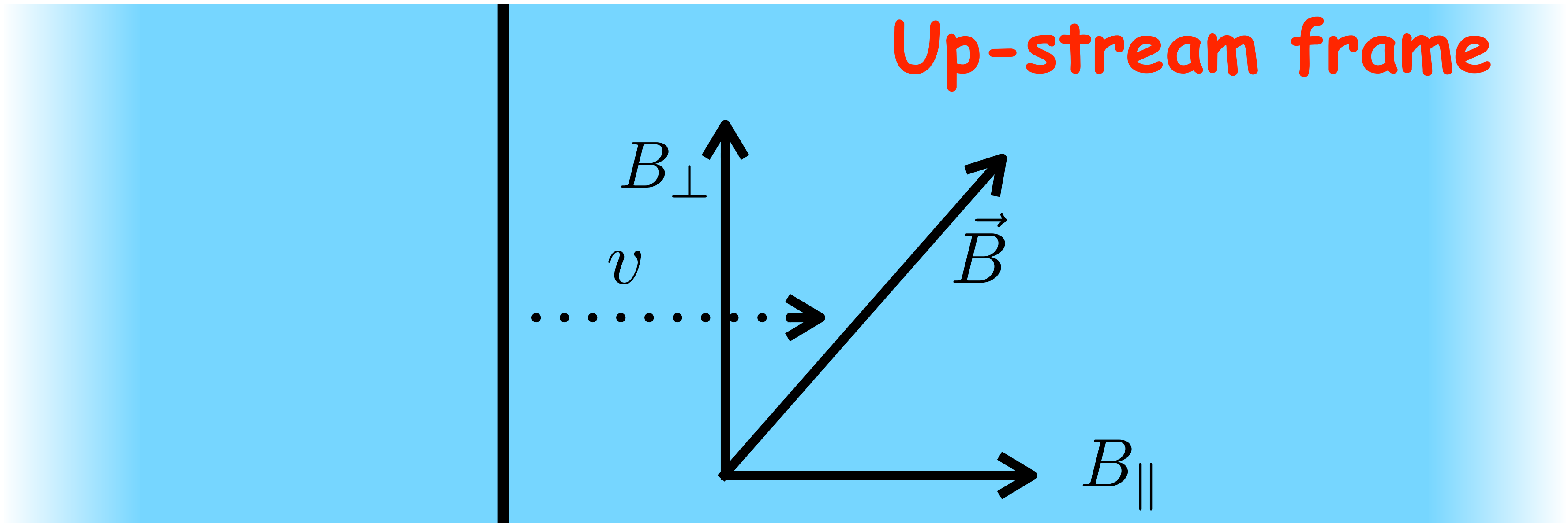

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a particle is overrun by the shock when it is deflected by an amount $\Delta\theta \sim 1/\Gamma_s$

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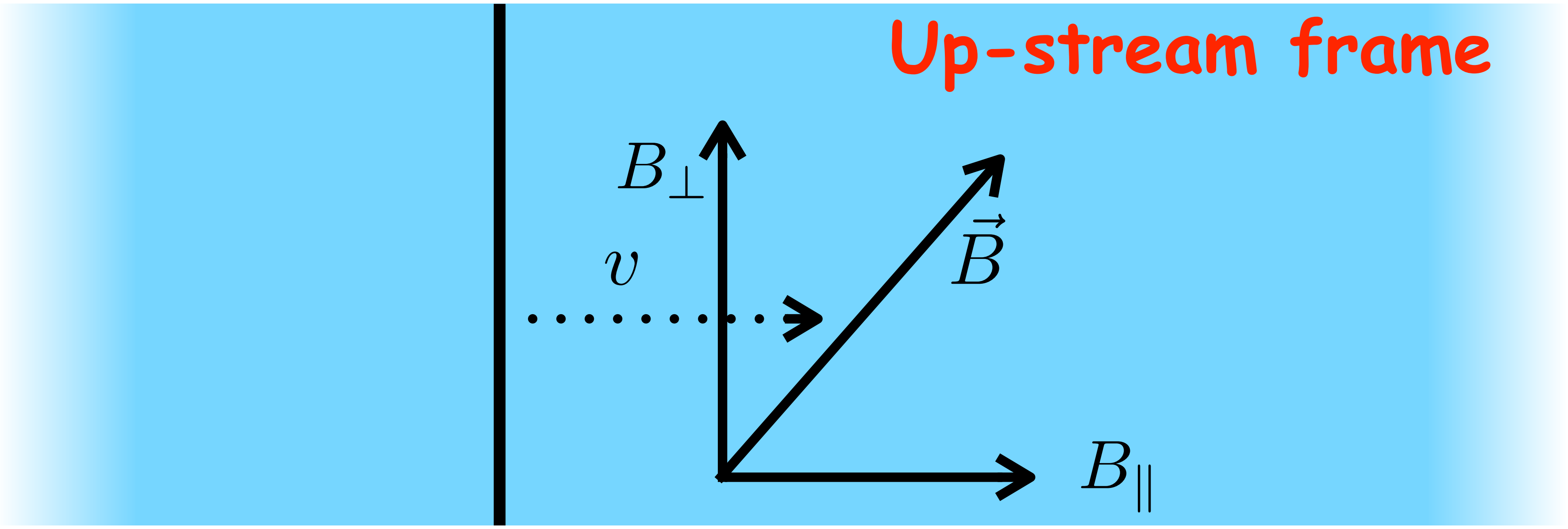

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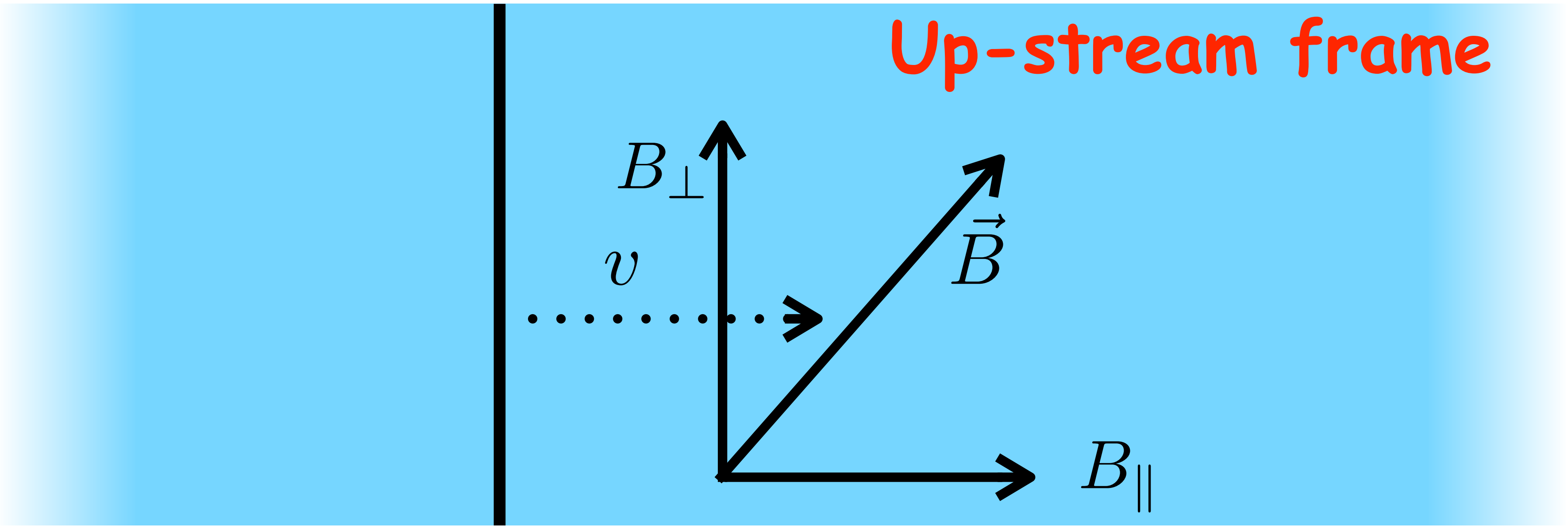

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residence time upstream —>

$$t_{up} \sim \frac{\Delta\vartheta}{2\pi}\tau_L \sim \frac{1}{\Gamma_s}\frac{E}{qB_\perp} = \frac{1}{\Gamma_s\Omega_\perp}$$

↖ gyration frequency

Acceleration rate

$$\frac{1}{E}\frac{\mathrm{d}E}{\mathrm{d}t} \sim \frac{1}{t_{up}}\frac{\Delta E}{E}$$

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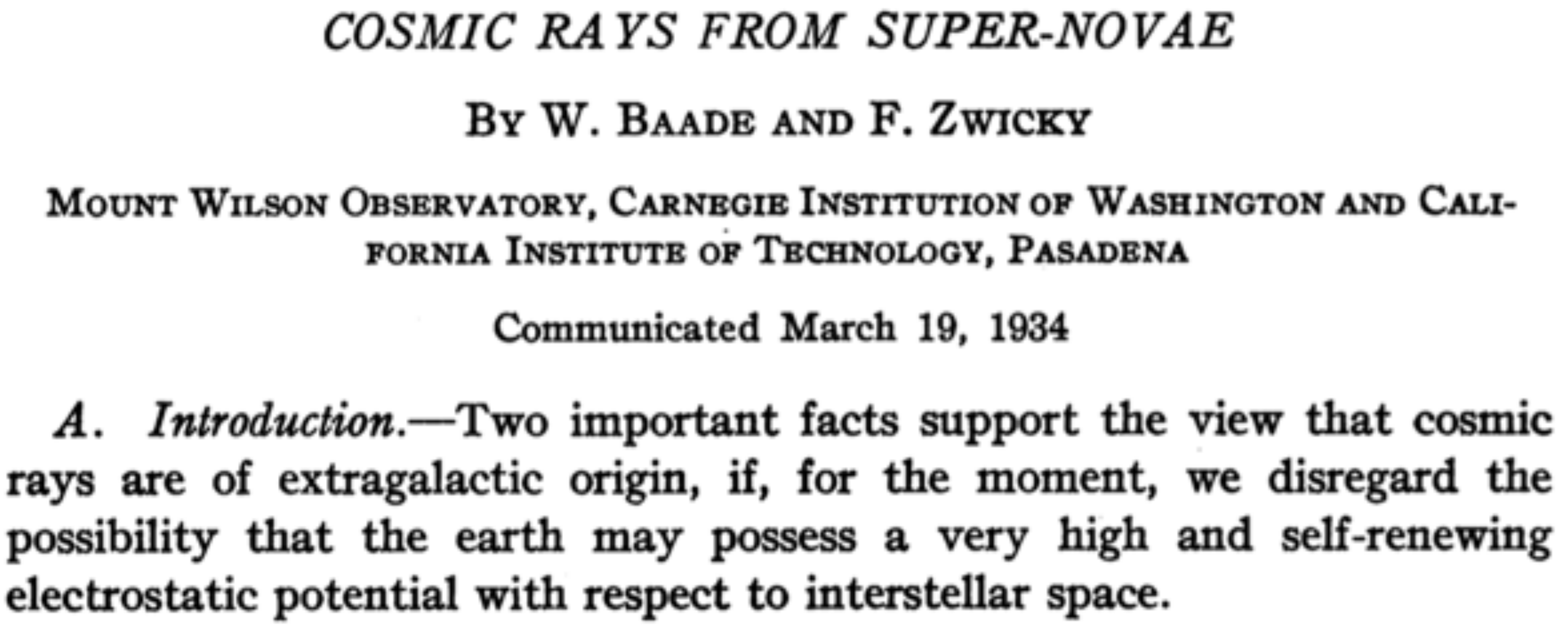
Next class



COSMIC RAYS FROM SUPER-NOVAE

BY W. BAADE AND F. ZWICKY

MOUNT WILSON OBSERVATORY, CARNEGIE INSTITUTION OF WASHINGTON AND CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA

Communicated March 19, 1934

A. Introduction.—Two important facts support the view that cosmic rays are of extragalactic origin, if, for the moment, we disregard the possibility that the earth may possess a very high and self-renewing electrostatic potential with respect to interstellar space.

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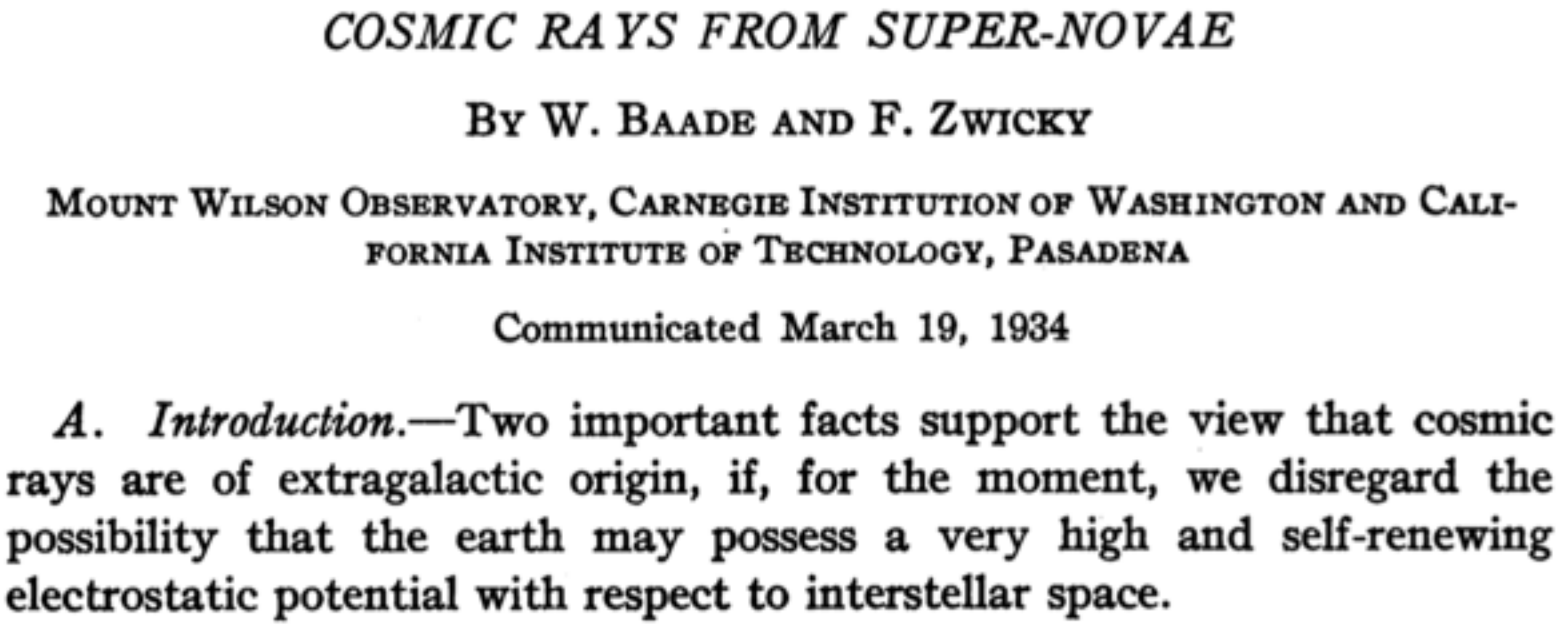

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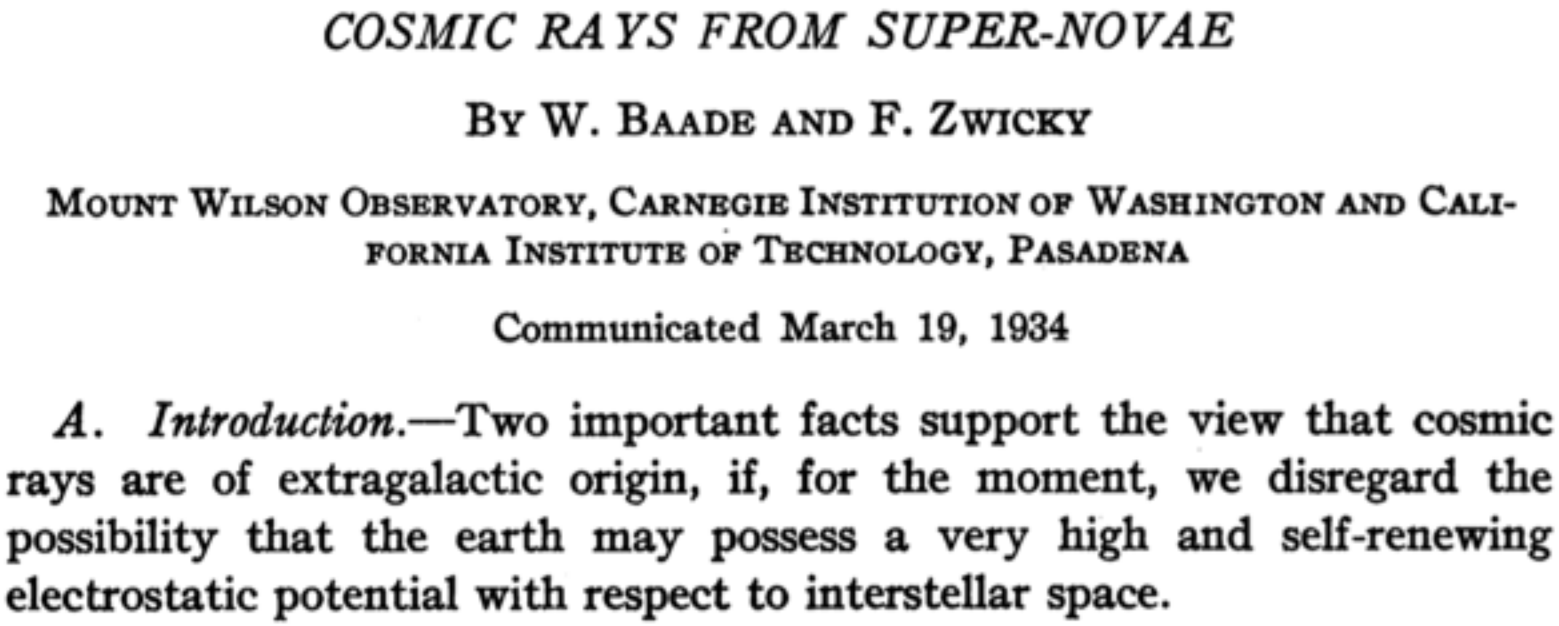

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and try to see if Zwicky was right!