Neutron star mergers
Lecture II

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Plan of the lectures

مصلى I: brief introduction to numerical relativity

مصلى II: brief review dynamics of merging binaries

مصلى III: brief overview of EOS constraints from mergers


V. Paschalidis, Classical Quantum Gravity 34, 084002 2017

The 3+1 splitting of the 4-dim spacetime represents an effective way to perform numerical solutions of the Einstein eqs.

Such a splitting amounts to projecting all 4-dim. tensors either on spatial hypersurfaces or along directions orthogonal to such hypersurfaces.

The 3-metric and the extrinsic curvature describe the properties of each slice.

Two functions, the lapse and the shift, tell how to relate coordinates between two slices: the lapse measures the proper time, while the shift measures changes in the spatial coords.
A number of tensor differential identities allow to cast the Einstein equations in a 3+1 split: this is the ADM formulation.

Einstein equations in the ADM formulation naturally split into evolution equations and constraint equations.

This is not very different from Maxwell equations, where there are also evolution and constraint equations.

The ADM eqs are ill posed and not suitable for numerics.

Alternative formulations (BSSNOK, CCZ4, Z4c) have been developed that are strongly hyperbolic and hence well-posed.
Both CCZ4, Z4c formulations make use of the constraint equations and can use additional evolution equations to damp the violations.

The hyperbolic evolution eqs. to solve are: \(6+6+(3+1) = 16\). We also “compute” \(3+1 = 4\) elliptic constraint eqs:

\[ \mathcal{H} \equiv (3)R + K^2 - K_{ij}K^{ij} = 0 \, , \quad \text{(Hamiltonian constraint)} \]

\[ \mathcal{M}^i \equiv D_j(K^{ij} - g^{ij}K) = 0 \, , \quad \text{(momentum constraints)} \]

NOTE: these eqs are not solved but only monitored to verify

\[ ||\mathcal{H}|| \simeq ||\mathcal{M}^i|| < \varepsilon \sim 10^{-4} - 10^{-2} \]

Four more equations are needed to set the gauges: lapse and shift.
The two-body problem: Newton vs Einstein
The two-body problem: Newton vs Einstein

Take two objects of mass \( m_1 \) and \( m_2 \) interacting only gravitationally

\[ \ddot{r} = -\frac{GM}{d_{12}^3} r \]

where \( M \equiv m_1 + m_2 \), \( r \equiv r_1 - r_2 \), \( d_{12} \equiv |r_1 - r_2| \).

In **Newtonian gravity** solution is analytic: there exist closed orbits (circular/elliptic) with

In **Einstein’s gravity** no analytic solution! No closed orbits: the system loses energy/angular momentum via gravitational waves.
The two-body problem in GR

- For BHs we know what to **expect**:

  $\text{BH} + \text{BH} \rightarrow \text{BH} + \text{GWs}$
The two-body problem in GR

• For BHs we know what to expect:
  \[ \text{BH} + \text{BH} \rightarrow \text{BH} + \text{GWs} \]

• For NSs the question is more subtle: the merger leads to an hyper-massive neutron star (HMNS), ie a metastable equilibrium:
  \[ \text{NS} + \text{NS} \rightarrow \text{HMNS} + \ldots \ ? \rightarrow \text{BH} + \text{torus} + \ldots \ ? \rightarrow \text{BH} + \text{GWs} \]
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• HMNS phase can provide clear information on EOS

• BH+torus system may tell us on the central engine of GRBs

artist impression (NASA)
The two-body problem in GR

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  \[
  \text{BH} + \text{BH} \rightarrow \text{BH} + \text{GWs}
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• For NSs the question is more subtle: the merger leads to an hyper-massive neutron star (HMNS), ie a metastable equilibrium:
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  \]

• ejected matter undergoes nucleosynthesis of heavy elements
The equations of numerical relativity

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}, \quad \text{(field equations)} \]
\[ \nabla_\mu T^{\mu\nu} = 0, \quad \text{(cons. energy/momentum)} \]
\[ \nabla_\mu (\rho u^\mu) = 0, \quad \text{(cons. rest mass)} \]
\[ p = p(\rho, \epsilon, Y_e, \ldots), \quad \text{(equation of state)} \]
\[ \nabla_\nu F^{\mu\nu} = I^\mu, \quad \nabla^*_\nu F^{\mu\nu} = 0, \quad \text{(Maxwell equations)} \]
\[ T_{\mu\nu} = T_{\mu\nu}^{\text{fluid}} + T_{\mu\nu}^{\text{EM}} + \ldots \quad \text{(energy – momentum tensor)} \]

In GR these equations do not possess an analytic solution in the regimes we are interested in
$M = 2 \times 1.35 \, M_\odot$

LS220 EOS
merger $\rightarrow$ HMNS $\rightarrow$ BH + torus
Quantitative differences are produced by:

- total mass (prompt vs delayed collapse)
Broadbrush picture

$M/M_{\text{max}}, q \approx 1$

- Binary ($\leq 1\, \text{kHz}$)
- Black hole + torus (5 – 6 kHz)
- Black hole (6 – 7 kHz)

- Binary ($\leq 1\, \text{kHz}$)
- HMNS (2 – 4 kHz)
- Black hole + torus (5 – 6 kHz)
- Black hole (6 – 7 kHz)

- Binary ($\leq 1\, \text{kHz}$)
- HMNS (2 – 4 kHz)
- Supramassive NS (1 – 2 kHz)
- NS (2 – 4 kHz)

- [10$^6$ – 10$^7$ yr]
- [1 ms – 1 s]
- [1 – 100 s]

FRB?
Quantitative differences are produced by:

- total mass (prompt vs delayed collapse)
- mass asymmetries (HMNS and torus)
Total mass: $3.37 M_\odot$; mass ratio: $0.80$;
the torii are generically more massive
the torii are generically more extended
the torii tend to stable quasi-Keplerian configurations
overall unequal-mass systems have all the ingredients needed to create a GRB
Quantitative differences are produced by:

- total mass (prompt vs delayed collapse)
- mass asymmetries (HMNS and torus)
- soft/stiff EOS (inspiral and post-merger)
- magnetic fields (equil. and EM emission)
- radiative losses (equil. and nucleosynthesis)
How to constrain the EOS from the GWs
Anatomy of the GW signal

binary black holes (2006)
Anatomy of the GW signal

Chirp signal

binary black holes (2006)
Anatomy of the GW signal

Chirp signal

black-hole ringdown

binary black holes (2006)
Anatomy of the GW signal

GNH3, $\bar{M} = 1.350M_\odot$
Anatomy of the GW signal

Inspiral: well approximated by PN/EOB; tidal effects important

Chirp signal

$GHN3, \tilde{M} = 1.350 M_\odot$
Anatomy of the GW signal

Merger: highly nonlinear but analytic description possible
Anatomy of the GW signal

Image of a graph showing the time evolution of the GW signal, with labels and annotations.

- **post-merger**: quasi-periodic emission of bar-deformed HMNS
Anatomy of the GW signal

Collapse-ringdown: signal essentially shuts off.
In frequency space

\[ \sqrt{S_n(f)} \text{ and } 2(f |\tilde{h}(f)|)^{1/2} \]

- Effectively point-particle
- Initial LIGO
- Tidal effects
- Advanced LIGO
- NS-NS merger
- BH-BH merger
- Post merger
- 100 Mpc

Read et al. (2013)
What we can do nowadays

Extracting information from the EOS

Extracting information from the EOS


There are lines! Logically not different from emission lines from stellar atmospheres. This is GW spectroscopy!
A spectroscopic approach to the EOS

A spectroscopic approach to the EOS


merger frequency
Understanding mode evolution

On a short timescale after the merger, it is possible to see the emergence of $f_1$, $f_2$, and $f_3$. 

![Graphs showing mode evolution over time with different frequencies and modes highlighted.](image)
Understanding mode evolution

On a long timescale after the merger, only $f_2$ survives.
A mechanical toy model for the $f_1, f_3$ peaks

- Consider disk with 2 masses moving along a shaft and connected via a spring $\sim$ HMNS with 2 stellar cores
- Let disk rotate and mass oscillate while conserving angular momentum

- If there is no friction, system will spin between: low freq ($f_1$, masses are far apart) and high ($f_3$, masses are close).
- If friction is present, system will spin asymptotically at $f_2 \sim (f_1 + f_3)/2$.
- Analytic model possible of post merger.
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Quasi-universal behaviour
Many other simulations have confirmed this (Bernuzzi+ 2014, Takami+ 2015, LR+2016).

“surprising” result: quasi-universal behaviour of GW frequency at amplitude peak (Read+2013).

Many other simulations have confirmed this (Bernuzzi+ 2014, Takami+ 2015, LR+2016).

Quasi-universal behaviour in the inspiral implies that once $f_{\text{max}}$ is measured, so is tidal deformability, hence $I, Q, M/R$.

\[ \Lambda = \frac{\lambda}{M^5} = \frac{16}{3} \kappa_2^T \] tidal deformability or Love number
Quasi-universal behaviour: *post-merger*

We have found quasi-universal behaviour: i.e., the properties of the spectra are only weakly dependent on the EOS. This has profound implications for the analytical modelling of the GW emission: “what we do for one EOS can be extended to all EOSs.”
Quasi-universal behaviour: post-merger

- Correlations with Love number found also for high frequency peak $f_2$.
- This and other correlations are weaker but equally useful.

- Important correlation also between compactness and deformability
Radius estimate from binary population

Bose, Chakravarti, LR, Sathyaprakash, Takami (2017)
Analytical modelling of postmerger waveform

- **Postmerger** appears hopeless but isn’t (Clark+14, 16; Bose+17)
Analytical modelling of postmerger waveform

- Knowledge of spectral properties provides **analytic ansatz**

\[
h(t) = \alpha \exp(-t/\tau_1) \left[ \sin(2\pi f_1 t) + \sin(2\pi (f_1 - f_1\epsilon) t) + \sin(2\pi (f_1 + f_1\epsilon) t) \right] + \\
\exp(-t/\tau_2) \sin(2\pi f_2 t + 2\pi \gamma_2 t^2 + \pi \beta_2).
\]
Analytical modelling of postmerger waveform

- Overall pretty decent fit in phase
- Fit in amplitude is less good but also less important
Analytical modelling of postmerger waveform

Good match is clear also in frequency space.

In summary: despite the complex signal, an **analytic** description of the full GW signal is now possible.
Even a small SNR counts

- Using **analytical modelling** performed **Fisher-matrix** analysis of GWs and **Monte-Carlo** simulation.
- Waveforms aligned at frequency, $f_2^c$. Standard frequency estimation yields value of $f_2^c$ and statistical spread.
- **Quasi-universal relation** between $f_2$ and compactness, and error-propagation, to deduce the error in radius.
- Employed 100 BNS signals injected in 100 uncorrelated timeseries of Gaussian noise with aLIGO sensitivity.
- Used information on $f_1$ and chirp **mass** from **inspiral**.
- Repeated over 900 experiments to build statistics.
Constraining the radius: MonteCarlo vs Fisher

- **Uniform** distribution in mass $[1.21, 1.38] M_{\odot}$ between 100 and 300 Mpc; isotropic distribution in space.
- Dashed lines for results of Fisher-matrix analysis with $N=50$
- Errors scale like $\sqrt{N}$
Constraining the radius: MonteCarlo vs Fisher

- **Gaussian** distribution in mass $[1.21, 1.38] \, M_\odot$ centred at $1.35 \, M_\odot$ with variance 0.05. Binaries are between 100 and 300 Mpc; isotropic distribution in space.

- dashed lines for results of Fisher-matrix analysis with $N=50$

- errors scale like $\sqrt{N}$
All in all

- **stiff EOSs:** \(|\Delta R/\langle R\rangle| < 10\% \) for \(N \sim 20\)
- **soft EOSs:** \(|\Delta R/\langle R\rangle| \sim 10\% \) for \(N \sim 50\)
- Discriminating stiff/soft EOSs will be possible even with moderate \(N\)
- Discriminating two-stiff/two-soft EOSs will be harder
- Very soft EOSs remain a challenge
- Golden binary: \(\text{SNR} \sim 6\) at 30 Mpc
  \(|\Delta R/\langle R\rangle| \lesssim 2\%\) at 90\% confidence
GW signal from binary neutron stars depends on a number of factors: mass, mass ratio, EOS, magnetic fields, neutrino transport.

Inspiral part is reasonably well understood and approximated with PN or EOB approaches; post-merger much more complex.

Spectra of post-merger shows clear "quasi-universal" peaks.

Unless binary very close, peaks have SNR ~ 1. However, multiple signals can be stacked and SNR will increase coherently.

Fisher-matrix and Monte-Carlo simulations can be performed combining information from inspiral and post-merger:

- **stiff** EOSs: $|\Delta R/\langle R\rangle| < 10\%$ for $N\sim 20$
- **soft** EOSs: $|\Delta R/\langle R\rangle| < 10\%$ for $N\sim 50$
- **very soft** EOS will be a challenge for aLIGO-Virgo (ET?)

Recap (I)