Neutron star mergers

Lecture I

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Plan of the lectures

- ✳Lecture I: **brief** introduction to numerical relativity
- ✳Lecture II: **brief** review dynamics of merging binaries
- ✳Lecture III: **brief** overview of EOS constraints from mergers

✳Alcubierre, *"Introduction to 3+1 Numerical Relativity"*, Oxford University Press, 2008

✳Baumgarte and Shapiro, *"Numerical Relativity: Solving Einstein's Equations on the Computer"*, Cambridge University Press, 2010

✳Gourgoulhon, *"3+1 Formalism in General Relativity"*, Lecture Notes in Physics, Springer 2012

✳**Rezzolla and Zanotti, "Relativistic Hydrodynamics", Oxford University Press, 2013**

The equations of numerical relativity

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \,, \qquad \text{(field equations)}$$

$$\nabla_\mu T^{\mu\nu} = 0 \,, \qquad \text{(cons. energy/momentum)}$$

$$\nabla_\mu (\rho u^\mu) = 0 \,, \qquad \text{(cons. rest mass)}$$

$$p = p(\rho, \epsilon, Y_e, \ldots) \,, \qquad \text{(equation of state)}$$

$$\nabla_\nu F^{\mu\nu} = I^\mu \,, \quad \nabla_\nu^* F^{\mu\nu} = 0 \,, \qquad \text{(Maxwell equations)}$$

$$T_{\mu\nu} = T_{\mu\nu}^{\text{fluid}} + T_{\mu\nu}^{\text{EM}} + \ldots \qquad \text{(energy − momentum tensor)}$$

In GR these equations do not possess an analytic solution in the nonlinear regimes we are interested in

Numerical Relativity: why so hard?…

✳ Coordinates (spatial and time) do not have a specific meaning

- gauge freedom introduces **complications.** Which part of the spacetime to cover?
- gauge conditions must **avoid singularities**
- gauge conditions must **counteract grid distorsions**

✳Einstein field equations are highly nonlinear

- essentially unknown in these regimes (well-posedeness not enough!...)

✳Physical singularities are the "butter-and-bread" of NR

- delicate techniques are needed to "excise" the troublesome region

✳No obviously "better" formulation of the Einstein equations

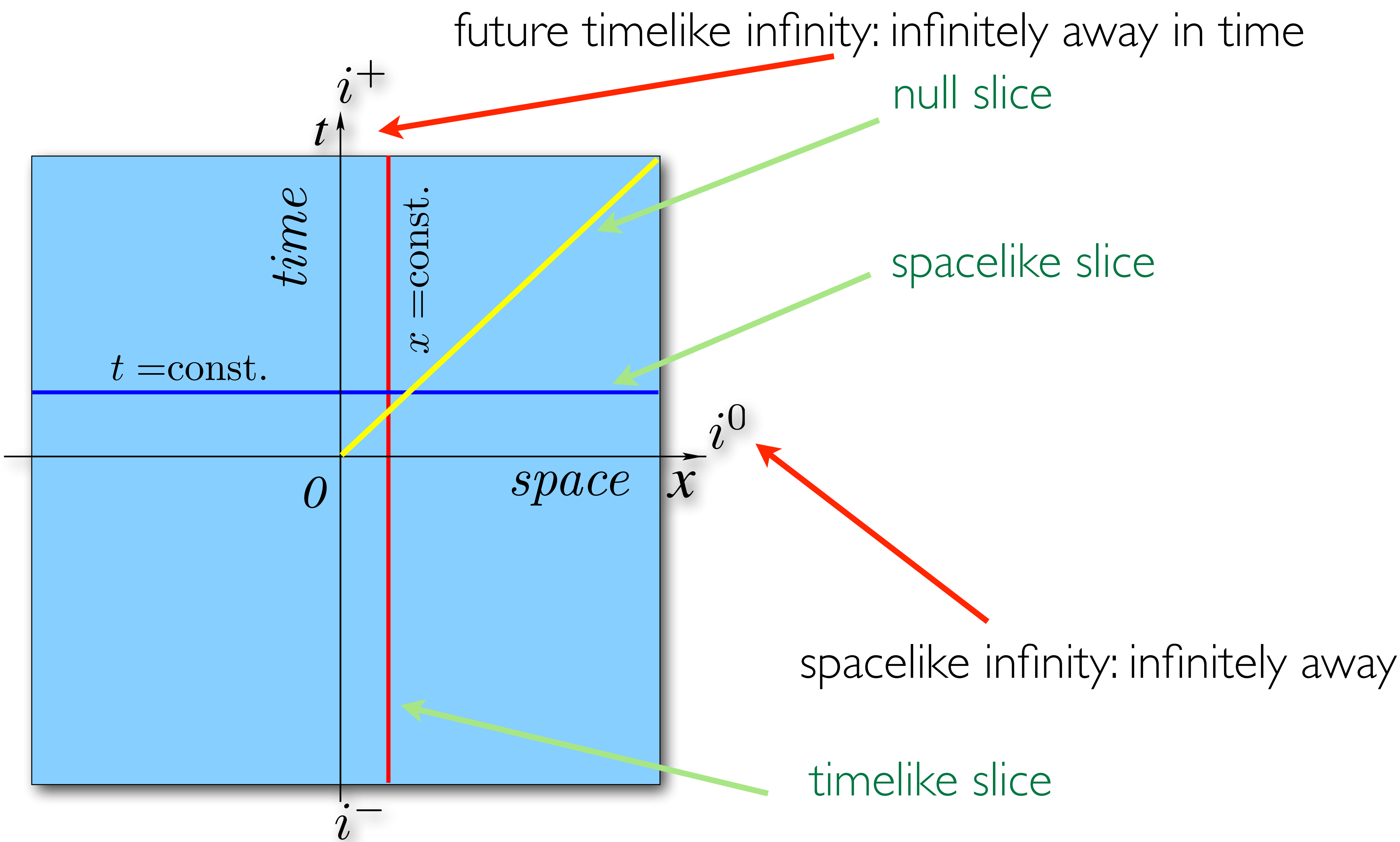
- ADM, conformal traceless decomposition, Z4, first-order hyperbolic, harmonic, …???

✳Simply lots of equations to solve: stretching supercomputers resources!

- large turn-around times make progress slow (2-3 weeks/simulation)
- implementations of AMR techniques is extremely problematic

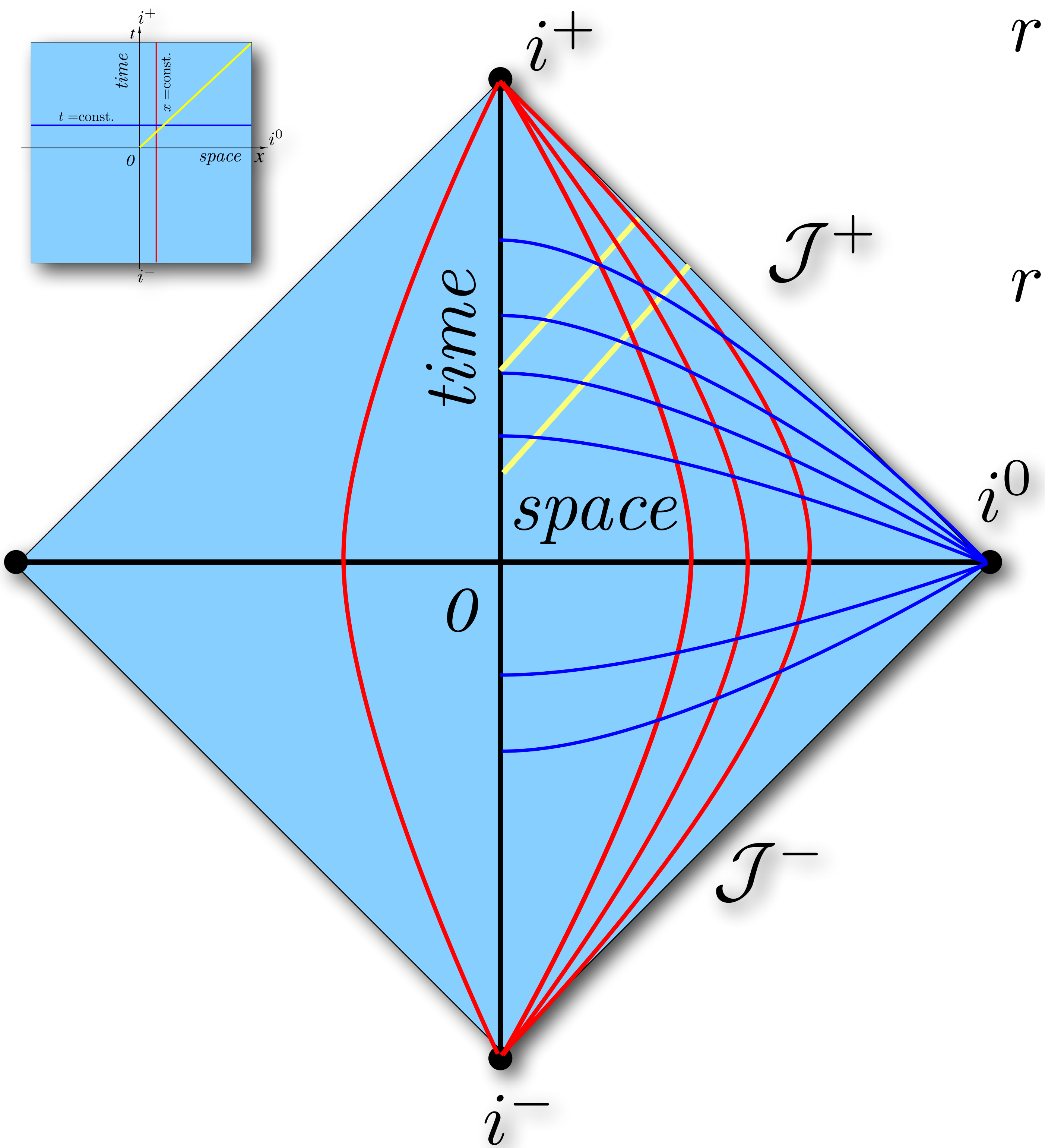
Which part of the spacetime to cover?…

Consider the Minkowski spacetime in Cartesian coordinates



Which part of the spacetime to cover?...

Consider now the same Minkowski spacetime in a **conformal** representation.



$$r \to r'(r) = \frac{r}{1+r};$$

$$r'(0) = 0; \;\; r'(\infty) = 1$$

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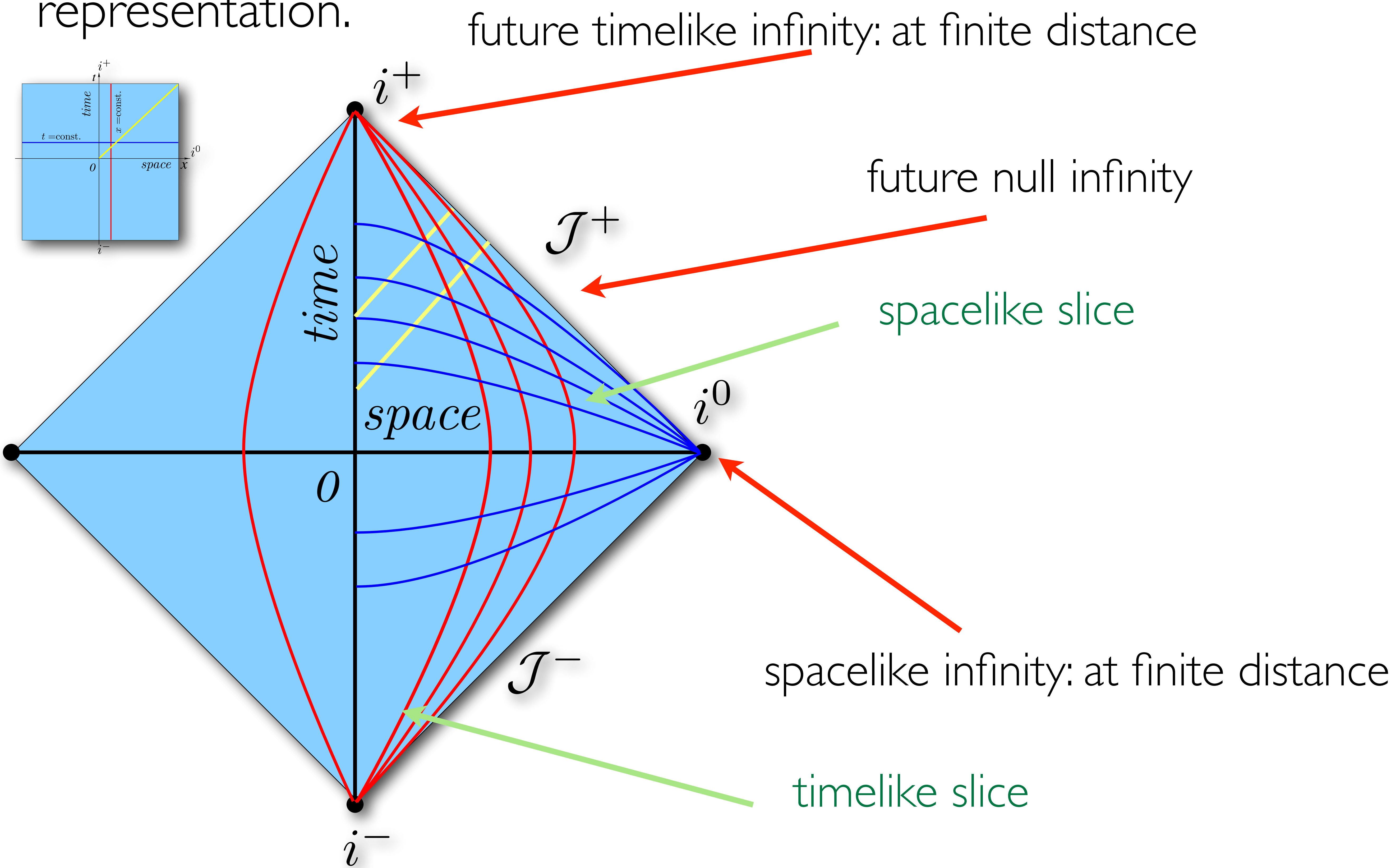
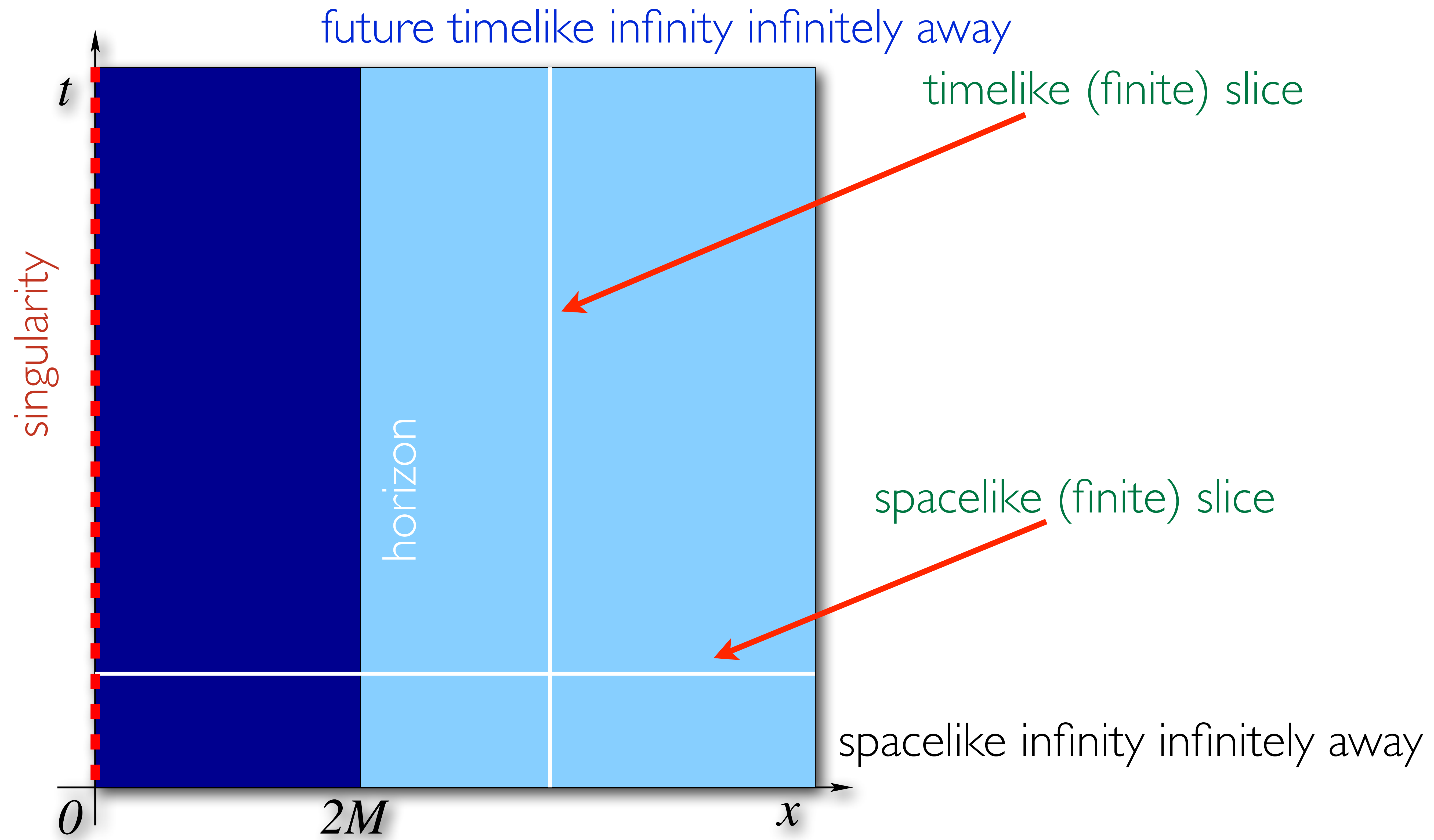


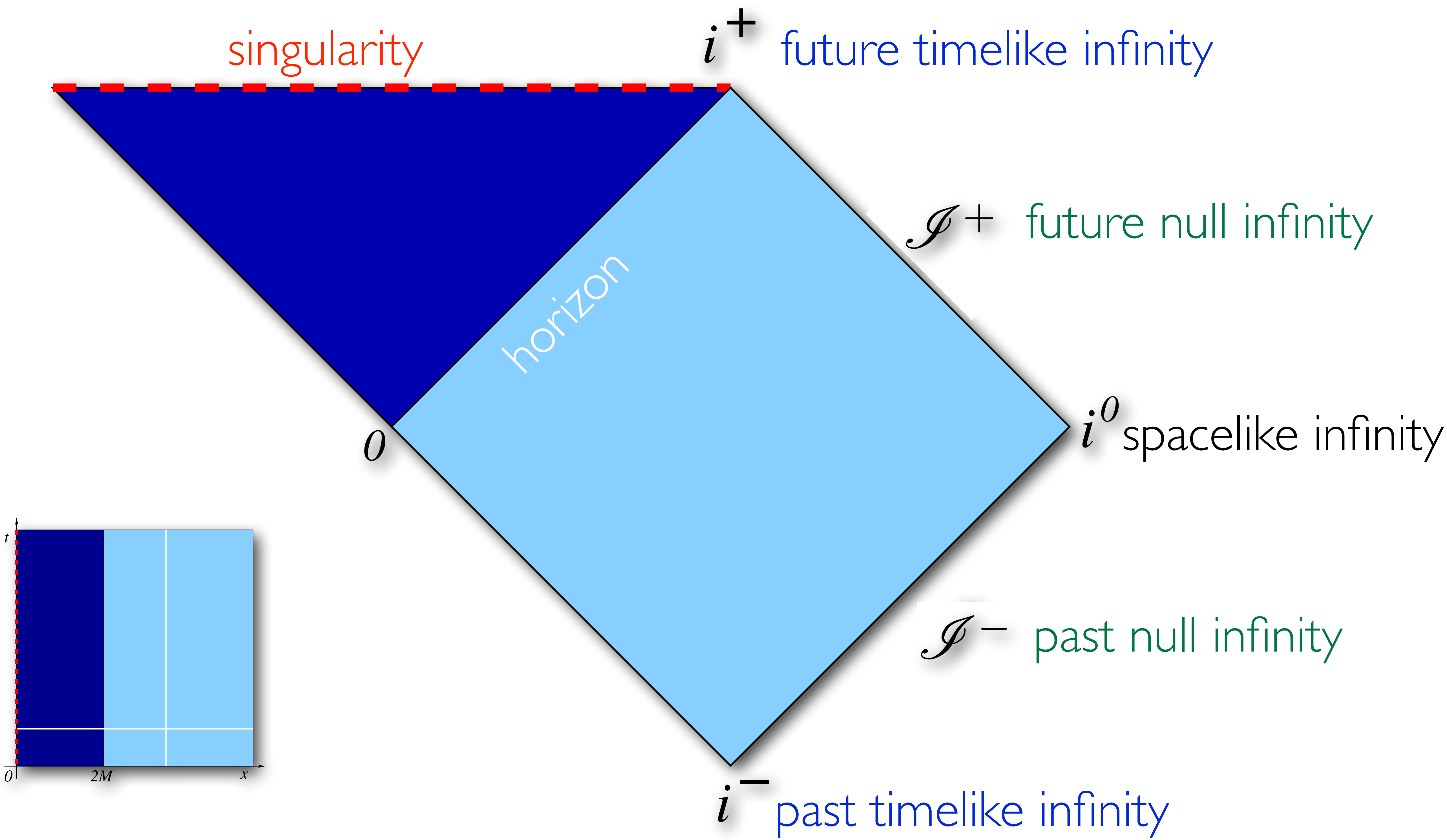
Which part of the spacetime to cover?...

Consider the simplest **black-hole** spacetime in Cartesian coordinates



Which part of the spacetime to cover?...

Consider the simplest black-hole spacetime: its **conformal** representation is given by a Carter-Penrose diagram



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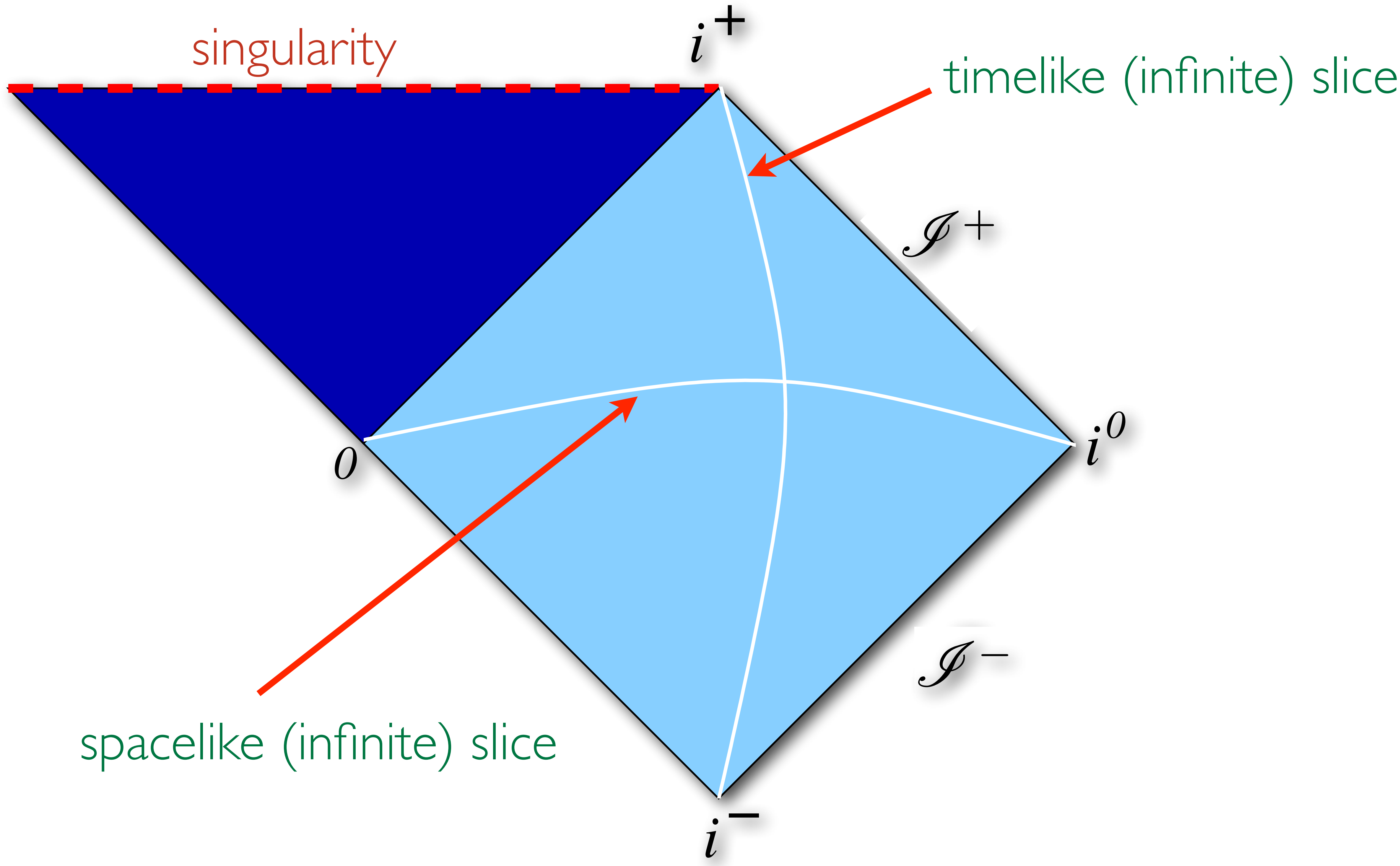
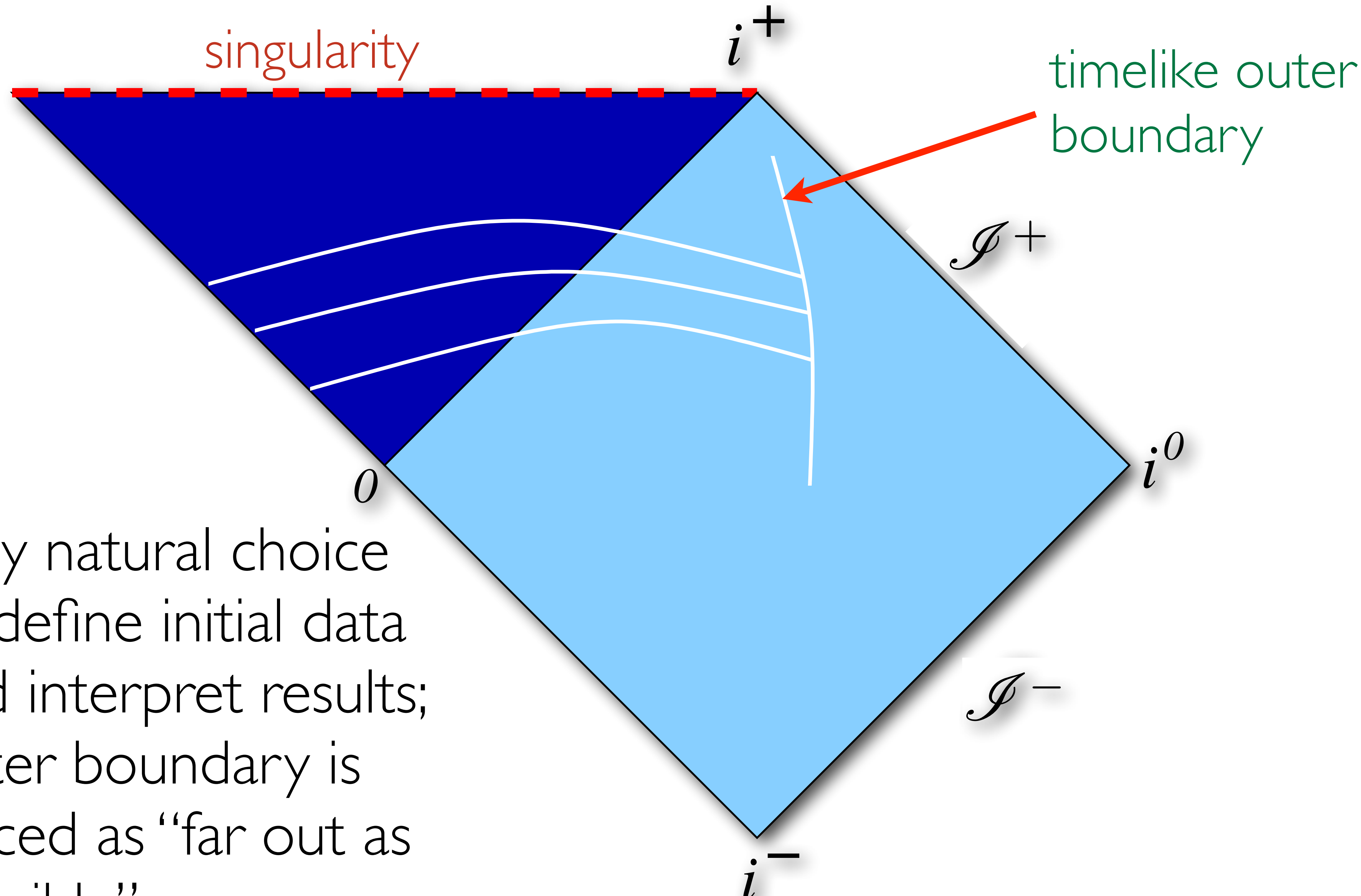


Spacelike finite slices

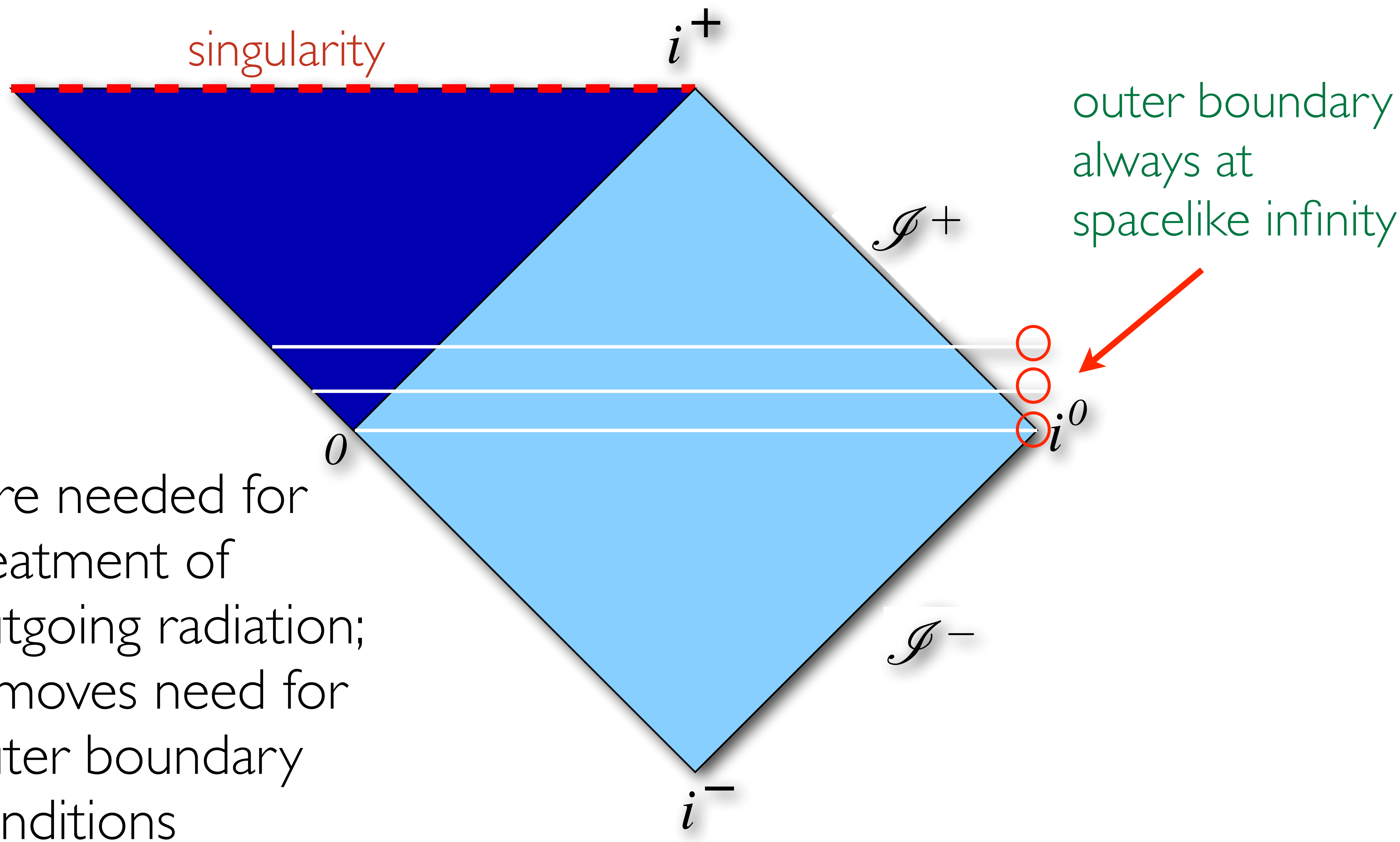
Most common discretization of the spacetime. Reminescent of fluid dynamics; introduces complications from outer boundary



very natural choice to define initial data and interpret results; outer boundary is placed as "far out as possible"

Spacelike infinite (conformal) slices

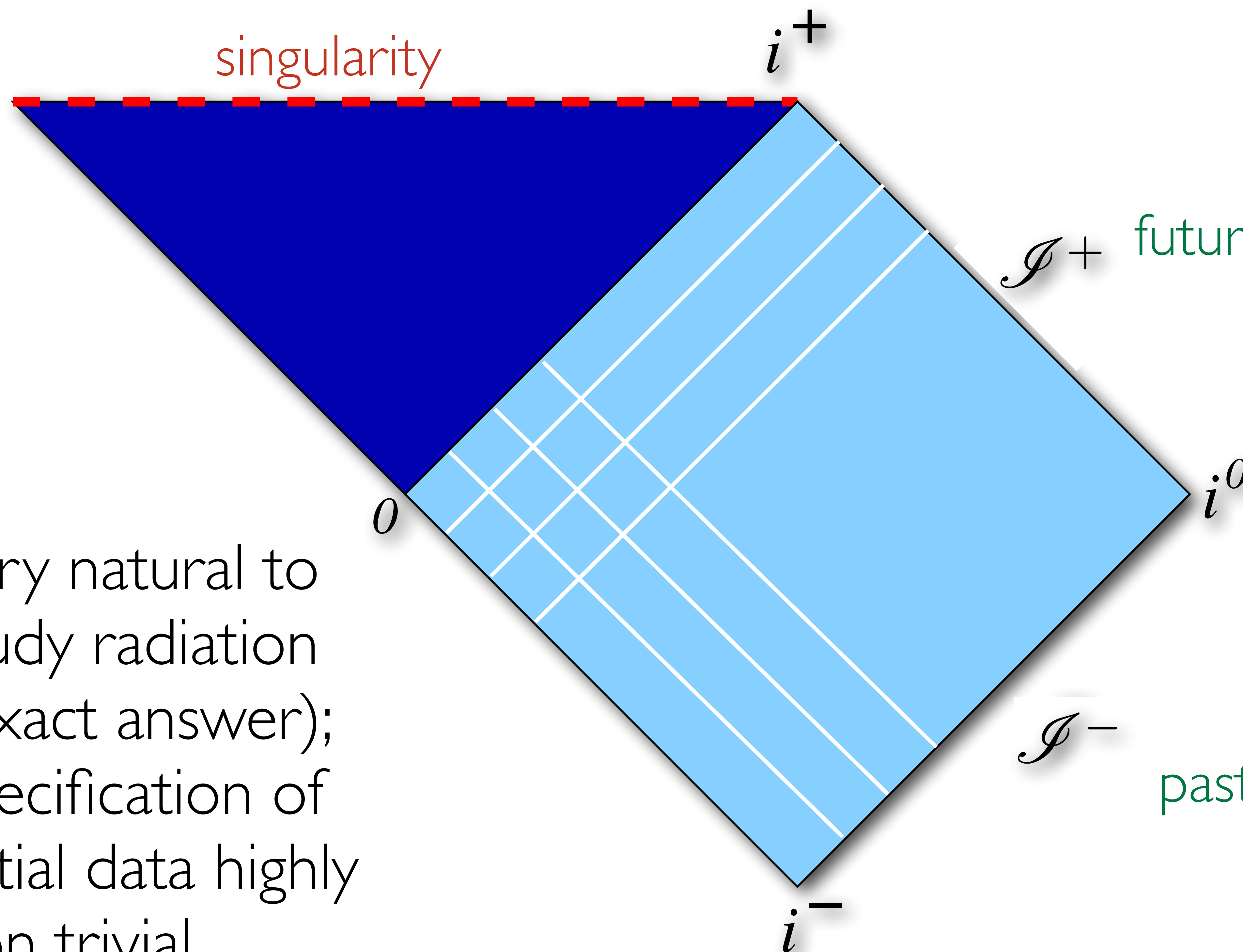
Not common discretization of the spacetime: spacelike infinity is included in the grid; requires suitable coord transformations



care needed for treatment of outgoing radiation; removes need for outer boundary conditions

Null (ingoing-outgoing) slices

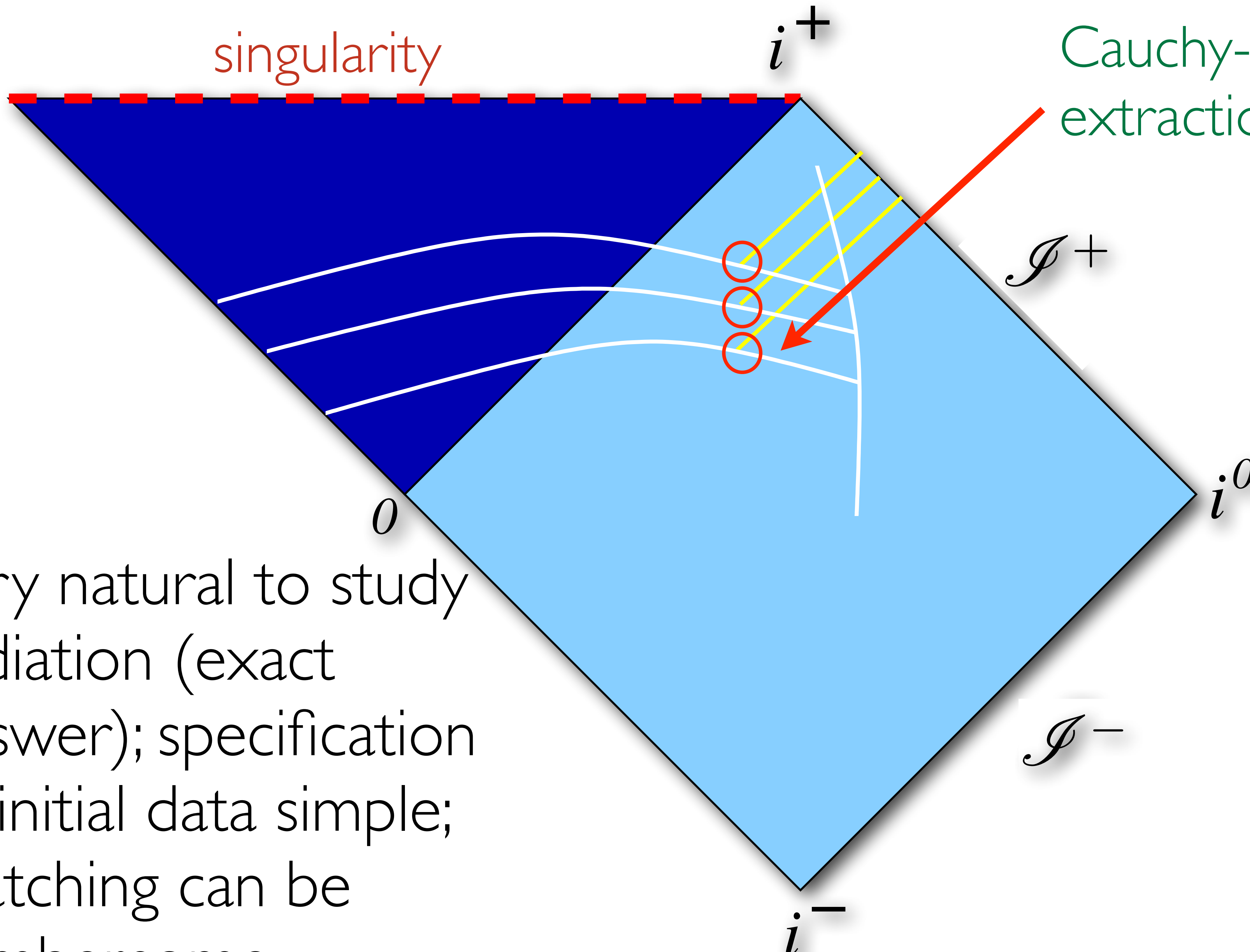
Not common discretization of the spacetime. Works well in 1D but not employed in higher dimensions



very natural to study radiation (exact answer); specification of initial data highly non trivial

spacelike-characteristic slices (CCE)

Combines advantages of spacelike slices with accurate description of outgoing radiation; tested in 3D linear regimes

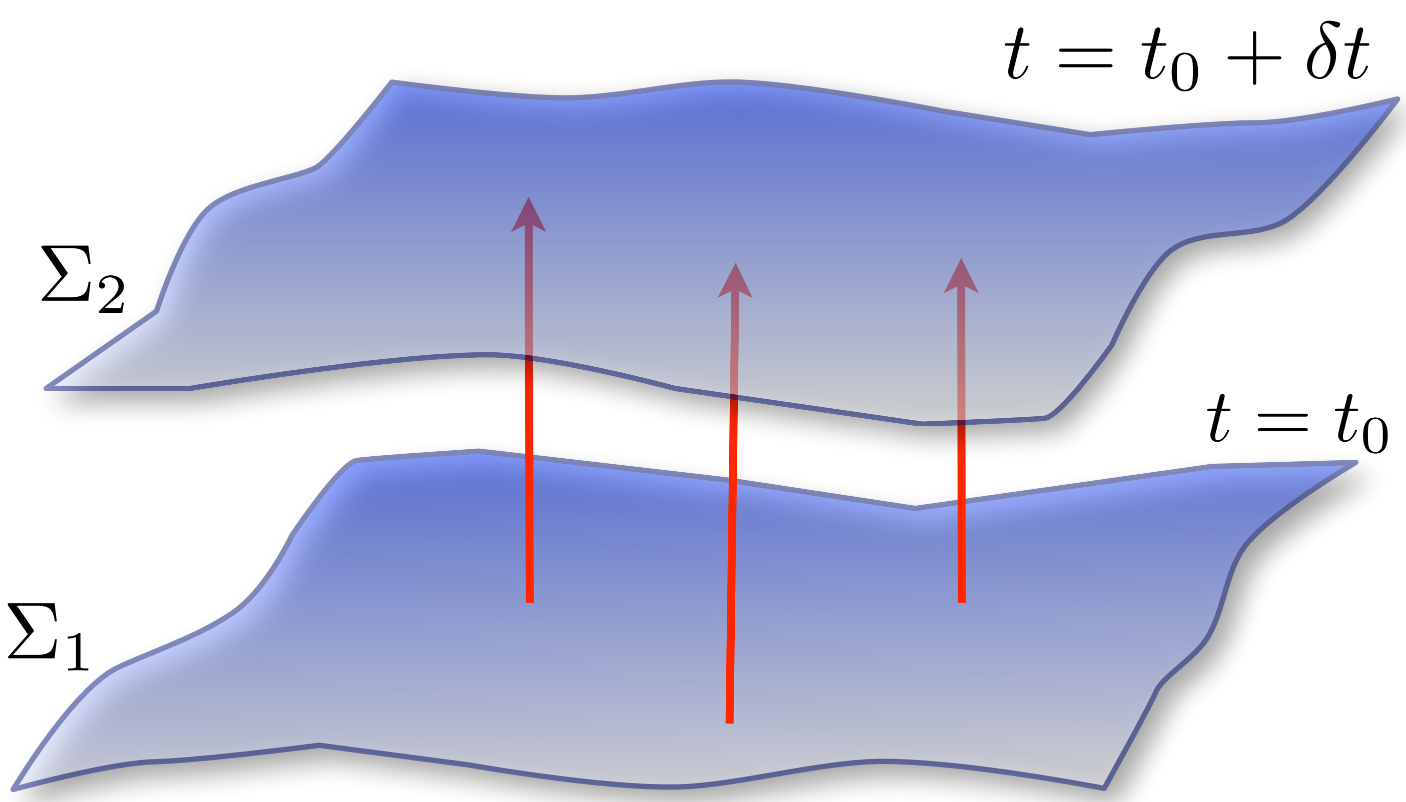


very natural to study radiation (exact answer); specification of initial data simple; matching can be cumbersome

3+1 splitting of spacetime

First step: foliate the 4D spacetime

Given a manifold $\mathcal{M}$ describing a spacetime with 4-metric $g_{\mu\nu}$ we want to foliate it via spacelike, three-dimensional hypersurfaces, i.e., $\Sigma_1, \Sigma_2, \ldots$ leveled by a scalar function. The time coordinate t is an obvious good choice.



Define therefore

$$\Omega_\mu \equiv \nabla_\mu t$$

such that

$$|\Omega|^2 \equiv g^{\mu\nu} \nabla_\mu t \nabla_\nu t = -\alpha^{-2}$$

This defines the "lapse" function which is strictly positive for spacelike hypersurfaces

$$\alpha(t, x^i) > 0$$

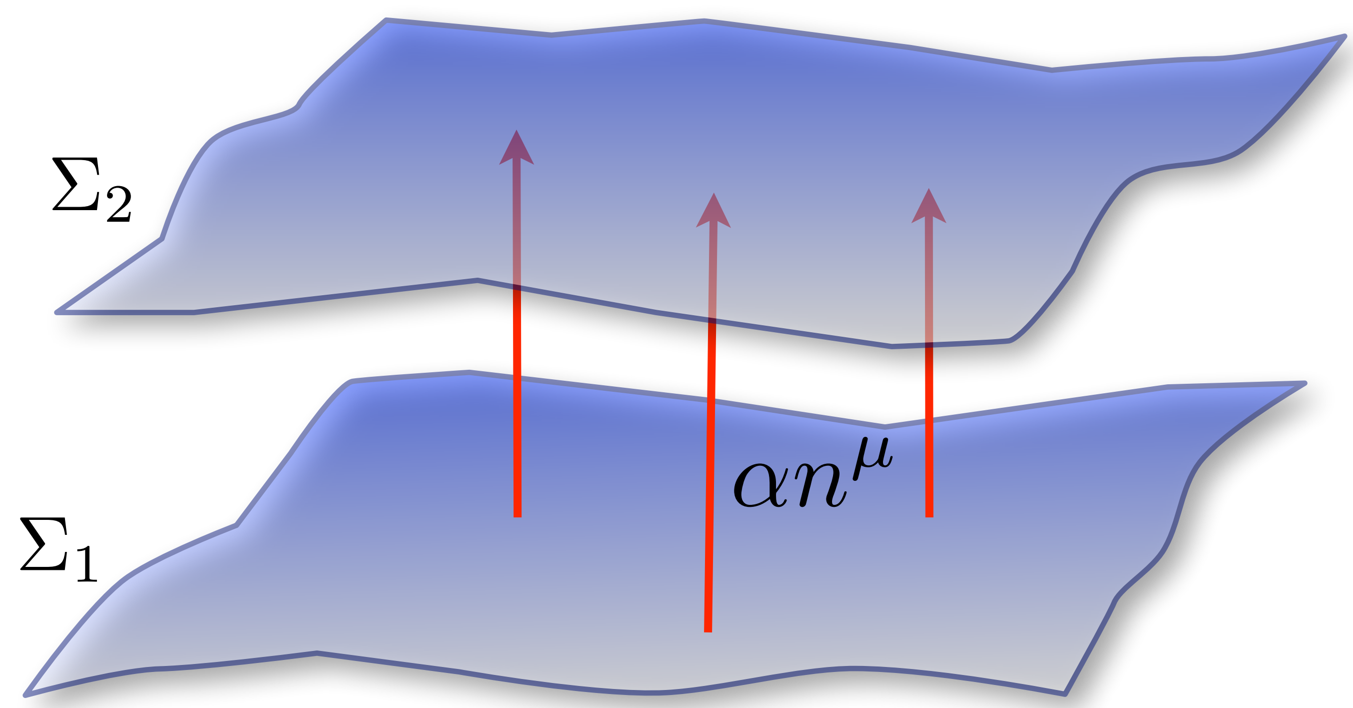
The lapse function allows then to do two important things:

i) define the unit normal vector to the hypersurface Σ

$$n^\mu \equiv -\alpha g^{\mu\nu}\Omega_\nu = -\alpha g^{\mu\nu}\nabla_\nu t$$

where

$$n^\mu n_\mu = -1$$



ii) define the spatial metric

$$\gamma_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu$$

Second step: decompose 4-dim tensors

$\boldsymbol{n}$ and $\boldsymbol{\gamma}$ provide two useful tools to decompose any 4-dim. tensor into a purely spatial part (hence in Σ) and a purely timelike part (hence orthogonal to Σ and aligned with $\boldsymbol{n}$).

The spatial part is obtained after contracting with the spatial projection operator

$$\gamma^{\mu}{}_{\nu} = g^{\mu\alpha}\gamma_{\alpha\nu} = g^{\mu}{}_{\nu} + n^{\mu}n_{\nu} = \delta^{\mu}{}_{\nu} + n^{\mu}n_{\nu}$$

while the timelike part is obtained after contracting with the timelike projection operator

$$N^{\mu}{}_{\nu} = -n^{\mu}n_{\nu}$$

where the two projectors are obviously orthogonal

$$\gamma^{\nu}{}_{\mu}N^{\mu}{}_{\nu} = 0$$

It is now possible to define the 3-dim covariant derivative of a spatial tensor. This is simply the projection on Σ of all the indices of the the 4-dim. covariant derivative

$$D_\alpha T^\beta_{\ \delta} = \gamma^\rho_{\ \alpha}\gamma^\beta_{\ \sigma}\gamma^\tau_{\ \delta}\nabla_\rho T^\sigma_{\ \tau}$$

which, as expected, is compatible with spatial metric

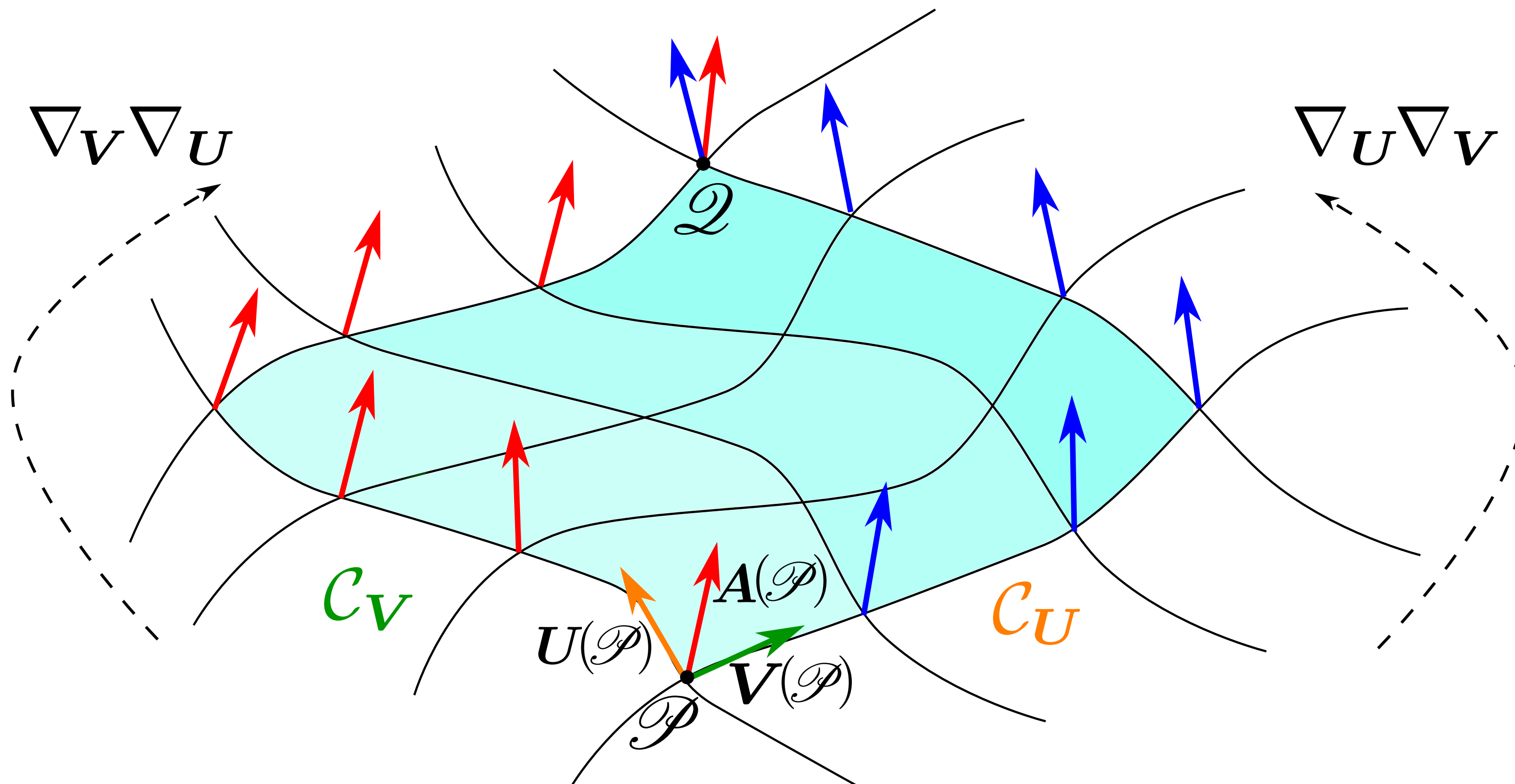
$$D_\alpha \gamma^\beta_{\ \delta} = 0$$

All of the 4-dim tensor algebra can be extended straightforwardly to the 3-dim. spatial slice, so that the 3-dim covariant derivative can be expressed in terms of the 3-dimensional connection coefficients:

$$^{(3)}\Gamma^\alpha_{\ \beta\delta} = \frac{1}{2}\gamma^{\alpha\mu}(\gamma_{\mu\beta,\delta} + \gamma_{\mu\delta,\beta} - \gamma_{\beta\delta\,\mu})$$

The 3-dim Riemann tensor associated with γ is defined can be defined in the same conceptual manner in which one defines the 4-dim Riemann tensor associated with $\boldsymbol{g}$

Given two vector fields $\boldsymbol{U}, \boldsymbol{V}$ and a covariant derivative, compute the difference in the parallel transport around a closed loop



to obtain:

$$2\nabla_{[\alpha}\nabla_{\beta]}W_\delta = R^{\mu}{}_{\delta\alpha\beta}W_\mu$$

Similarly, the 3-dim Riemann tensor associated with $\boldsymbol{\gamma}$ is defined via the double 3-dimensional covariant derivative of any spatial vector $\boldsymbol{W}$, ie

$$2D_{[\alpha}D_{\beta]}W_\delta = {}^{(3)}R^\mu{}_{\delta\alpha\beta}W_\mu$$

where

$${}^{(3)}R^\mu{}_{\delta\alpha\beta}n_\mu = 0 \qquad \text{and} \qquad 2T_{[\alpha\beta]} = T_{\alpha\beta} - T_{\beta\alpha}$$

More explicitly, the 3-dim Riemann tensor can be written in terms of the 3-dim connection coefficients as

$${}^{(3)}R^\alpha{}_{\beta\gamma\delta} = {}^{(3)}\Gamma^\alpha{}_{\beta\delta,\gamma} - {}^{(3)}\Gamma^\alpha{}_{\beta\gamma,\delta} + {}^{(3)}\Gamma^\mu{}_{\beta\delta}{}^{(3)}\Gamma^\alpha{}_{\mu\gamma} - {}^{(3)}\Gamma^\mu{}_{\beta\gamma}{}^{(3)}\Gamma^\alpha{}_{\mu\delta}$$

Also, the 3-dim contractions of the 3-dim Riemann tensor, i.e. the 3-dim Ricci tensor the 3-dim Ricci scalar are respectively given by

$${}^{(3)}R_{\alpha\beta} = {}^{(3)}R^\delta{}_{\alpha\delta\beta} \qquad\qquad {}^{(3)}R = {}^{(3)}R^\delta_\delta$$

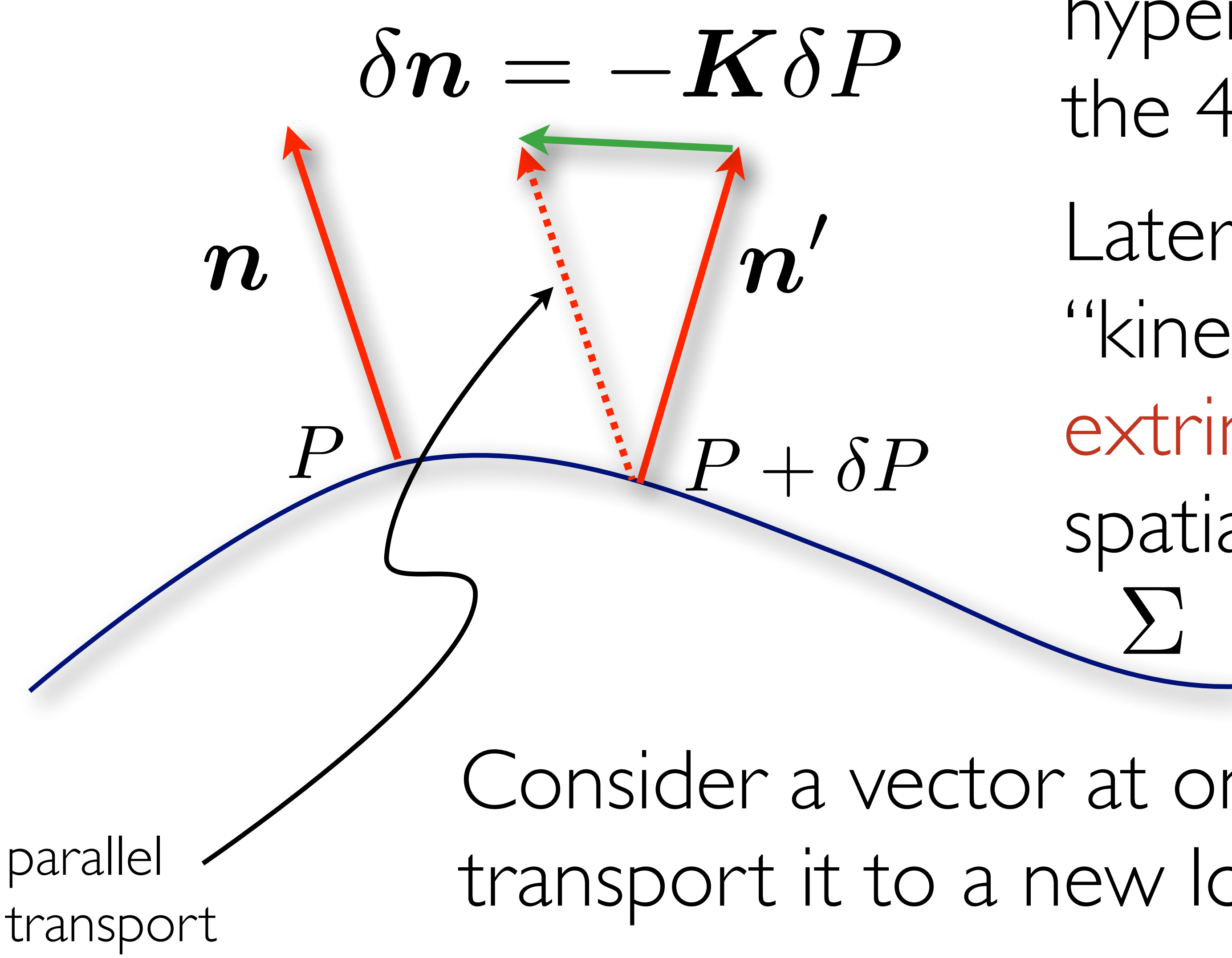
It is important not to confuse the 3-dim Riemann tensor ${}^{(3)}R^{\mu}{}_{\delta\alpha\beta}$ with the corresponding 4-dim one $R^{\mu}{}_{\delta\alpha\beta}$

${}^{(3)}R^{\mu}{}_{\delta\alpha\beta}$ is a 4-dimensional tensor but it is purely spatial (spatial derivatives of spatial metric $\boldsymbol{\gamma}$)

$R^{\mu}{}_{\delta\alpha\beta}$ is a full 4-dimensional tensor containing also time derivatives of the full 4-dim metric $\boldsymbol{g}$

The information present in $R^{\mu}{}_{\delta\alpha\beta}$ and "missing" in ${}^{(3)}R^{\mu}{}_{\delta\alpha\beta}$ can be found in another spatial tensor: the extrinsic curvature.
As we shall see, this information is indeed describing the time evolution of the spatial metric

More geometrically, the extrinsic curvature measures the changes in the normal vector under parallel transport



Hence it measures how the 3-dim hypersurface is "bent" with respect to the 4-dim spacetime

Later on we will discuss also a "kinematical" interpretation of the extrinsic curvature in terms of the spatial metric

Consider a vector at one position P and parallel-transport it to a new location $P + \delta P$

The difference in the two vectors is proportional to the extrinsic curvature and this can be positive or negative

$$K_{\mu\nu} := -\gamma^{\lambda}{}_{\mu} \nabla_{\lambda} n_{\nu}$$

Since the extrinsic curvature measures the bending of the spacelike hypersurface, two more equivalent definitions exists for the extrinsic curvature:

2) in terms of the acceleration of normal observers:

$$K_{\mu\nu} := -\nabla_\mu n_\nu - n_\mu a_\nu = -\nabla_\mu n_\nu - n_\mu n^\lambda \nabla_\lambda n_\nu$$

3) in terms of the Lie derivative of the spatial metric:

$$K_{\mu\nu} := -\frac{1}{2}\mathcal{L}_{\boldsymbol{n}}\gamma_{\mu\nu}$$

1) in terms of the projection of the parallel transported normal

$$K_{\mu\nu} := -\gamma^\lambda{}_\mu \nabla_\lambda n_\nu$$

Finding a direction for evolutions

Note that the unit normal $\boldsymbol{n}$ to a spacelike hypersurface Σ is not the natural time derivative. This is because $\boldsymbol{n}$ is not dual to the surface 1-norm $\boldsymbol{\Omega}$, i.e.

$$n^\mu \Omega_\mu = n^\mu \nabla_\mu t = -\alpha \Omega^\mu \Omega_\mu = \frac{1}{\alpha}$$

We need therefore to find a new vector along which to carry out the time evolutions and that is dual to the surface 1-norm. The most general of such a vector is easily defined as

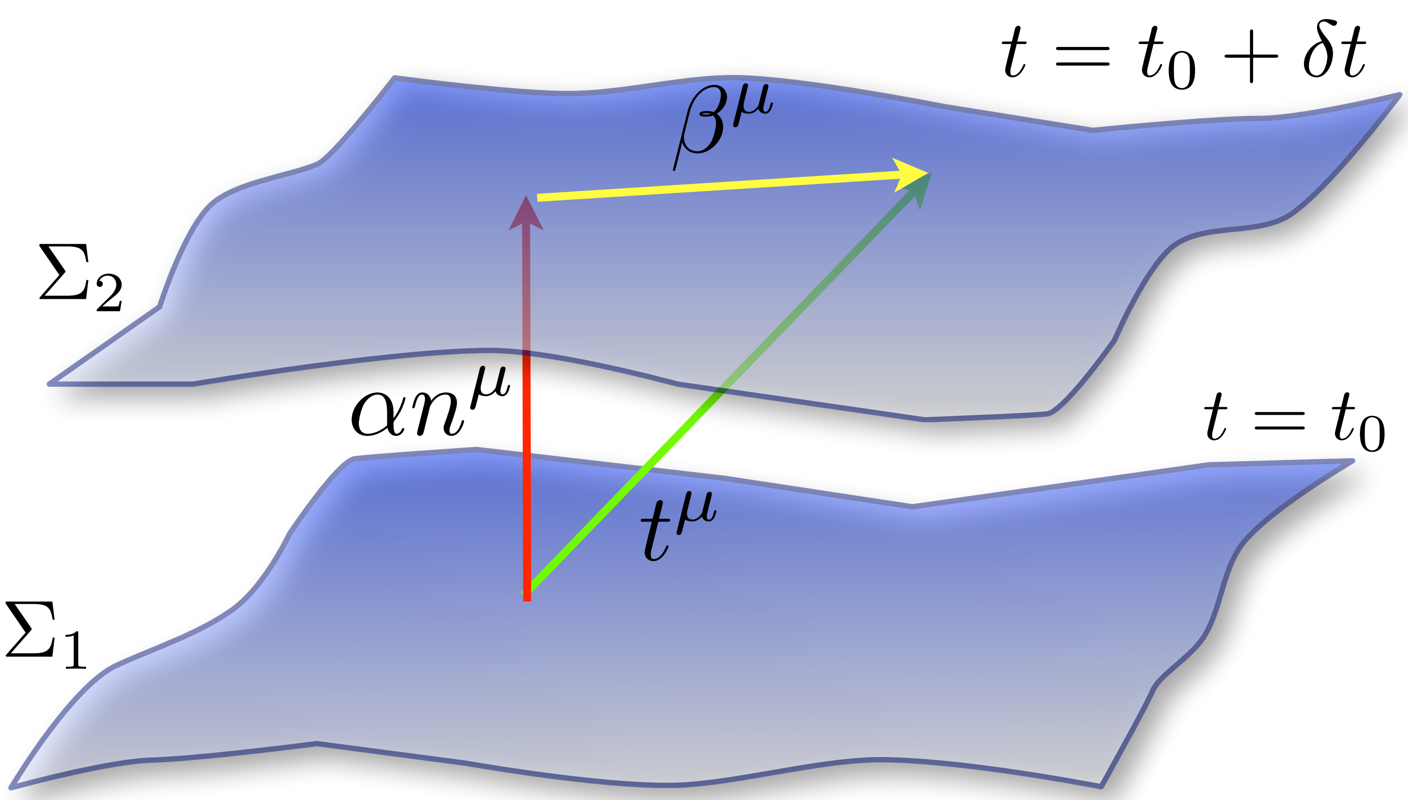
$$t^\mu \equiv \alpha n^\mu + \beta^\mu$$

where $\boldsymbol{\beta}$ is any spatial "shift" vector.

Clearly now the two tensors are dual to each other, ie

$$t^\mu \Omega_\mu = \alpha n^\mu \Omega_\mu + \beta^\mu \Omega_\mu = \alpha/\alpha = 1$$

Because the vector t^μ is dual to the 1-form Ω_μ, we are guaranteed that the integral curves of t^μ are naturally parametrized by the time coordinate.



Stated differently, all infinitesimal vectors t^μ originating on one hypersurface Σ_1 would end up on the same hypersurface Σ_2

This is not guaranteed for translations along Ω^μ

A more intuitive description of the lapse function α and of the shift vector β^μ will be presented once we introduce a coordinate basis

Note that t^μ is not necessarily timelike if the shift is superluminal

$$t^\mu t_\mu = -\alpha^2 + \beta^\mu \beta_\mu \lesseqgtr 0$$

With this definition we can revise the Lie derivative along the unit normal $\mathcal{L}_{\boldsymbol{n}}$. Since

$$\alpha\mathcal{L}_{\boldsymbol{n}} = \mathcal{L}_{\boldsymbol{t}} - \mathcal{L}_{\boldsymbol{\beta}}$$

the Ricci equation we have encountered before: $K_{\alpha\beta} = -\frac{1}{2}\mathcal{L}_{\boldsymbol{n}}\gamma_{\alpha\beta}$ can now be rewritten as

$$\mathcal{L}_{\boldsymbol{t}}\gamma_{\mu\nu} = -2\alpha K_{\mu\nu} + \mathcal{L}_{\boldsymbol{\beta}}\gamma_{\mu\nu} \qquad (*)$$

Once again, this a clear expression that the extrinsic curvature can be seen as the rate of change of the spatial metric, i.e.

$$K_{\mu\nu} \propto -\frac{1}{\alpha}\mathcal{L}_{\boldsymbol{t}}\gamma_{\mu\nu}$$

Finally, note that the Ricci equations (*) are definitions and not pieces of the Einstein eqs, although this is sometimes confused

Selecting a coordinate basis

So far we have dealt with tensor eqs and not specified a coordinate basis with unit vectors $\boldsymbol{e}_j$. Doing so can be useful to simplify equations and to highlight the "spatial" nature of $\boldsymbol{\gamma}$ and $\boldsymbol{K}$

The choice in this case in very simple. We want:
i) three of them have to be purely spatial, i.e.

$$n_\mu(\boldsymbol{e}_j)^\mu = 0 \qquad e.g. \qquad (\boldsymbol{e}_1)^\mu = (0,1,0,0)$$

ii) the fourth one has to be along the vector $\boldsymbol{t}$, i.e.

$$(\boldsymbol{e}_0)^\mu = t^\mu = (1,0,0,0)$$

As a result:

$$\mathcal{L}_{\boldsymbol{t}} = \partial_t$$

i.e. the Lie derivative along $\boldsymbol{t}$ is a simple partial derivative

$$n_j = n_\mu (e_j)^\mu = 0 \quad \text{but} \quad n_0 \neq 0$$

i.e. the space covariant components of a **timelike** vector are zero; only the covariant time component survives

$$n_\mu \beta^\mu = \beta^0 n_0 = 0 \quad \Longrightarrow \quad \beta^0 = 0 \quad \Longrightarrow \quad \beta^\mu = (0, \beta^j)$$

i.e. the time contravariant component of a **spacelike** vector is zero; only the spatial contravariant components survive

Putting things together and bearing in mind that $n_\mu n^\mu = -1$

$$\boxed{n^\mu = \frac{1}{\alpha}(1, -\beta^i)\,;} \qquad \boxed{n_\mu = (-\alpha, 0, 0, 0)}$$

Because for any spatial tensor $T^{\mu 0} = 0$ the contravariant components of the metric in a 3+1 split are

$$g^{\mu\nu} = \begin{pmatrix} -1/\alpha^2 & \beta^i/\alpha^2 \\ \beta^i/\alpha^2 & \gamma^{ij} - \beta^i\beta^j/\alpha^2 \end{pmatrix}$$

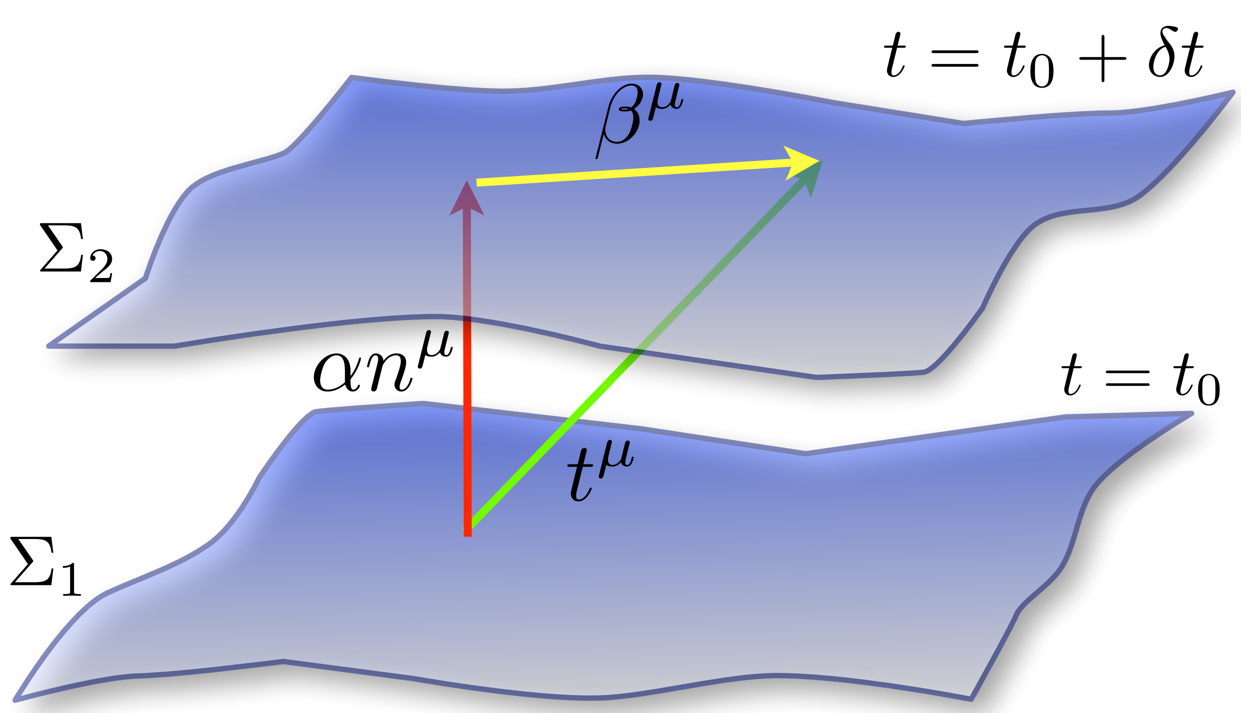
Similarly, since $g_{ij} = \gamma_{ij}$ the covariant components are

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ij} \end{pmatrix}$$

Note that $\gamma^{ik}\gamma_{kj} = \delta^i{}_j$ (i.e. γ^{ij}, γ_{ij} are inverses) and thus they can be used to raise/lower the indices of **spatial** tensors

We can now have a more intuitive interpretation of the lapse, shift and spatial metric. Using the expression for the covariant 4-dim covariant metric, the line element is given

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(\alpha^2 - \beta^i\beta_i)dt^2 + 2\beta_i dx^i dt + \gamma_{ij}dx^i dx^j$$



Hence:

the lapse measures proper time between two adjacent hypersurfaces

$$\mathrm{d}\tau^2 = -\alpha^2(t, x^j)\mathrm{d}t^2$$

the shift relates spatial coordinates between two adjacent hypersurfaces

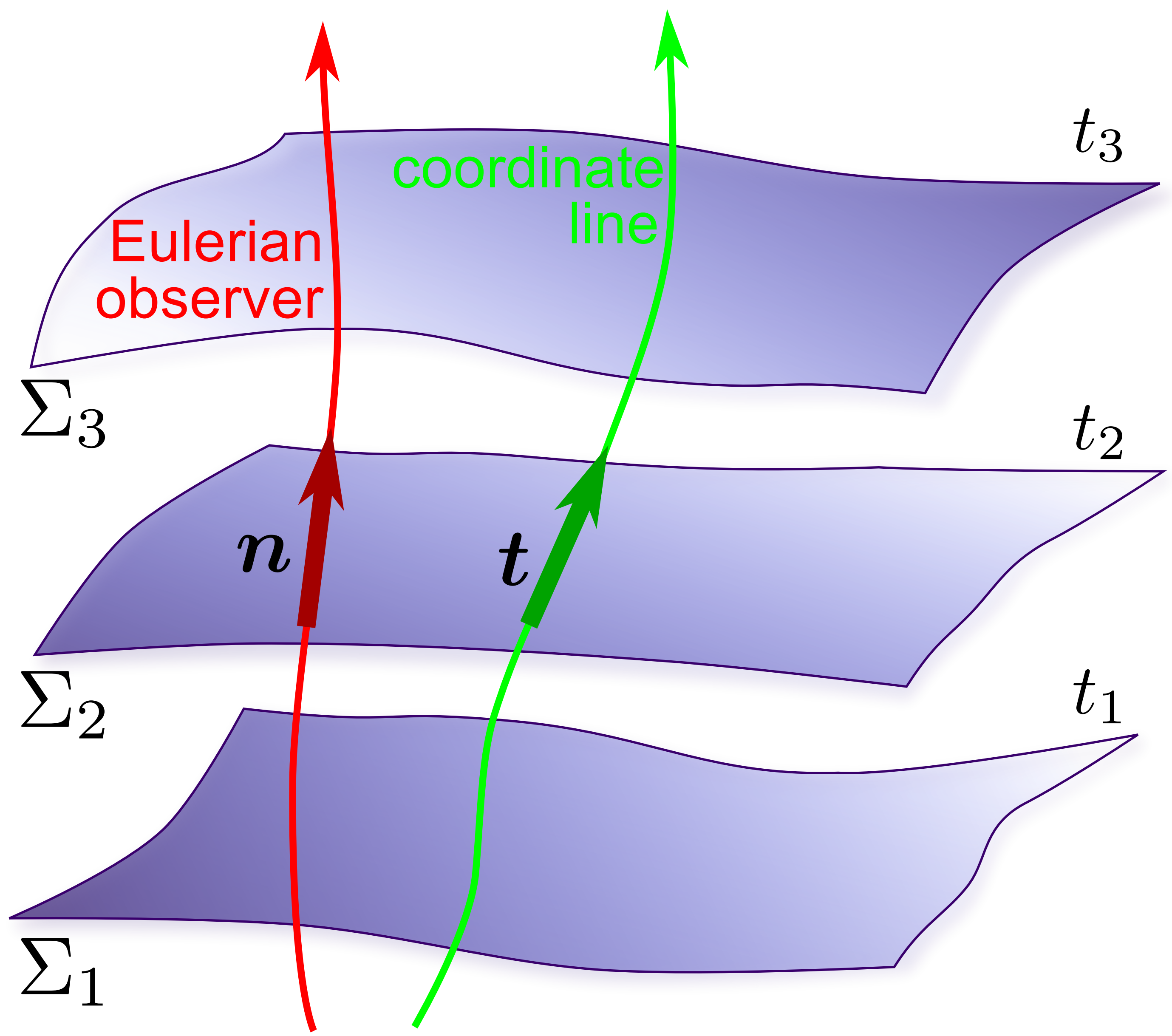
$$x^i_{t_0+\delta t} = x^i_{t_0} - \beta^i(t, x^j)\mathrm{d}t$$

the spatial metric measures distances between points on every hypersurface

$$\mathrm{d}l^2 = \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

We can now also distinguish between a normal line and a coordinate line.

Both are worldlines but the first one tells us about the evolution of the normal to the hypersurface, while the second one tells us about the evolution of a point in the coordinate chart



Recap (I)

☑ The 3+1 splitting of the 4-dim spacetime represents an effective way to perform numerical solutions of the Einstein eqs.

☑ Such a splitting amounts to projecting all 4-dim. tensors either on spatial hypersurfaces or along directions orthogonal to such hypersurfaces.

☑ The 3-metric and the extrinsic curvature describe the properties of each slice.

☑ Two functions, the lapse and the shift, tell how to relate coordinates between two slices: the lapse measures the proper time, while the shift measures changes in the spatial coords.

Decomposing the Einstein equations

- So far we have just played with differential geometry. No mention has been made of the Einstein equations.

- The 3+1 splitting naturally "splits" the Einstein equations into:
  - a set which is fully defined on each spatial hypersurfaces (and does not involve therefore time derivatives).
  - a set which instead relates quantities (i.e. spatial metric and extrinsic curvature) between two adjacent hypersurfaces.

- The first set is usually referred to as the "constraint" equations, while the second one as the "evolution" equation

Next, we need to decompose the Einstein equations in the spatial and timelike parts.

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}$$

and to do this we need to define a few identities

First we decompose the 4-dim Riemann tensor $R_{\alpha\beta\mu\nu}$ projecting all indices to obtain the Gauss equations

$$^{(3)}R_{\alpha\beta\gamma\delta} + K_{\alpha\gamma}K_{\beta\delta} - K_{\alpha\delta}K_{\beta\gamma} = \gamma^{\mu}{}_{\alpha}\gamma^{\nu}{}_{\beta}\gamma^{\rho}{}_{\delta}\gamma^{\sigma}{}_{\gamma}\ R_{\mu\nu\sigma\rho}$$

Next, we make 3 spatial projections and a timelike one to obtain the Codazzi equations

$$D_{\alpha}K_{\beta\gamma} - D_{\beta}K_{\alpha\gamma} = \gamma^{\rho}{}_{\beta}\gamma^{\mu}{}_{\alpha}\gamma^{\nu}{}_{\gamma}n^{\sigma}R_{\mu\nu\sigma\rho}$$

Finally we take 2 spatial projections and 2 timelike ones to obtain the Ricci equations

$$\mathcal{L}_{\boldsymbol{n}} K_{\alpha\beta} = n^{\delta} n^{\gamma} \gamma^{\mu}{}_{\alpha} \gamma^{\nu}{}_{\beta} R_{\nu\delta\mu\gamma} - \frac{1}{\alpha} D_{\alpha} D_{\beta} \alpha - K^{\gamma}{}_{\beta} K_{\alpha\gamma}$$

where the second derivative of the lapse has been introduced via the identity

$$a_{\mu} = D_{\mu} \ln \alpha$$

Another important identity which will be used in the following is

$$D_{\mu} U^{\nu} = \gamma_{\mu}{}^{\rho} \nabla_{\rho} U^{\nu} + K_{\mu\rho} U^{\rho} n^{\nu}$$

and which holds for any spatial vector $\boldsymbol{U}$ $(U^{\mu} n_{\mu} = 0)$

The evolution part of the Einstein equations

We are now ready to express the missing piece of the 3+1 decomposition and derive the evolution part of the Einstein eqs.

We need suitable projections of the right-hand-side of the Einstein equations and in particular the two spatial ones, ie

$$\gamma^{\mu}{}_{\alpha}\gamma^{\nu}{}_{\beta}G_{\mu\nu} = 8\pi S_{\alpha\beta} \equiv 8\pi\gamma^{\mu}{}_{\alpha}\gamma^{\nu}{}_{\beta}T_{\mu\nu}$$

where the energy-momentum tensor of a perfect fluid is:

$$T_{\mu\nu} = (e+p)u_{\mu}u_{\nu} + pg_{\mu\nu} = h\rho u_{\mu}u_{\nu} + pg_{\mu\nu}$$

with

ρ : rest-mass density

p : pressure

ϵ : specific internal energy

$e = \rho(1+\epsilon)$: total energy density

$h = \dfrac{e+p}{\rho}$: specific enthalpy

$S \equiv S^{\mu}{}_{\mu}$

Since $n^\mu u_\mu = 1$, (the two vectors are parallel and unit vectors) the energy density measured by the normal observers will be given by the double timelike projection

$$e = n^\mu n^\nu T_{\mu\nu}$$

Similarly, the momentum density (i.e. the extension of the Newtonian mass current) will be given by the mixed time and spatial projection

$$j_\mu = -\gamma^\alpha{}_\mu n^\beta T_{\alpha\beta} = -(h\rho u_\mu + p n_\mu)$$

Just as a reminder, the fully spatial projection of the energy-momentum tensor was already introduced as

$$S_{\mu\nu} = \gamma^\alpha{}_\mu \gamma^\beta{}_\nu T_{\alpha\beta}$$

The (ADM) Einstein eqs in 3+1

In such a foliation, we can write the Einstein eqs in the 3+1 splitting of spacetime in a set of evolution and constraint equations as:

$$\boldsymbol{\gamma} \cdot \boldsymbol{\gamma} \cdot (\text{Einstein eqs}) \; + \; \text{Ricci eqs} \Longrightarrow$$

$$\boxed{\begin{aligned}\partial_t K_{ij} = -D_i D_j \alpha + \alpha(R_{ij} - 2K_{ik}K^{kj} + KK_{ij}) \\ -8\pi\alpha(R_{ij} - \frac{1}{2}\gamma_{ij}(S-e)) + \mathcal{L}_{\boldsymbol{\beta}} K_{ij}\end{aligned}}$$

[6 eqs]

$$\boxed{\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_{\boldsymbol{\beta}} \gamma_{ij}}$$

[6 eqs]

These are 12 hyperbolic, first-order in time, second-order in space, nonlinear partial differential equations: evolution equations

The constraint equations (I)

We first time-project twice the left-hand-side of the Einstein equations to obtain

$$2n^\mu n^\nu G_{\mu\nu} = {}^{(3)}R + K^2 - K_{\mu\nu}K^{\mu\nu}$$

Doing the same for the right-hand-side, using the Gauss eqs contracted twice with the spatial metric and the definition of the energy density we finally reach the form of the Hamiltonian constraint equation

$$\boxed{{}^{(3)}R + K^2 - K_{\mu\nu}K^{\mu\nu} = 16\pi e}$$

Note that this is a single elliptic equation (hence not containing time derivative) which should be satisfied everywhere on the spatial hypersurface Σ

The constraint equations (II)

Similarly, with a mixed time-space projection of the left-hand-side of the Einstein equations we obtain

$$-\gamma^{\mu}{}_{\alpha} n^{\nu} G_{\mu\nu} = -{}^{(3)}R_{\alpha\nu} n^{\nu} + \frac{1}{2} n_{\alpha} R$$

Doing the same for the right-hand-side, using the contracted Codazzi equations and the definition of the momentum density we finally reach the form of the momentum constraint equations

$$\boxed{D_{\nu} K^{\nu}{}_{\mu} - D_{\mu} K = 8\pi j_{\mu}}$$

which are also 3 elliptic equations.

The 4 constraint equations are the necessary and sufficient integrability conditions for the embedding of the spacelike hypersurfaces $(\Sigma, \gamma_{\mu\nu}, K_{\mu\nu})$ in the 4-dim. spacetime $(\mathcal{M}, g_{\mu\nu})$

The (ADM) Einstein eqs in 3+1

Similarly

$$\boldsymbol{n} \cdot \boldsymbol{n} \cdot (\text{Einstein eqs}) \; + \; \text{Gauss eqs} \Longrightarrow$$

$$R + K^2 - K_{ij}K^{ij} = 16\pi e$$

Hamiltonian Constraint (HC) **[1 eq]**

$$\boldsymbol{\gamma} \cdot \boldsymbol{n} \cdot (\text{Einstein eqs}) \; + \; \text{Codazzi eqs} \Longrightarrow$$

$$D_j K^j{}_i - D_i K = 8\pi j_i$$

Momentum Constraints (MC) **[3 eqs]**

These are 1+3 elliptic (second-order in space), nonlinear partial differential equations: constraint equations

The (ADM) Einstein eqs in 3+1

All together we have:

$$\partial_t K_{ij} = -D_i D_j \alpha + \alpha(R_{ij} - 2K_{ik}K^{kj} + KK_{ij}) \\ -8\pi\alpha(R_{ij} - \frac{1}{2}\gamma_{ij}(S-e)) + \mathcal{L}_{\boldsymbol{\beta}} K_{ij}$$ [6]

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_{\boldsymbol{\beta}} \gamma_{ij}$$ [6]

$$R + K^2 - K_{ij}K^{ij} = 16\pi e$$ [1]

$$D_j K^j{}_i - D_i K = 8\pi j_i$$ [3]

These 6+6 (+3+1) eqs are also known as the ADM equations. In practice, only the evolution eqs are solved and the constraints are instead monitored (more later)

ADM vs Maxwell

The ADM eqs may appear as rather cryptic and simply complicated. However, it is easy to see analogies with the Maxwell eqs. and make the equations less cryptic.
The relevant quantities in this case are the electric and magnetic fields $\boldsymbol{E}, \boldsymbol{B}$, the charge density ρ_e and the charge current density $\boldsymbol{J}$

Then also the Maxwell equations split into **evolution** equations

$$\partial_t \boldsymbol{E} = \nabla \times \boldsymbol{B} - 4\pi \boldsymbol{J}\,, \quad \Longleftrightarrow \quad \partial_t E_i = \epsilon_{ijk} D^j B^k - 4\pi J_i\,,$$

$$\partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E}\,, \quad \Longleftrightarrow \quad \partial_t B_i = -\epsilon_{ijk} D^j E^k$$

and **constraint** equations

$$\nabla \cdot \boldsymbol{E} = 4\pi \rho_e\,, \quad \Longleftrightarrow \quad \partial_i E^i = 4\pi \rho_e\,,$$

$$\nabla \cdot \boldsymbol{B} = 0\,, \quad \Longleftrightarrow \quad \partial_i B^i = 0\,,$$

Also for the Maxwell eqs it is possible to show that **if** the constraints are satisfied initially, then the evolution eqs preserve this property. To further highlight the analogies let's introduce the vector potential

$$A_\mu = (\Phi, A_i)\,, \qquad \text{such that} \qquad B_i = \epsilon_{ijk} D^j A^k$$

and so that the Maxwell **evolution** equations become

$$\partial_t A_i = -E_i - D_i \Phi\,,$$

$$\partial_t E_i = -D^j D_j A_i + D_i D^j A_j - 4\pi J_i$$

to be compared with the ADM **evolution** eqs

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_{\boldsymbol{\beta}} \gamma_{ij}$$

$$\begin{aligned}\partial_t K_{ij} = & -D_i D_j \alpha + \alpha(R_{ij} - 2K_{ik}K^{kj} + KK_{ij}) \\ & -8\pi\alpha(R_{ij} - \frac{1}{2}\gamma_{ij}(S - e)) + \mathcal{L}_{\boldsymbol{\beta}} K_{ij}\end{aligned}$$

It is then possible to make the associations

$$\begin{aligned} \Phi &\longleftrightarrow \beta_i \\ A_i &\longleftrightarrow \gamma_{ij} \\ E_i &\longleftrightarrow K_{ij} \end{aligned}$$

and realize that the RHSs of the evolution equation of A_i/γ_{ij} involve a field variable E_i/K_{ij} and the spatial derivatives of a gauge quantity Φ/β_i

Similarly, the RHS of the evolution equation of E_i/K_{ij} involve matter sources as well as second spatial derivatives of the second field variable A_i/γ_{ij}

Indeed, the similarities between the ADM eqs and the Maxwell eqs written in terms of the vector potential (i.e. as in previous slide) are so large that they suffer of the same problems/instabilities (more later)

In practice, the ADM are essentially never used!

These equations are perfectly alright mathematically but not in a form that is well suited for numerical implementation.

Indeed the system can be shown to be weakly hyperbolic and hence "ill-posed"

In practice, numerical instabilities rapidly appear that destroy the solution exponentially

However, the stability properties of numerical implementations can be improved by introducing certain new auxiliary functions and rewriting the ADM equations in terms of these functions.

The same is done for the ADM eqs and new evolution variables are introduced to obtain a set of eqs that is strongly hyperbolic and hence well-posed (doesn't blow up).

$$\phi = \frac{1}{12}\ln(\det(\gamma_{ij})) = \frac{1}{12}\ln(\gamma),$$

ϕ : conformal factor

$$\tilde{\gamma}_{ij} = e^{-4\phi}\gamma_{ij},$$

$\tilde{\gamma}_{ij}$: conformal 3-metric

$$K = \gamma^{ij}K_{ij},$$

K : trace of extrinsic curvature

$$\tilde{A}_{ij} = e^{-4\phi}(K_{ij} - \frac{1}{3}\gamma_{ij}K),$$

$\tilde{A}_{ij}$: trace-free conformal extrinsic curvature

$$\Gamma^i = \gamma^{jk}\Gamma^i_{jk}$$

$$\tilde{\Gamma}^i = \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk}$$

$\tilde{\Gamma}^i$:"Gammas"

are our new evolution variables

The ADM equations are then rewritten as

$$\mathcal{D}_t \tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} \;, \qquad \text{where } \mathcal{D}_t \equiv \partial_t - \mathcal{L}_\beta$$

$$\mathcal{D}_t \phi = -\frac{1}{6}\alpha K \;,$$

$$\mathcal{D}_t \tilde{A}_{ij} = e^{-4\phi}\left[-\nabla_i \nabla_j \alpha + \alpha \left(R_{ij} - S_{ij}\right)\right]^{\mathrm{TF}} + \alpha \left(K \tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}^l_j\right) \;,$$

$$\mathcal{D}_t K = -\gamma^{ij}\nabla_i \nabla_j \alpha + \alpha \left[\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2 + \frac{1}{2}\left(\rho + S\right)\right] \;,$$

$$\begin{aligned}\mathcal{D}_t \tilde{\Gamma}^i = {} & -2\tilde{A}^{ij}\partial_j \alpha + 2\alpha \left(\tilde{\Gamma}^i_{jk}\tilde{A}^{kj} - \frac{2}{3}\tilde{\gamma}^{ij}\partial_j K - \tilde{\gamma}^{ij}S_j + 6\tilde{A}^{ij}\partial_j \phi\right) \\ & -\partial_j \left(\beta^l \partial_l \tilde{\gamma}^{ij} - 2\tilde{\gamma}^{m(j}\partial_m \beta^{i)} + \frac{2}{3}\tilde{\gamma}^{ij}\partial_l \beta^l\right) \;.\end{aligned}$$

These equations are also known as the BSSNOK equations or more simply the conformal traceless formulation of the Einstein equations.

Although not self evident, the BSSNOK equations are strongly hyperbolic with a structure which is resembling the 1st-order in time, 2nd-order in space formulation

$$\Box\phi = 0 \quad \Longleftrightarrow \begin{cases} \partial_t \phi = \psi \\ \\ \partial_t \psi = \partial^i \partial_i \phi \end{cases}$$

scalar wave equation

$$\begin{cases} \partial_t \tilde{\gamma}_{ij} \propto \tilde{A}_{ij} \\ \\ \partial_t \tilde{A}_{ij} \propto D^i D_i \tilde{\gamma}_{ij} \end{cases}$$

BSSNOK

The BSSNOK is a widely used formulation of the Einstein eqs and used to simulate black holes and neutron stars. Other formulations have been recently suggested that have even better properties, e.g. CCZ4, Z4c.

Recap (II)

- ☑ A number of tensor differential identities allow to cast the Einstein equations in a 3+1 split: this is the ADM formulation.

- ☑ Einstein equations in the ADM formulation naturally split into evolution equations and constraint equations.

- ☑ This is not very different from Maxwell equations, where there are also evolution and constraint equations.

- ☑ The ADM eqs are ill posed and not suitable for numerics.

- ☑ Alternative formulations (BSSNOK, CCZ4, Z4c) have been developed that are strongly hyperbolic and hence well-posed.

Recap (III)

☑ Both CCZ4, Z4c formulations make use of the constraint equations and can use additional evolution equations to damp the violations

☑ The hyperbolic evolution eqs. to solve are: 6+6+(3+1+1) = 17. We also "compute" 3+1=4 elliptic constraint eqs

$$\mathcal{H} \equiv {}^{(3)}R + K^2 - K_{ij}K^{ij} = 0 \; , \quad \text{(Hamiltonian constraint)}$$

$$\mathcal{M}^i \equiv D_j(K^{ij} - g^{ij}K) = 0 \; , \quad \text{(momentum constraints)}$$

NOTE: these eqs are not solved but only monitored to verify

$$||\mathcal{H}|| \simeq ||\mathcal{M}^i|| < \varepsilon \sim 10^{-4} - 10^{-2}$$